General Mathematics Vol. 13, No. 2 (2005), 95–104

Estimations of the Error for Two-point Formula via Pre-Grüss Inequality

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

Generalization of estimation of two-point formula is given, by using pre-Grüss inequality.

2000 Mathematical Subject Classification: 26D10, 41A55.

Key words and phrases: Pre-Grüss inequality, generalization, two-point formula.

1 Introduction

In the recent paper [4] N. Ujević use the generalization of pre-Grüss inequality to derive some better estimations of the error for Simpson's quadrature rule. In fact, he proved the next as his main result:

Theorem 1. *If* $g, h, \Psi \in L_2(0,1)$ *and* $\int_0^1 \Psi(t)dt = 0$ *then we have*

$$|S_\Psi(g,h)| \le S_\Psi(g,g)^{1/2} S_\Psi(h,h)^{1/2}, \tag{1}$$

where

$$S_\Psi(g,h) = \int_0^1 g(t)h(t)dt - \int_0^1 g(t)dt \int_0^1 h(t)dt - \int_0^1 g(t)\Psi_0(t)dt \int_0^1 h(t)\Psi_0(t)dt$$

and $\Psi_0(t) = \Psi(t)/\|\Psi\|_2$.

Further, he gave some improvements of the Simpson's inequality. For example he get:

Theorem 2. *Let* $I \subset \mathbb{R}$ *be a closed interval and* $a, b \in IntI,\ a < b$. *If* $f : I \to \mathbb{R}$ *is an absolutely continuous function with* $f' \in L_2(a,b)$ *then we have*

$$\left| \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int_a^b f(t)dt \right| \le \frac{(b-a)^{3/2}}{6} K_1, \tag{2}$$

where

$$K_1^2 = \|f'\|_2^2 - \frac{1}{b-a}\left(\int_a^b f'(t)dt\right)^2 - \left(\int_a^b f'(t)\Psi_0(t)dt\right)^2 \tag{3}$$

and $\Psi(t) = t - \frac{a+b}{2}$, $\Psi_0(t) = \Psi(t)/\|\Psi\|_2$.

In this paper using the Theorem 1 we will give the similar result for Euler two-point formula and for functions whose derivative of order $n,\ n \ge 1$, is from $L_2(0,1)$ space. We will use interval $[0,1]$ because of simplicity and since it involves no loss in generality.

2 Estimations of the error for Euler two-point formula

In the recent paper [3] the following identity, named Euler two-point formula, has been proved. For $n \geq 1$, $x \in \left[0, \frac{1}{2}\right]$ and every $t \in [0,1]$ we have

$$\int_0^1 f(t)\mathrm{d}t = D(x) - T_n(x) + R_n(x) \tag{4}$$

where

$$D(x) = \frac{1}{2}\left[f(x) + f(1-x)\right],$$

$T_0(x) = 0$ and

$$T_m(x) = \frac{1}{2}\sum_{k=1}^{m} \frac{\tilde{B}_k(x)}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right], \tag{5}$$

for $1 \leq m \leq n$ and $x \in \left[0, \frac{1}{2}\right]$, while

$$\tilde{B}_k(x) = B_k(x) + B_k(1-x),\ k \geq 1,$$

$$R_n(x) = \frac{1}{2(n!)}\int_0^1 G_n^x(t)\, f^{(n)}(t)dt$$

and

$$G_n^x(t) = B_n^*(x-t) + B_n^*(1-x-t),\ t \in \mathbb{R}.$$

The identity holds for every function $f : [0,1] \to \mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$. The functions $B_k(t)$ are the Bernoulli polynomials, $B_k = B_k(0)$ are the Bernoulli numbers, and $B_k^*(t)$, $k \geq 0$, are periodic functions of period 1, related to the Bernoulli polynomials as

$$B_k^*(t) = B_k(t),\ 0 \leq t < 1 \quad \text{and} \quad B_k^*(t+1) = B_k^*(t),\ t \in \mathbb{R}.$$

The Bernoulli polynomials $B_k(t)$, $k \geq 0$ are uniquely determined by the following identities

$$B_k'(t) = kB_{k-1}(t),\ k \geq 1;\ B_0(t) = 1,\ B_k(t+1) - B_k(t) = kt^{k-1},\ k \geq 0.$$

For some further details on the Bernoulli polynomials and the Bernoulli numbers see for example [1] or [2]. We have $B_0^*(t) = 1$ and $B_1^*(t)$ is a discontinuous function with a jump of -1 at each integer. It follows that $B_k(1) = B_k(0) = B_k$ for $k \geq 2$, so that $B_k^*(t)$ are continuous functions for $k \geq 2$. We get

$$B_k^{*\prime}(t) = kB_{k-1}^*(t),\ k \geq 1 \tag{6}$$

for every $t \in \mathbb{R}$ when $k \geq 3$, and for every $t \in \mathbb{R} \setminus \mathbb{Z}$ when $k = 1, 2$.

Theorem 3. *If $f : [0,1] \to \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous function with $f^{(n)} \in L_2(0,1)$ then we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) + T_n(x) \right| \leq \frac{1}{2} \left[\frac{2(-1)^{n-1}}{(2n)!} \left[B_{2n} + B_{2n}(1-2x) \right] \right]^{1/2} K, \tag{7}$$

where

$$K^2 = \|f^{(n)}\|_2^2 - \left(\int_0^1 f^{(n)}(t)dt \right)^2 - \left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt \right)^2. \tag{8}$$

For n even

$$\Psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right] \\ -1, & t \in \left(\frac{1}{2}, 1\right] \end{cases},$$

while for n odd we have

$$\Psi(t) = \begin{cases} t + \frac{B_{n+1}\left(\frac{1}{2}+x\right)}{2\left(B_{n+1}(x) - B_{n+1}\left(\frac{1}{2}+x\right)\right)}, & t \in \left[0, \frac{1}{2}\right], \\ t + \frac{B_{n+1}\left(\frac{1}{2}+x\right) - 2B_{n+1}(x)}{2\left(B_{n+1}(x) - B_{n+1}\left(\frac{1}{2}+x\right)\right)}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Proof. It is not difficult to verify that

$$\int_0^1 G_n(t)dt = 0, \tag{9}$$

$$\int_0^1 \Psi(t)dt = 0, \tag{10}$$

$$\int_0^1 G_n(t)\Psi(t)dt = 0. \tag{11}$$

From (4), (9) and (11) it follows that

$$\begin{aligned} \int_0^1 f(t)\mathrm{d}t - D(x) + T_n(x) &= \frac{1}{2(n!)} \int_0^1 G_n^x(t) f^{(n)}(t)dt - \\ &- \frac{1}{2(n!)} \int_0^1 G_n^x(t)dt \int_0^1 f^{(n)}(t)dt - \\ &- \frac{1}{2(n!)} \int_0^1 G_n^x(t)\Psi_0(t)dt \int_0^1 f^{(n)}(t)\Psi_0(t)dt = \\ &= \frac{1}{2(n!)} S_\Psi(G_n^x, f^{(n)}). \end{aligned} \tag{12}$$

Using (12) and (1) we get

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) + T_n(x) \right| \leq \frac{1}{2(n!)} S_\Psi(G_n^x, G_n^x)^{1/2} S_\Psi(f^{(n)}, f^{(n)})^{1/2}. \tag{13}$$

We also have (see [3])

$$\begin{aligned} S_\Psi(G_n^x, G_n^x) &= \|G_n^x\|_2^2 - \left(\int_0^1 G_n^x(t)dt \right)^2 - \left(\int_0^1 G_n^x(t)\Psi_0(t)dt \right)^2 = \\ &= (-1)^{n-1} \frac{2(n!)^2}{(2n)!} \left[B_{2n} + B_{2n}(1-2x) \right] \end{aligned} \tag{14}$$

and

(15)

$$S_\Psi(f^{(n)}, f^{(n)}) = \|f^{(n)}\|_2^2 - \left(\int_0^1 f^{(n)}(t)dt \right)^2 - \left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt \right)^2 = K^2.$$

From (13)-(15) we easily get (7).

Remark 1. *Function $\Psi(t)$ can be any function witch satisfies conditions $\int_0^1 \Psi(t)dt = 0$ and $\int_0^1 G_n^x(t)\Psi(t)dt = 0$. Because $G_n^x(1-t) = (-1)^n G_n^x(t)$ (see [3]), for n even we can take function $\Psi(t)$ such that $\Psi(1-t) = -\Psi(t)$. For n odd we have to calculate $\Psi(t)$ and with not lost in generality in our theorem we take the form* $\Psi(t) = \begin{cases} t+a, & t \in \left[0, \frac{1}{2}\right], \\ t+b, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$

Remark 2. *For $n = 1$ in Theorem 3 we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) \right| \le \frac{1}{2} \left[\frac{1}{3} - 2x + 4x^2 \right]^{1/2} K, \tag{16}$$

while

$$\Psi(t) = \begin{cases} t + \frac{1-12x^2}{24x-6}, & t \in \left[0, \frac{1}{2}\right], \\ t + \frac{12x^2-24x+5}{24x-6}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Also, for $n = 2$ we have

$$\left| \int_0^1 f(t)\mathrm{d}t - D(x) \right| \le \frac{1}{2} \left[\frac{1}{180} - \frac{x^2}{3} + \frac{4x^3}{3} - \frac{4x^4}{3} \right]^{1/2} K, \tag{17}$$

while

$$\Psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right], \\ -1, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

If in Theorem 3 we choose $x = 0, 1/2, 1/3, 1/4$ we get inequality related to the trapezoid, the midpoint, the two-point Newton-Cotes and the two-point MacLaurin formula:

Corollary 1. *If $f : [0,1] \to \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous function with $f^{(n)} \in L_2(0,1)$ then we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - \frac{1}{2}[f(0) + f(1)] + T_n(0) \right| \le \left[\frac{(-1)^{n-1}}{(2n)!} B_{2n} \right]^{1/2} K, \tag{18}$$

where $T_0(0) = 0$,

$$T_n(0) = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!} \left[f^{(2k-1)}(1) - f^{(2k-1)}(0) \right]$$

and

$$K^2 = \|f^{(n)}\|_2^2 - \left(\int_0^1 f^{(n)}(t)dt \right)^2 - \left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt \right)^2 .$$

For n *even*

$$\Psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right], \\ -1, & t \in \left(\frac{1}{2}, 1\right], \end{cases}$$

while for n *odd we have*

$$\Psi(t) = \begin{cases} t + \frac{2^{-n}-1}{4-2^{1-n}}, & t \in \left[0, \frac{1}{2}\right], \\ t + \frac{2^{-n}-3}{4-2^{1-n}}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Remark 3. *For* $n = 1$ *in Corollary 1 we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - \frac{1}{2}[f(0) + f(1)] \right| \le \frac{K}{2\sqrt{3}},$$

while

$$\Psi(t) = \begin{cases} t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right], \\ t - \frac{5}{6}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Corollary 2. *If* $f : [0,1] \to \mathbb{R}$ *is such that* $f^{(n-1)}$ *is absolutely continuous function with* $f^{(n)} \in L_2(0,1)$ *then we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - f\left(\frac{1}{2}\right) + T_n\left(\frac{1}{2}\right) \right| \le \left[\frac{(-1)^{n-1}}{(2n)!} B_{2n} \right]^{1/2} K, \tag{19}$$

where $T_0\left(\frac{1}{2}\right) = 0$,

$$T_n\left(\frac{1}{2}\right) = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(2^{1-2k} - 1)B_{2k}}{(2k)!} \left[f^{(2k-1)}(1) - f^{(2k-1)}(0) \right]$$

and

$$K^2 = \|f^{(n)}\|_2^2 - \left(\int_0^1 f^{(n)}(t)dt\right)^2 - \left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt\right)^2 .$$

For n even

$$\Psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right], \\ -1, & t \in \left(\frac{1}{2}, 1\right], \end{cases}$$

while for n odd we have

$$\Psi(t) = \begin{cases} t + \frac{1}{2^{1-n}-4}, & t \in \left[0, \frac{1}{2}\right], \\ t + \frac{3-2^{1-n}}{2^{1-n}-4}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Remark 4. *For $n = 1$ in Corollary 2 we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - f\left(\frac{1}{2}\right) \right| \le \frac{K}{2\sqrt{3}},$$

while

$$\Psi(t) = \begin{cases} t - \frac{1}{3}, & t \in \left[0, \frac{1}{2}\right], \\ t - \frac{2}{3}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Corollary 3. *If $f : [0,1] \to \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous function with $f^{(n)} \in L_2(0,1)$ then we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) \right] + T_n\left(\frac{1}{3}\right) \right| \le \tag{20}$$

$$\le \frac{1}{2}\left[\frac{(-1)^{n-1}}{(2n)!}(1 + 3^{1-2n})B_{2n} \right]^{1/2} K,$$

where $T_0\left(\frac{1}{3}\right) = 0$,

$$T_n\left(\frac{1}{3}\right) = \frac{1}{2}\sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(3^{1-2k}-1)B_{2k}}{(2k)!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0) \right]$$

and

$$K^2 = \|f^{(n)}\|_2^2 - \left(\int_0^1 f^{(n)}(t)dt\right)^2 - \left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt\right)^2.$$

For n *even*

$$\Psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right], \\ -1, & t \in \left(\frac{1}{2}, 1\right], \end{cases}$$

while for n *odd we have*

$$\Psi(t) = \begin{cases} t + \frac{1-2^n}{2^{2+n}-2}, & t \in \left[0, \frac{1}{2}\right], \\ t + \frac{1-3\cdot 2^n}{2^{2+n}-2n}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Remark 5. *For* $n = 1$ *in Corollary 3 we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right]\right| \le \frac{K}{6},$$

while

$$\Psi(t) = \begin{cases} t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right], \\ t - \frac{5}{6}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Corollary 4. *If* $f : [0,1] \to \mathbb{R}$ *is such that* $f^{(2m-1)}$ *is absolutely continuous function with* $f^{(2m)} \in L_2(0,1)$ *then we have*

(21)
$$\left|\int_0^1 f(t)\mathrm{d}t - \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right] + T_{2m}\left(\frac{1}{4}\right)\right| \le \left[\frac{-2^{-4m}}{(4m)!}B_{4m}\right]^{1/2} K,$$

where $T_0\left(\frac{1}{4}\right) = 0$,

$$T_{2m}\left(\frac{1}{4}\right) = \sum_{k=1}^{m} \frac{2^{-2k}(2^{1-2k}-1)B_{2k}}{(2k)!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]$$

and

$$K^2 = \|f^{(2m)}\|_2^2 - \left(\int_0^1 f^{(2m)}(t)dt\right)^2 - \left(\int_0^1 f^{(2m)}(t)\Psi_0(t)dt\right)^2,$$

while

$$\Psi(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right], \\ -1, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

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