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Grüss Type Inequalities for Forward Difference of Vectors in Inner Product Spaces

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Dedicated to Professor Dumitru Acu on the occasion of his 60th birthday

Abstract

Some Grüss type inequalities for two sequences of vectors in terms of the forward difference are given. An application for the Jensen inequality for convex functions defined on inner product spaces is also pointed out.

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1 Introduction

In [1], we have proved the following generalisation of the Grüss inequality.

Theorem 1. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be an inner product space over* $\mathbb{K}$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$ *and* $e \in H$, $\|e\| = 1$. *If* $\phi, \Phi, \gamma, \Gamma \in \mathbb{K}$ *and* $x, y \in H$ *are such that*

$$\operatorname{Re} \langle \Phi e - x, x - \phi e \rangle \geq 0 \quad \textit{and} \quad \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0,$$

hold, then we have the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma| .$$

The constant $\frac{1}{4}$ *is the best possible.*

A Grüss type inequality for sequences of vectors in inner product spaces was pointed out in [2].

Theorem 2. *Let* H *and* $\mathbb{K}$ *be as in Theorem 1 and* $x_i \in H$, $a_i \in \mathbb{K}$, $p_i \geq 0$ $(i = 1, \dots, n)$ $(n \geq 2)$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $a, A \in \mathbb{K}$ *and* $x, X \in H$ *are such that:*

$$\operatorname{Re} \left[(A - a_i) \left(\overline{a_i} - \overline{a} \right) \right] \geq 0, \quad \operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0$$

for any $i \in \{1, \dots, n\}$, *then we have the inequality*

$$0 \leq \left\| \sum_{i=1}^{n} p_i a_i x_i - \sum_{i=1}^{n} p_i a_i \cdot \sum_{i=1}^{n} p_i x_i \right\| \leq \frac{1}{4} |A - a| \|X - x\| .$$

The constant $\frac{1}{4}$ *is best possible.*

A complementary result for two sequences of vectors in inner product spaces is the following result that has been obtained in [3].

Theorem 3. *Let* H *and* $\mathbb{K}$ *be as above,* $x_i, y_i \in H$, $p_i \geq 0$ $(i = 1, \dots, n)$ $(n \geq 2)$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $x, X, y, Y \in H$ *are such that:*

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \quad \textit{and} \quad \operatorname{Re} \langle Y - y_i, y_i - y \rangle \geq 0$$

for all $i \in \{1, \ldots, n\}$, *then we have the inequality*

$$0 \le \left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \right| \le \frac{1}{4} \|X - x\| \|Y - y\| .$$

The constant $\frac{1}{4}$ *is best possible.*

In the general case of normed linear spaces, the following Grüss type inequality in terms of the forward difference is known, see [4].

Theorem 4. *Let* $(E, \|\cdot\|)$ *be a normed linear space over* $\mathbb{K} = \mathbb{C}, \mathbb{R}$, $x_i \in E$, $\alpha_i \in \mathbb{K}$ *and* $p_i \ge 0 \quad (i = 1, \ldots, n)$ *such that* $\sum_{i=1}^{n} p_i = 1$. *Then we have the inequality*

$$(1) \qquad 0 \le \left\| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \cdot \sum_{i=1}^{n} p_i x_i \right\| \le$$

$$\le \max_{1 \le j \le n-1} |\Delta \alpha_j| \max_{1 \le j \le n-1} \|\Delta x_j\| \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i \right)^2 \right],$$

where $\Delta \alpha_j = \alpha_{j+1} - \alpha_j$ *and* $\Delta x_j = x_{j+1} - x_j \; (j = 1, \ldots, n-1)$ *are the forward differences of the vectors having the components* α_j *and* $x_j \; (j = 1, \ldots, n-1)$, *respectively.*

The inequality (1) is sharp in the sense that the multiplicative constant $C = 1$ *in the right hand side cannot be replaced by a smaller one.*

An important particular case is the one where all the weights are equal, giving the following corollary [4].

Corollary 1. *Under the above assumptions for* $\alpha_i, x_i \; (i = 1, \ldots, n)$ *we have the inequality*

$$(2) \qquad 0 \le \left\| \frac{1}{n} \sum_{i=1}^{n} \alpha_i x_i - \frac{1}{n} \sum_{i=1}^{n} \alpha_i \cdot \frac{1}{n} \sum_{i=1}^{n} x_i \right\| \le$$

$$\leq \frac{n^2-1}{12} \max_{1\leq j\leq n-1} |\Delta\alpha_j| \max_{1\leq j\leq n-1} \|\Delta x_j\| .$$

The constant $\frac{1}{12}$ is best possible.

Another result of this type was proved in [6].

Theorem 5. *With the assumptions of Theorem 4, one has the inequality*

$$0 \leq \left\| \sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \cdot \sum_{i=1}^{n} p_i x_i \right\| \leq \tag{3}$$

$$\leq \frac{1}{2} \sum_{j=1}^{n-1} |\Delta\alpha_j| \sum_{j=1}^{n-1} \|\Delta x_j\| \sum_{i=1}^{n} p_i (1-p_i) .$$

The constant $\frac{1}{2}$ is best possible.

As a useful particular case, we have the following corollary [6].

Corollary 2. *If α_i, x_i $(i = 1, \ldots, n)$ are as in Theorem 4, then*

$$0 \leq \left\| \frac{1}{n} \sum_{i=1}^{n} \alpha_i x_i - \frac{1}{n} \sum_{i=1}^{n} \alpha_i \cdot \frac{1}{n} \sum_{i=1}^{n} x_i \right\| \leq$$

$$\leq \frac{1}{2} \left(1 - \frac{1}{n}\right) \sum_{i=1}^{n-1} |\Delta\alpha_i| \sum_{i=1}^{n-1} \|\Delta x_i\| .$$

The constant $\frac{1}{2}$ is the best possible.

Finally, the following result is also known [5].

Theorem 6. *With the assumptions in Theorem 4, we have the inequality:*

$$0 \leq \left\| \sum_{i=1}^{n} p_i\alpha_i x_i - \sum_{i=1}^{n} p_i\alpha_i \cdot \sum_{i=1}^{n} p_i x_i \right\| \leq \tag{4}$$

$$\leq \left(\sum_{j=1}^{n-1}|\Delta\alpha_j|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{n-1}\|\Delta x_j\|^q\right)^{\frac{1}{q}}\sum_{1\leq i<j\leq n}(j-i)\,p_ip_j,$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

The constant $c=1$ *in the right hand side of (4) is sharp.*

The case of equal weights is embodied in the following corollary [5].

Corollary 3. *With the above assumptions for* α_i, x_i $(i=1,\dots,n)$ *one has*

$$0\leq\left\|\frac{1}{n}\sum_{i=1}^{n}\alpha_ix_i-\frac{1}{n}\sum_{i=1}^{n}\alpha_i\cdot\frac{1}{n}\sum_{i=1}^{n}x_i\right\|\leq$$

$$\leq\frac{n^2-1}{6n}\left(\sum_{j=1}^{n-1}|\Delta\alpha_j|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{n-1}\|\Delta x_j\|^q\right)^{\frac{1}{q}},$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

The constant $\frac{1}{6}$ *is the best possible.*

The main aim of this section is to establish some similar bounds for the absolute value of the difference

$$\sum_{i=1}^{n}p_i\left\langle x_i,y_i\right\rangle-\left\langle\sum_{i=1}^{n}p_ix_i,\sum_{i=1}^{n}p_iy_i\right\rangle$$

provided that x_i, y_i $(i=1,\dots,n)$ are vectors in an inner product space H, and $p_i\geq 0$ $(i=1,\dots,n)$ with $\sum_{i=1}^{n}p_i=1$.

2 The Main Results

We assume that $(H,\langle\cdot,\cdot\rangle)$ is an inner product space over $\mathbb{K}$, $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$. The following discrete inequality of Grüss type holds.

Theorem 7. *If $x_i, y_i \in H$, $p_i \geq 0$ $(i = 1, \dots, n)$ with $\sum_{i=1}^{n} p_i = 1$, then one has the inequalities:*

$$\left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \right| \leq \tag{5}$$

$$\leq \begin{cases} \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i \right)^2 \right] \max_{k=1,\dots,n-1} \|\Delta x_k\| \max_{k=1,\dots,n-1} \|\Delta y_k\| ; \\ \\ \left[\sum_{1 \leq j < i \leq n} p_i p_j (i-j) \right] \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}} \\ \hfill \text{if } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{1}{2} \left[\sum_{i=1}^{n} p_i (1 - p_i) \right] \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\| . \end{cases}$$

All the inequalities in (5) are sharp.

The following particular case for equal vectors holds.

Corollary 4. *With the assumptions of Theorem 7, one has the inequalities*

$$0 \leq \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \leq$$

$$\leq \begin{cases} \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i \right)^2 \right] \max_{k=1,\dots,n-1} \|\Delta x_k\|^2 ; \\ \\ \sum_{1 \leq j < i \leq n} p_i p_j (i-j) \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}} \\ \hfill \text{if } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{1}{2} \sum_{i=1}^{n} p_i (1 - p_i) \left(\sum_{k=1}^{n-1} \|\Delta x_k\| \right)^2 . \end{cases}$$

The following particular case for equal weights may be useful in practice.

Corollary 5. *If $x_i, y_i \in H$ $(i = 1, \ldots, n)$, then one has the inequalities:*

$$\left| \frac{1}{n} \sum_{i=1}^{n} \langle x_i, y_i \rangle - \left\langle \frac{1}{n} \sum_{i=1}^{n} x_i, \frac{1}{n} \sum_{i=1}^{n} y_i \right\rangle \right| \leq$$

$$\leq \begin{cases} \dfrac{n^2-1}{12} \max\limits_{k=1,\ldots,n-1} \|\Delta x_k\| \max\limits_{k=1,\ldots,n-1} \|\Delta y_k\| \,; \\ \\ \dfrac{n^2-1}{6n} \left(\sum\limits_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left(\sum\limits_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}} \\ \qquad\qquad\qquad \text{if } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \dfrac{n-1}{2n} \sum\limits_{k=1}^{n-1} \|\Delta x_k\| \sum\limits_{k=1}^{n-1} \|\Delta y_k\| \,. \end{cases}$$

The constants $\frac{1}{12}$, $\frac{1}{6}$ *and* $\frac{1}{2}$ *are best possible.*

In particular, the following corollary holds.

Corollary 6. *If $x_i \in H$ $(i = 1, \ldots, n)$, then one has the inequality*

$$0 \leq \frac{1}{n} \sum_{i=1}^{n} \|x_i\|^2 - \left\| \frac{1}{n} \sum_{i=1}^{n} x_i \right\|^2 \leq$$

$$\leq \begin{cases} \dfrac{n^2-1}{12} \max\limits_{k=\overline{1,n}} \|\Delta x_k\|^2 \,; \\ \\ \dfrac{n^2-1}{6n} \left(\sum\limits_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left(\sum\limits_{k=1}^{n-1} \|\Delta x_k\|^q \right)^{\frac{1}{q}} \\ \qquad\qquad\qquad \text{if } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \dfrac{n-1}{2n} \left(\sum\limits_{k=1}^{n-1} \|\Delta x_k\| \right)^2 . \end{cases}$$

The constants $\frac{1}{12}$, $\frac{1}{6}$ *and* $\frac{1}{2}$ *are best possible.*

3 Proof of the Main Result

It is well known that, the following identity holds in inner product spaces:

$$\sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle = \tag{6}$$

$$= \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \langle x_i - x_j, y_i - y_j \rangle = \sum_{1 \le j < i \le n} p_i p_j \langle x_i - x_j, y_i - y_j \rangle .$$

We observe, for $i > j$, we can write that

$$x_i - x_j = \sum_{k=j}^{i-1} \Delta x_k, \qquad y_i - y_j = \sum_{k=j}^{i-1} \Delta y_k. \tag{7}$$

Taking the modulus in (6) and by the use of (7) and Schwarz′s inequality in inner product spaces, i.e., we recall that $|\langle z, u \rangle| \le \|z\| \|u\|$, $z, u \in H$, we have:

$$\left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \right| \le \sum_{1 \le j < i \le n} p_i p_j |\langle x_i - x_j, y_i - y_j \rangle| \le$$

$$\le \sum_{1 \le j < i \le n} p_i p_j \|x_i - x_j\| \|y_i - y_j\| = \sum_{1 \le j < i \le n} p_i p_j \left\| \sum_{k=j}^{i-1} \Delta x_k \right\| \left\| \sum_{l=j}^{i-1} \Delta y_l \right\| \le$$

$$\le \sum_{1 \le j < i \le n} p_i p_j \sum_{k=j}^{i-1} \|\Delta x_k\| \sum_{l=j}^{i-1} \|\Delta y_l\| := M.$$

It is obvious that

$$\sum_{k=j}^{i-1} \|\Delta x_k\| \le (i-j) \max_{k=j,\ldots,i-1} \|\Delta x_k\| \le (i-j) \max_{k=1,\ldots,n} \|\Delta x_k\|$$

and

$$\sum_{k=j}^{i-1} \|\Delta y_k\| \leq (i-j) \max_{k=j,\ldots,i-1} \|\Delta y_k\| \leq (i-j) \max_{k=1,\ldots,n} \|\Delta y_k\|,$$

giving that

$$M \leq \sum_{1 \leq j < i \leq n} p_i p_j (i-j)^2 \cdot \max_{k=1,\ldots,n} \|\Delta x_k\| \max_{k=1,\ldots,n} \|\Delta y_k\|,$$

and since

$$\sum_{1 \leq j < i \leq n} p_i p_j (i-j)^2 = \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (i-j)^2 = \sum_{i=1}^{n} p_i i^2 - \left(\sum_{i=1}^{n} i p_i \right)^2,$$

the first inequality in (5) is proved.

Using Hölder′s discrete inequality, we can state that

$$\sum_{k=j}^{i-1} \|\Delta x_k\| \leq (i-j)^{\frac{1}{q}} \left(\sum_{k=j}^{i-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \leq (i-j)^{\frac{1}{q}} \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}}$$

and

$$\sum_{k=j}^{i-1} \|\Delta y_k\| \leq (i-j)^{\frac{1}{p}} \left(\sum_{k=j}^{i-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}} \leq (i-j)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}},$$

for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, giving that:

$$M \leq \sum_{1 \leq j < i \leq n} p_i p_j (i-j) \cdot \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}}$$

and the second inequality in (5) is proved.

Also, observe that

$$\sum_{k=j}^{i-1} \|\Delta x_k\| \leq \sum_{k=1}^{n-1} \|\Delta x_k\| \quad \text{and} \quad \sum_{k=j}^{i-1} \|\Delta y_k\| \leq \sum_{k=1}^{n-1} \|\Delta y_k\|$$

and thus

$$M \le \sum_{1\le j<i\le n} p_i p_j (i-j) \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\| .$$

Since

$$\sum_{1\le j<i\le n} p_i p_j = \frac{1}{2}\left[\sum_{i,j=1}^{n} p_i p_j - \sum_{k=1}^{n} p_k^2\right] = \frac{1}{2}\left(1-\sum_{k=1}^{n} p_k^2\right) = \frac{1}{2}\sum_{i=1}^{n} p_i (1-p_i),$$

the last part of (5) is also proved.

Now, assume that the first inequality in (5) holds with a constant $c > 0$, i.e.,

$$\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \le$$

$$\le c\left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right] \max_{k=1,\dots,n-1} \|\Delta x_k\| \max_{k=1,\dots,n-1} \|\Delta y_k\|$$

and choose $n = 2$ to get

$$p_1 p_2 |\langle x_2 - x_1, y_2 - y_1\rangle| \le c p_1 p_2 \|x_2 - x_1\| \|y_2 - y_1\| \tag{8}$$

for any $p_1, p_2 > 0$ and $x_1, x_2, y_1, y_2 \in H$.

If in (8) we choose $y_2 = x_2$, $y_1 = x_1$ and $x_2 \neq x_1$, then we deduce $c \ge 1$, which proves the sharpness of the constant in the first inequality in (5).

In a similar way one may show that the other two inequalities are sharp, and the theorem is completely proved.

4 A Reverse for Jensen′s Inequality

Let $(H; \langle\cdot,\cdot\rangle)$ be a real inner product space and $F : H \to \mathbb{R}$ a Fréchet differentiable convex function on H. If $\nabla F : H \to H$ denotes the gradient operator associated to F, then we have the inequality

$$F(x) - F(y) \geq \langle \nabla F(y), x - y \rangle$$

for each $x, y \in H$.

The following result holds.

Theorem 8. *Let $F : H \to \mathbb{R}$ be as above and $z_i \in H$, $i \in \{1, \dots, n\}$. If $q_i \geq 0 \quad (i \in \{1, \dots, n\})$ with $\sum_{i=1}^{n} q_i = 1$, then we have the following reverse of Jensen′s inequality*

$$0 \leq \sum_{i=1}^{n} q_i F(z_i) - F\left(\sum_{i=1}^{n} q_i z_i\right) \leq \tag{9}$$

$$\leq \begin{cases} \left[\sum_{i=1}^{n} i^2 q_i - \left(\sum_{i=1}^{n} i q_i\right)^2\right] \max\limits_{k=1,\dots,n-1} \|\Delta(\nabla F(z_i))\| \max\limits_{k=1,\dots,n-1} \|\Delta z_i\|; \\ \\ \left[\sum_{1 \leq j < i \leq n} q_i q_j (i - j)\right] \left(\sum_{i=1}^{n-1} \|\Delta(\nabla F(z_i))\|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n-1} \|\Delta z_i\|^q\right)^{\frac{1}{q}} \\ \qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{1}{2}\left[\sum_{i=1}^{n} q_i (1 - q_i)\right] \sum_{i=1}^{n-1} \|\Delta(\nabla F(z_i))\| \sum_{i=1}^{n-1} \|\Delta z_i\|. \end{cases}$$

Proof. We know, see for example [3, Eq. (4.4)], that the following reverse of Jensen′s inequality for Fréchet differentiable convex functions

$$0 \leq \sum_{i=1}^{n} q_i F(z_i) - F\left(\sum_{i=1}^{n} q_i z_i\right) \leq \tag{10}$$

$$\leq \sum_{i=1}^{n} q_i \langle \nabla F(z_i), z_i \rangle - \left\langle \sum_{i=1}^{n} q_i \nabla F(z_i), \sum_{i=1}^{n} q_i z_i \right\rangle$$

holds.

Now, if we apply Theorem 7 for the choices $x_i = \nabla F(z_i), y_i = z_i$ and $p_i = q_i \ (i = 1, ..., n)$, then we may state

$$\left|\sum_{i=1}^{n} q_i \langle \nabla F(z_i), z_i\rangle - \left\langle \sum_{i=1}^{n} q_i \nabla F(z_i), \sum_{i=1}^{n} q_i z_i \right\rangle\right| \leq \tag{11}$$

$$\leq \begin{cases} \left[\sum_{i=1}^{n} i^2 q_i - \left(\sum_{i=1}^{n} i q_i\right)^2\right] \max_{k=1,\dots,n-1} \|\Delta(\nabla F(z_k))\| \max_{k=1,\dots,n-1} \|\Delta z_k\|; \\ \left[\sum_{1\leq j<i\leq n} q_i q_j (i-j)\right] \left(\sum_{k=1}^{n-1} \|\Delta(\nabla F(z_k))\|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta z_k\|^q\right)^{\frac{1}{q}} \\ \qquad \text{if } p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2}\left[\sum_{i=1}^{n} q_i (1 - p_i)\right] \sum_{k=1}^{n-1} \|\Delta(\nabla F(z_k))\| \sum_{k=1}^{n-1} \|\Delta z_k\|. \end{cases}$$

Finally, on making use of the inequalities (10) and (11), we deduce the desired result (9).

The unweighted case may useful in application and is incorporated in the following corollary.

Corollary 7. *Let $F : H \to \mathbb{R}$ be as above and $z_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequalities*

$$0 \leq \frac{1}{n}\sum_{i=1}^{n} F(z_i) - F\left(\frac{1}{n}\sum_{i=1}^{n} z_i\right) \leq$$

$$\leq \begin{cases} \dfrac{n^2-1}{12} \max\limits_{k=1,\dots,n-1} \left\|\Delta\left(\nabla F\left(z_k\right)\right)\right\| \max\limits_{k=1,\dots,n-1} \left\|\Delta z_k\right\|; \\[2ex] \dfrac{n^2-1}{6n} \left(\sum\limits_{k=1}^{n-1} \left\|\Delta\left(\nabla F\left(z_k\right)\right)\right\|^p\right)^{\frac{1}{p}} \left(\sum\limits_{k=1}^{n-1} \left\|\Delta z_k\right\|^q\right)^{\frac{1}{q}} \\ \hfill \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \dfrac{n-1}{2n} \sum\limits_{k=1}^{n-1} \left\|\Delta\left(\nabla F\left(z_k\right)\right)\right\| \sum\limits_{k=1}^{n-1} \left\|\Delta z_k\right\|. \end{cases}$$

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