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Rate of approximation for Bézier variant of Bleimann-Butzer and Hahn operators

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Dedicated to Professor Emil C. Popa on his 60th anniversary

Abstract

In the present paper we obtain the rate of convergence for the Bézier variant of the Bleimann-Butzer and Hahn operators $L_{n,\alpha}$, for functions of bounded variation. We consider the case when $0 < \alpha < 1$.

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1 Introduction

In the year 1980, Bleimann, Butzer and Hahn [5] introduced an interesting sequence of positive linear operators defined on the space of real functions on the infinite interval $[0, \infty)$ by

$$(1) \qquad L_n(f, x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n-k-1}\right), \; x \in [0, \infty), \; n \in \mathbb{N}$$

where

$$p_{n,k}(x) = \binom{n}{k} \frac{x^k}{(1+x)^n}$$

The approximation properties of the Bleimann-Butzer and Hahn operators (BBH operators) were studied by many researchers see e.g. [1], [2], [3], [5] and [6] etc. Very recently Srivastava and Gupta [8], introduced the Bézier variant of the BBH operators, which is defined as

$$(2) \qquad L_{n,\alpha}(f,x) = \sum_{k=0}^{\infty} Q_{n,k}^{\infty}(x) f\left(\frac{k}{n-k+1}\right), x \in [0,\infty),\ n \in \mathbb{N}$$

where $Q_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$ and $J_{n,k}(x) = \sum_{j=k}^{\alpha} p_{n,j}(x)$. Some basic properties of $J_{n,k}(x)$ can be found in [8]. It is easily verified that $L_{n,\alpha}(f,x)$ are positive linear operators. The authors [8] have obtained the rate of convergence for functions of bounded variation for the case whenever $\alpha \geq 1$. As a special case $\alpha = 1$, $L_{n,\alpha}(f,x)$ reduce to the operators $L_{n,1}(f,x) \equiv L_n(f,x)$ defined by (1). The other case $0 < \alpha < 1$ of the operators defined by (2) is also equally important. In the present paper we extend the study in this direction and obtain teh rate of convergence for functions of bounded variation, for the operators $L_{n,\alpha}$, $0 < \alpha < 1$.

Our main theorem is stated as:

Theorem 1. *Let f be a function of bounded variation on every finite subinterval of* $[0,\infty)$. *Let* $f(t) = O(t^r)$ *for some* $r \in \mathbb{N}$ *as* $t \to \infty$. *Then for* $x \in (0,\infty)$, $0 < \alpha < 1$ *and n sufficiently large, we have*

$$\left| L_{n,\alpha}(f,x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right| \leq$$

$$\leq \frac{7(1+x)^2}{(n+2)x} \sum_{k=1}^{n} V_{n-x/\sqrt{k}}^{x+x/\sqrt{k}}(f_x) + \frac{|1-x|}{6\sqrt{2\pi(n+1)x}} |f(x+) - f(x-)| +$$

$$+ \frac{1+x}{\sqrt{2enx}} \varepsilon_n(x) |f(x) - f(x-)| + \frac{C(\alpha,f,x,r)}{n^m},\ m > r$$

where

$$\varepsilon_n(x) = \begin{cases} 1, & if (n+1)p_x \in \mathbb{N} \\ 0, & otherwise \end{cases},$$

$$f_x(t) = \begin{cases} f(t) - f(x-), & 0 \le t < x \\ 0, & t = x \\ f(t) - f(x+), & x < t < \infty \end{cases}$$

and $V_a^b(f_x)$ *is the total variation of* f_x *on* $[a, b]$.

We recall the Lebesque-Stieltjes integral representation

$$L_{n,\alpha}(f, x) = \int_0^\infty f(t) d_t(K_{n,\alpha}(x, t)),$$

where

$$K_{n,\alpha}(x, t) = \begin{cases} \sum\limits_{k \le (n-k+1)t} Q_{n,k}^{(\alpha)}(x), & 0 < t < \infty \\ 0, & t = 0 \end{cases}.$$

Also we define

$$H_{n,\alpha}(x, t) = \begin{cases} 1 - K_{n,\alpha}(x, t), & 0 < t < \infty \\ 0, & t = 0 \end{cases}.$$

2 Auxiliary Results

In this section we give certain results, which are necessary to prove the main results.

Lemma 1. [3] *For all* $x \in [0, \infty)$ *and* $n \ge 1$ *we have the following inequality:*

$$L_n((t - x)^2, x) \le \frac{3x(1 + x)^2}{n + 2}.$$

Lemma 2. *For $x \in (0,\infty)$, we have*

$$\left|\sum_{k/(n-k+1)>x} p_{n,k}(x) - \frac{1}{2}\right| \le \frac{|1-x|}{6\sqrt{2\pi(n+1)x}}.$$

Proof. Apart from at most one term $p_{n,k}(x)$ when $k/(n-k+1)=x$, using the transform

$$p_{n,k}(x) = \binom{n}{k}\frac{x^k}{(1+x)^n} = \binom{n}{k}\left(\frac{x}{1+x}\right)^k\left(\frac{1}{1+x}\right)^{n-k} =$$

$$= \binom{n}{k} y^k(1-y)^{n-k} \equiv b_{n,k}(y),$$

with $y = x/(1+x)$, we have

$$\sum_{k/(n-k+1)>x} p_{n,k}(x) = \sum_{k>(n+1)y} b_{n,k}(y)$$

this is approximately equal to

$$\frac{B_y((n+1)y, n-(n+1)y+1)}{B((n+1)y, n-(n+1)y+1)}$$

with the incomplete Beta function

$$B_y(a,b) = \int_0^y t^{a-1}(1-t)^{b-1}dt,\ a>0,\ b>0.$$

We have

$$\left|1-2\sum_{k/(n-k+1)>x} p_{n,k}(x)\right| = \left|1-2\sum_{k(n+1)>y} b_{n,k}(y)\right| =$$

$$= \left|1-2\frac{1}{B((n+1)y, n-(n+1)y+1)}\int_0^y t^{(n+1)y-1}(1-t)^{n-(n+1)y}dt\right|$$

Now we estimate the right hand side as follows:

Let $y \in (0,1)$, we have

$$\frac{B((n+1)y,(n+1)(1-y))}{B((n+1)y,(n+1)(1-y))} = 1 - \frac{I_y(n+1)}{B((n+1)y,(n+1)(1-y))}$$

where

$$I_y(n+1) = \int_y^1 t^{(n+1)y-1}(1-t)^{(n+1)(1-y)-1}dt = \int_y^1 g(t)e^{(n+1)h_y(t)}dt$$

with

$$g(t) = (t(1-t))^{-1}$$

$$h_y(t) = y\log t + (1-y)\log(1-t).$$

Since

$$h'_y = \frac{-(t+y)}{t(1-t)} < 0 \quad (y<t<1) \text{ and } h'_y(y) = 0$$

h_y is strictly decreasing on $(y,1)$.

Furthermore $h''_y(t) = \dfrac{-(t-y)^2 + y(1-y)}{t^2(1-t)^2} < 0$ and $h''_y(y) = -(y(1-y))^{-1} \neq 0$.

Thus it is well known (e.g. $I_y(n)$ meets the assumptions of [7, Th. 1, Kap. 3]) that there holds the complete asymptotic expansion

$$I_y(n+1) \sim \frac{1}{2}e^{(n+1)h_y}\sum_{k=0}^{\infty}\frac{\Gamma((k+1)/2)}{(n+1)^{(k+1)/2}}$$

$$(y^y(1-y)^{1-y})\left[\frac{a_0\sqrt{\pi}}{2(n+1)^{1/2}} + \frac{a_1}{2(n+1)} + \frac{a_2}{4(n+1)^{3/2}} + O(n^{-2})\right],$$

for $n \to \infty$, with the coefficients

$$a_k = \frac{1}{k!}\left(\left(\frac{d}{dt}\right)^k\left\{g(t)\left(\frac{t-y}{\sqrt{h_y(y)-h_y(t)}}\right)^{k+1}\right\}\right)_{t=1/y}$$

By direct calculation we obtain the explicit expressions

$$a_0=\frac{\sqrt{2}}{\sqrt{y(1-y)}},\ a_1=\frac{-2(2-2y)}{3y(1-y)},\ a_2=\frac{1-y+y^2}{3\sqrt{2}(y(1-y))^{3/2}}$$

Thus we have

$$I_y(n+1)=(y^y(1-y)^{1-y})^n\cdot$$

$$\cdot\left[\frac{\sqrt{\pi}}{\sqrt{2n\pi(1-y)}}-\frac{1-2y}{3ny(1-y)}+\frac{\sqrt{\pi}}{12\sqrt{2}}\cdot\frac{1-y+y^2}{(y(1-y))^{3/2}}+O(n^{-2})\right].$$

Also by Stirling′s formula, we have

$$\frac{1}{B((n+1)y,(n+1)(1-y))}=\frac{\Gamma(n+1)}{\Gamma((n+1)x)}=\frac{1}{\sqrt{2\pi}}(y^y(1-y)^{1-y})^{-n}\cdot$$

$$\cdot\left[\sqrt{(n+1)y(1-y)}-\frac{1-y+y^2}{12\sqrt{ny(1-y)}}+\frac{(1-y+y^2)^2}{299n^{3/2}(y(1-y))^{3/2}}+O(n^{-5/2})\right].$$

Combining the both asymptotic expansions, we have

$$\frac{I_y(n+1)}{B((n+1)y,(n+1)(1-y))}=\frac{1}{2}-\frac{1-2y}{3\sqrt{2\pi(n+1)y(1-y)}}+O(n^{-3/2}).$$

Thus

$$\left|\sum_{k/(n-k+1)>x}p_{n,k}(x)-\frac{1}{2}\right|\le\frac{|1-x|}{6\sqrt{2\pi(n+1)x}}+O(n^{-3/2}).$$

Lemma 3. *For all $x\in(0,\infty)$, $\alpha\le 1$ and $k\in\mathbb{N}$, there holds*

$$Q_{n,k}^{(\alpha)}(x)\le\alpha p_{n,k}(x)<\frac{(1+x)}{\sqrt{2enx}}.$$

Proof. For the Bernstein functions the optimum bound was obtained by Zeng [8], and Bastien and Rogalski [4] in a problem posed by the author, which is as follows:

$$\binom{n}{k}y^k(1-y)^{n-k}\le\frac{1}{\sqrt{2eny(1-y)}},\quad 0<y<1.$$

Substituting $y=\frac{x}{1+x}$, we get the required result.

Lemma 4. *For $0 < \alpha < 1$ and $0 < x < t < \infty$, we have*

$$H_{n,\alpha}(x,t) \leq \frac{C}{n^m (t-x)^m},$$

where C is a positive constant that depends on x but independent of n.

Proof. Since $0 < x < t < \infty$, so $\dfrac{\left|\dfrac{k}{n-k+1} - x\right|}{|t-x|} \geq 1$, for $k \geq nt$. Thus for $m \in \mathbb{N}$, we have

$$H_{n,\alpha}(x,t) = 1 - K_{n,\alpha}(x,t) = 1 - \sum_{k \leq (n-k+1)t} Q_{n,k}^{(\alpha)}(x) \leq \sum_{k \geq (n-k+1)t} Q_{n,k}^{(\alpha)}(x) \leq$$

$$\leq \left(\sum_{k \geq (n-k+1)t} \frac{\left|\dfrac{k}{n-k+1}\right|^{2m/\alpha}}{(t-x)^{2m/\alpha}} p_{n,k}(x) \right)^{\alpha} \leq$$

$$\leq \frac{1}{(t-x)^{2m}} \left(\sum_{k=0}^{\infty} \left| \frac{k}{n-k+1} - x \right|^{2m/\alpha} p_{n,k}(x) \right)^{\alpha}.$$

For all conjugate $p, q \geq 1$, i.e. $1/p + 1/q = 1$, we have

$$\left(\sum_{k=0}^{\infty} \left| \frac{k}{n-k+1} - x \right|^{2m/\alpha} p_{n,k}(x) \right)^{\alpha} =$$

$$= \left(\sum_{k=0}^{\infty} \left| \frac{k}{n-k+1} - x \right|^{2m/\alpha} p_{n,k}^{1/p}(x) p_{n,k}^{1/q}(x) \right)^{\alpha} \leq$$

$$\leq \left(\sum_{k=0}^{\infty} \left| \frac{k}{n-k+1} - x \right|^{2mp/\alpha} p_{n,k}(x) \right)^{\alpha/p}$$

since $\left(\sum\limits_{k=0}^{\infty} p_{n,k}(x) \right)^{\alpha/q} = 1.$

Choosing $p = \frac{q}{m}\left[\frac{m}{\alpha}+1\right]$ we have that $2mp/\alpha$ is an even positive integer. By the well known result $B_{n,1}(\psi_x^{2r},x) = O(n^{-r})$, as $n \to \infty$ $(r = 1, 2, 3, ...)$ we obtain

$$\left(\sum_{k=0}^{\infty}\left|\frac{k}{n-k+1}-x\right|^{2mp/\alpha} p_{n,k}(x)\right)^{\alpha/p} = (L_{n,1}(\psi_x^{2mp/\alpha},x))^{\alpha/p} = O(n^{-m})$$

as $n \to \infty$.

This completes the proof of Lemma 4.

3 Proof of main Theorem

Our main theorem is stated as:

Proof. We have

$$f(t) = 2^{-\alpha}f(x+)+(1-2^{-\alpha})f(x-)+g_x(t)+2^{-\alpha}(f(f+)-f(x-))sign^{(\alpha)}(t)+$$
$$+(f(x)-2^{-\alpha}f(x+)-(1-2^{-\alpha})f(x-))\delta_x(t),$$

where

$$sign^{(\alpha)}(t-x) := \begin{cases} 2^{\alpha}-1, & \text{if } t > x \\ 0, & \text{if } t = x \\ -1, & \text{if } t < x \end{cases} \quad \text{and } \delta_x(t) \begin{cases} 1, & \text{if } x = t \\ 0, & \text{if } x \neq t \end{cases}$$

Therefore

$$\left|L_{n,\alpha}(f,x) - \frac{1}{2^{\alpha}}f(x+) - \left(1-\frac{1}{2^{\alpha}}\right)f(x-)\right| \leq |L_{n,\alpha}(f_x,x)| + \tag{3}$$
$$+\left|\frac{f(x+)-f(x-)}{2^{\alpha}}L_{n,\alpha}(sign^{(\alpha)}(t-x),x)+\right.$$
$$\left.+\left[f(x)-\frac{1}{2^{\alpha}}f(x+)-\left(1-\frac{1}{2^{\alpha}}\right)f(x-)\right]L_{n,\alpha}(\delta_x,x)\right|.$$

We first estimate

$$L_{n,\alpha}(sign^{(\alpha)}(t-x),x) = 2^\alpha \sum_{k>(n-k+1)x} Q_{n,k}^{(\alpha)}(x) - 1 + e_n(x)Q_{n,k'}^{(\alpha)}(x) =$$

$$= 2^\alpha \left(\sum_{k>(n-k+1)x} p_{n,k}(x) \right)^\alpha - 1 + \varepsilon_n(x) Q_{n,k'}^{(\alpha)}(x)$$

and

$$L_{n,\alpha}(\delta_x, x) = \varepsilon_n(x) Q_{n,k'}^{(\alpha)}(x).$$

Hence, we have

$$(4) \qquad \left| \frac{f(x+)-f(x-)}{2^\alpha} L_{n,\alpha}(sign^\alpha(t-x),x) + \right.$$

$$\left. + \left[f(x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right] L_{n,\alpha}(\delta_x, x) \right| =$$

$$= \left| \frac{f(x+)-f(x-)}{2^\alpha} \left[2^\alpha \left(\sum_{k>(n-k+1)x} p_{n,k}(x) \right)^\alpha - 1 \right] + [f(x) - f(x-)] \varepsilon_n Q_{n,k}^{(\alpha)}(x) \right|$$

By mean value theorem, we have

$$\left| \left(\sum_{j>(n-j+1)x} p_{n,j}(x) \right)^\alpha - \frac{1}{2^\alpha} \right| = \alpha(\xi_{n,j}(x))^{\alpha-1} \left| \sum_{j>(n-j+1)x} p_{n,j}(x) - \frac{1}{2} \right|$$

where $\xi_{n,j}(x)$ lies between $\frac{1}{2}$ and $\sum_{j>(n-j+1)x} p_{n,j}(x)$. In view of Lemma 2, it is observed that for n sufficiently large, the intermediate point $\zeta_{n,j}$ is arbitrary close to 1/2 i.e.

$$\zeta_{n,j} = \frac{1}{2+\varepsilon}$$

with an arbitrary small $|\varepsilon|$. Then we have

$$\alpha(\xi_{n,j}(x))^{\alpha-1} \leq \alpha(2+\varepsilon)^{1-\alpha}.$$

The later expression is positive and strictly increasing for $\alpha \in (0,1)$, since

$$\frac{\partial}{\partial \alpha}\alpha(2+\varepsilon)^{1-\alpha} = (2+\varepsilon)^{1-\alpha}[1-\alpha \log(2+\varepsilon)] > 0,$$

for sufficiently small $|\varepsilon|$. This it takes maximum value at $\alpha = 1$. This implies

$$\alpha(\xi_{n,j}(x))^{\alpha-1} \leq 1.$$

Hence

$$\left| \left(\sum_{j>(n-j+1)x} p_{n,j}(x) \right)^{\alpha} - \frac{1}{2^{\alpha}} \right| \leq \frac{|1-x|}{6\sqrt{2\pi(n+1)x}}. \tag{5}$$

Also we have

$$Q_{n,k'}^{(\alpha)}(x) = J_{n,k'}^{\alpha}(x) - J_{n,k'+1}^{\alpha}(x) = \alpha(\xi_{n,k'})^{\alpha-1} p_{n,k'}(x),$$

where $J_{n,k'+1}(x) < \xi_{n,k'}(x) < J_{n,k'}(x)$. Thus by Lemma 3, we have

$$Q_{n,k'}^{(\alpha)}(x) \leq \frac{1+x}{\sqrt{2enx}}. \tag{6}$$

Combining the estimates of (4), (5) and (6), we have

$$\left| \frac{f(x+)-f(x-)}{2^{\alpha}} L_{n,\alpha}(sign(t-x),x) + \right. \tag{7}$$

$$\left. + \left[f(x) - \frac{1}{2^{\alpha}} f(x+) - \left(1-\frac{1}{2^{\alpha}}\right) f(x-) \right] L_{n,\alpha}(\delta_x, x) \right| \leq$$

$$\leq \frac{|1-x|}{6\sqrt{2\pi(n+1)x}} |f(x+)-f(x-)| + \frac{1+x}{\sqrt{2enx}} \varepsilon_n(x) |f(x)-f(x-)|.$$

Next we estimate $L_{n,\alpha}(f_x, x)$. We decompose the integral into four parts as follows:

$$L_{n,\alpha}(f_x,x) = \int_0^{\infty} f_x(t) d_t(K_{n,\alpha}(x,t)) = \tag{8}$$

$$= \left(\int\limits_{I_1} + \int\limits_{I_2} + \int\limits_{I_3} + \int\limits_{I_4} \right) f_x(t) d_t(K_{n,\alpha}(x,t)) = E_1 + E_2 + E_3 + E_4 \text{ say,}$$

where $I_1 = [0, x - x/\sqrt{n}]$, $I_2 = [x - x/\sqrt{n}, x + x/\sqrt{n}]$, $I_3 = [x + x/\sqrt{n}, 2x]$ and $I_4 = [2x, \infty]$. We first estimate E_2. For $t \in [x - x/\sqrt{n}, x + x/\sqrt{n}]$, we have

$$|f_x(t)| = |f_x(t) - f_x(x)| \leq V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(f_x)$$

and therefore

$$|E_2| \leq V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(f_x) \int\limits_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} d_t(K_{n,\alpha}(x,t)).$$

Since $\int\limits_a^b d_t(K_{n,\alpha}(x,t)) \leq 1$ for $(a,b) \subset [0,\infty)$, therefore

$$|E_2| \leq V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(f_x) \leq \frac{1}{n} \sum_{k=1}^{n} V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(f_x). \tag{9}$$

Next, we estimate E_1, writing $y = x - x/\sqrt{n}$ and using Lebesque-Stieltjes integration by parts, we have

$$E_1 = \int\limits_0^y f_x(t) d_t(K_{n,\alpha}(x,t)) = f_x(y) K_{n,\alpha}(x,y) - \int\limits_0^y K_{n,\alpha}(x,t) d_t(f_x(t))$$

Since $|f_x(y)| \leq V_y^x(f_x)$, it follows that

$$|E_1| \leq V_y^x(f_x) K_{n,\alpha}(x,y) + \int\limits_0^y K_{n,\alpha}(x,t) d_t(-V_t^x(f_x)).$$

Applying Lemma 1, we have

$$K_{n,\alpha}(x,t) \leq \frac{3x(1+x)^2}{(n+2)}, \quad 0 \leq t < x. \tag{10}$$

Using the inequality (10), we get

$$|E_1| \le V_y^x(f_x)\frac{3x(1+x)^2}{(n+2)(x-y)^2} + \frac{3x(1+x)^2}{(n+2)}\int_0^y \frac{1}{(x-t)^2} d_t(-V_t^x(f_x)).$$

Integrating by parts the last term, we have

$$\int_0^y \frac{1}{(x-t)^2} d_t(-V_t^x(f_x)) = -\frac{V_y^x(f_x)}{(x-y)^2} + \frac{V_0^x(f_x)}{x^2} + 2\int_0^y V_t^x(f_x)\frac{dt}{(x-t)^3}.$$

Hence

$$|E_1| \le \frac{3x(1+x)^2}{(n+2)}\left[\frac{V_0^x(f_x)}{x^2} + 2\int_0^y \frac{V_t^x(f_x)}{(x-t)^3}\right].$$

Now replacing the variable y in the last integral by $x - x/\sqrt{u}$, we get

$$\text{(11)} \qquad |E_1| \le \frac{6(1+x)^2}{(n+2)x}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^x(f_x).$$

Using the similar method to estimate E_3, we get

$$\text{(12)} \qquad |E_3| \le \frac{6(1+x)^2}{(n+2)x}\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(f_x).$$

Finally, by the assumption that $f_x(t) = 0(t^r)$, $r \in \mathbb{N}, t \to \infty$, we have

$$|f_x(t)| \le Mt^r \le M\left(\frac{t-x}{x}\right)^r, \text{ for } t \ge 2x.$$

Now

$$|E_4| = \left|\int_{2x}^{\infty} f_x(t)K_{n,\alpha}(x,t)\right| \le \int_{2x}^{\infty} |f_x(t)| d_t K_{n,\alpha}(x,t) \le$$

$$\le Mx^{-r}\int_0^{\infty} (t-x)^r d_t K_{n,\alpha}(x,t) \le -Mx^{-r}\int_{2x}^{\infty} (t-x)^r d_t(1 - K_{n,\alpha}(x,t)) =$$

$$= -Mx^{-r}\int_{2x}^{\infty}(t-x)^r d_t(H_{n,\alpha}(x,t)) \le -Mx^{-r}\int_{0}^{\infty}(t-x)^r d_t(1-K_{n,\alpha}(x,t)) =$$

$$= Mx^{-r}\lim_{R\to\infty}\left(-(t-x)^r H_{n,\alpha}(x,t)|_{2x}^{R} + \int_{2x}^{R} H_{n,\alpha}(x,t)d_t(t-x)^r\right) =$$

$$= Mx^{-r}\lim_{R\to\infty}\left(-(t-x)^r H_{n,\alpha}(x,t)|_{2x}^{R} + \int_{2x}^{R} H_{n,\alpha}(x,t)d_t(t-x)^{r-1}dt\right) =$$

$$= Mx^{-r}\lim_{R\to\infty}\left(-(t-x)^r \frac{C}{n^m(t-x)^m}|_{2x}^{R} + \frac{rC}{n^m}\int_{2x}^{R}(t-x)^{r-m-1}dt\right) =$$

$$= M\frac{C}{n^m x^m} + \frac{rC}{n^m(m-r)x^{m-r}},\ m > r.$$

By combining the estimates given by (3), (7) - (9) and (11) to (13), we obtain the desired result. This completes the proof of Theorem.

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