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Section of Euler summability method

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Abstract

In this paper, we determine sections of Euler summability method, using random variables which follow Bernoulli′s law and the central limit theorem.

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1 Introduction

The paper aims to obtain some regular transformation of the sequences of real numbers, see (12), (13), (19), (22), (24), (28), (30), (32), which are sections of Euler's summability method.

Let $A = \|a_{n,k}\|_{n,k\in\mathbb{N}}$ be a real elements matrix.

A sequence $(s_n)_{n\in\mathbb{N}}$ is said to be A **- summable to the value** $s \in \mathbb{R}$ if each of the series $\sigma_n = \sum_{k=0}^{\infty} a_{n,k} \cdot s_k$, $n = 0, 1, ...$ is convergent and if $\sigma_n \to s$ for $n \to \infty$.

The method A is called regular if each convergent sequence is A - summable to its limit.

Theorem 1. (Toeplitz) *(see [2]) The summation method A is regular if and only if*

(1) $$\lim_{n\to\infty} a_{n,k} = 0 \ , \ \text{for every } k \text{ natural},$$

(2) $$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = 1,$$

(3) $$\sum_{k=0}^{\infty} |a_{n,k}| \le M \ , \ \text{for every } n \text{ natural},$$

M being a constant independent of n.

Remark 1.1. *If the elements of matrix A are positive and $\sum_{k=0}^{\infty} a_{n,k} = 1$, for every n natural, then conditions (2), (3) are verified.*

Euler summability method is obtained for the matrix $A = \|a_{n,k}\|_{\substack{n \in \mathbb{N} \\ k=\overline{0,n}}}$, where

$$a_{n,k} = \binom{n}{k} \alpha^k (1-\alpha)^{n-k} \ , \quad \alpha \in (0,1).$$

In the followings we will use the next notations:

$M(f)$ represents the expectation of a random variable f

$D^2(f)$ represents the variance (dispersion) of a random variable f.

$[x]$ represents the whole part of x

We remind that the whole part verifies:

i) $[m+x] = m + [x]$, for $m \in \mathbb{Z}$, $x \in \mathbb{R}$

ii) $[-x] = -1 - [x]$, for $x > 0$.

Now, we recall the well-known Lyapunov central limit theorem and a result on the distribution functions of a random variables.

Theorem 2. (Lyapunov) *(see [1], [3]) Let* $(f_n)_{n\in\mathbb{N}}$ *a sequence of independent random variables. Let us suppose that* $M_k = M(f_k)$, $D_k^2 = D^2(f_k)$, $H_k = \sqrt[3]{M(|f_k - M_k|^3)}$ *exists for every* k *natural. Note with*

$$S_n = \sqrt{D_1^2 + ... + D_n^2}, K_n = \sqrt[3]{H_1^3 + ... + H_n^3},$$

$$\beta_n = f_1 + ... + f_n, \beta_n^* = \frac{\beta_n - M(\beta_n)}{D(\beta_n)}$$

and with $F_{n,\beta_n^*}(x)$ *the distribution function of variable* β_n^**. Thus, if*

$$\lim_{n\to\infty} \frac{K_n}{S_n} = 0 \tag{4}$$

we have

$$\lim_{n\to\infty} F_{n,\beta_n^*}(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{-t^2/2} dt \ , \ \text{for every } x \text{ real.} \tag{5}$$

Function $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$ represent the standard normal distribution function.

Theorem 2 is also true in the case when the independent random variables have the same distribution.

Theorem 3. *If the random variables* η_1 *and* η_2 *verify the condition* $\eta_2 = a\eta_1 + b$ *with* a, b *real, then* $f_{\eta_2}(t) = f_{\eta_1}(at)e^{ibt}$ *where* $f_{\eta_2}(t)$ *and* $f_{\eta_1}(t)$ *denote the characteristic functions of the variables* η_2 *and* η_1*.*

If the random variables η_1 and η_2 verify the condition $\eta_2 = a \cdot \eta_1 + b$ with a, b real, $a > 0$, then the distribution functions of these random variables verify

$$F_{\eta_2}(x) = F_{\eta_1}\left(\frac{x-b}{a}\right). \tag{6}$$

2 Principal results

If the independent random variables $f_1, ..., f_n$ have the distribution $\begin{pmatrix} 1 & 0 \\ p & q \end{pmatrix}$, $q = 1 - p$, $p \in (0,1)$, then the random variable $\beta_n = f_1 + ... + f_n$ has the distribution $\begin{pmatrix} k \\ \binom{n}{k} p^k \cdot q^{n-k} \end{pmatrix}_{k=\overline{0,n}}$ and the distribution function $F_{n,\beta_n}(t) = \sum_{k=0}^{n} \binom{n}{k} p^k \cdot q^{n-k} \theta(t-k)$, where $\theta(x) = \begin{cases} 0\ , & x < 0 \\ 1\ , & x \geq 0 \end{cases}$ represents Heaviside′s function.

From Theorem 2 and (6), we get

$$\lim_{n\to\infty} F_{n,\beta_n^*}(t) = \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \cdot \theta(np + t\sqrt{npq} - k) = \tag{7}$$

$$= \Phi(t)\ , \quad \text{for all } t \geq 0$$

or

$$\lim_{n\to\infty} \sum_{k=0}^{[np+t\sqrt{npq}]} \binom{n}{k} p^k \cdot q^{n-k} = \Phi(t)\ , \quad \text{for all } t \geq 0. \tag{8}$$

Next, (8) is generalized.

Let the independent random variables $f_1, ..., f_n$ with the distributions $\begin{pmatrix} 1 & 0 \\ p_k & q_k \end{pmatrix}$, $q_k = 1 - p_k$, $p_k \in (0,1)$, $k = \overline{1,n}$, and $\lim_{n\to\infty} \sum_{k=1}^{n} p_k = \infty$.

The random variable $\beta_n = f_1 + ... + f_n$ has the distribution $\begin{pmatrix} k \\ \frac{P_n(0)}{k!} \end{pmatrix}_{k=\overline{0,n}}$, where $P_n(x) = (p_1 x + q_1)...(p_n x + q_n)$.

From Theorem 2 and (6) we get

$$\lim_{n\to\infty} F_{n,\beta_n^*}(t) = \tag{9}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{P_n^{(k)}(0)}{k!}\theta\left(t\sqrt{\sum_{i=1}^{n} p_i q_i} + \sum_{i=1}^{n} p_i - k\right) = \Phi(t)\ , \quad \text{for all } t \geq 0$$

or

$$\lim_{n\to\infty} \sum_{k=0}^{\left[t\sqrt{\sum_{i=1}^{n} p_i q_i} + \sum_{i=1}^{n} p_i\right]} \frac{1}{k!} P_n^{(k)}(0) = \Phi(t) \text{ for all } t \text{ real positive.} \tag{10}$$

Remark 2.1. *As the function $\Phi(t)$ is bounded for $x \geq 0$, we can consider*

$$\lim_{n\to\infty} \frac{F_{n,\beta_n^*}(t)}{\Phi(t)} = 1\ ,\ \text{for all } t \text{ real positive.} \tag{11}$$

Using Theorem 1, (8) and (10) we can build the following regular summation methods of the sequence of real numbers $(s_n)_{n\in\mathbb{N}}$:

$$T_n^{(1)}(t) = \sum_{k=0}^{[np+t\sqrt{npq}]} c_{n,k}^{(1)}(t) \cdot s_k, \tag{12}$$

where $c_{n,k}^{(1)}(t) = \dfrac{1}{\Phi(t)} \dbinom{n}{k} p^k \cdot (1-p)^{n-k}$, $p \in (0,1)$, $t \in \left[0, \sqrt{\dfrac{nq}{p}}\right]$,

$$T_n^{(2)}(t) = \sum_{k=0}^{\left[t\sqrt{\sum_{i=1}^{n} p_i q_i} + \sum_{i=1}^{n} p_i\right]} c_{n,k}^{(2)}(t) \cdot s_k, \tag{13}$$

where $c_{n,k}^{(2)}(t) = \dfrac{1}{\Phi(t)} \cdot \dfrac{P^{(k)}(0)}{k!}$, $P(x) = \displaystyle\prod_{i=1}^{n} (p_i x + q_i)$, $p_i \in (0,1)$, $q_i = 1 - p_i$

for $i = \overline{1,n}$ and $t \in \left[0, \dfrac{n - \sum_{i=1}^{n} p_i}{\sqrt{\sum_{i=1}^{n} p_i q_i}}\right]$, $\lim_{n\to\infty} \displaystyle\sum_{k=1}^{n} p_k = \infty$.

Next, we will study the case $t = 0$; transformations from (12) and (13) become:

$$T_n^{(1)}(0) = 2 \cdot \sum_{k=0}^{[np]} \binom{n}{k} p^k \cdot (1-p)^{n-k} s_k \ , \quad p \in (0,1), \tag{14}$$

$$T_n^{(2)}(0) = 2 \cdot \sum_{k=0}^{\left[\sum\limits_{i=1}^{n} p_i\right]} \frac{P^{(k)}(0)}{k!} \cdot s_k \ , \quad P(x) = \prod_{i=1}^{n} (p_i x + q_i) \ , \tag{15}$$
$$p_i \in (0,1) \ , q_i = 1 - p_i \text{ for } i = \overline{1,n} \text{ and } \lim_{n\to\infty} \sum_{k=1}^{n} p_k = \infty.$$

From (7), for $t = 0$, we have

$$\begin{cases} \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \cdot \theta(np-k) = \dfrac{1}{2} \\ \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \binom{n}{k} p^{n-k} \cdot q^k \cdot \theta(np-n+k) = \dfrac{1}{2}, \end{cases} \tag{16}$$

or

$$\begin{cases} \lim\limits_{n\to\infty} \sum\limits_{k=0}^{[np]} \binom{n}{k} p^k q^{n-k} = \dfrac{1}{2} \\ \lim\limits_{n\to\infty} \sum\limits_{k=[nq]+1}^{n} \binom{n}{k} p^{n-k} \cdot q^k = \dfrac{1}{2} \ , \quad nq \notin \mathbb{Z} \ , \quad q \in (0,1) \\ \lim\limits_{n\to\infty} \sum\limits_{k=nq}^{n} \binom{n}{k} p^{n-k} \cdot q^k = \dfrac{1}{2} \ , \quad nq \in \mathbb{Z} \ , \quad q \in (0,1) \end{cases} . \tag{17}$$

We consider the following cases:

Case 1. $[np] < [nq]$ where $nq \notin \mathbb{Z}$, $q \in (0,1)$, $p = 1 - q$. From the properties i) and ii) we get

$$p < \frac{n-1}{2n}. \tag{18}$$

Using Theorem 1, (17) and (18), we get the following regular transformation of the sequence of real numbers $(s_n)_{n\in\mathbb{N}}$:

$$T_n^{(3)}(\alpha_n) = \sum_{k=0}^{[n\alpha_n]} c_{n,k}^{(3)}(\alpha)\cdot s_k + \sum_{k=[n(1-\alpha_n)]+1}^{n} c_{n,k}^{(3)}(\alpha)\cdot s_k \quad \text{where} \tag{19}$$

$$c_{n,k}^{(3)}(\alpha) = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} \alpha_n^k(1-\alpha_n)^{n-k} & , \ k=\overline{0,[n\alpha_n]} \\ 0 & , \ k=\overline{[n\alpha_n]+1,[n(1-\alpha_n)]} \\ \begin{pmatrix} n \\ k \end{pmatrix} \alpha_n^{n-k}(1-\alpha_n)^{k} & , \ k=\overline{[n(1-\alpha_n)]+1,n} \ , \end{cases}$$

$\alpha_n \in \left(0, \dfrac{n-1}{2n}\right)$ and $n(1-\alpha_n)\notin\mathbb{Z}$.

Case 2. $[np]=[nq]$, $p\neq\dfrac{1}{2}$, $nq\notin\mathbb{Z}$, $q\in(0,1)$, $p=1-q$. From properties i) and ii), we get

$$[np]=\frac{n-1}{2}. \tag{20}$$

Equality from (20) is valid if n is odd number, i.e. $n=2m+1$, $m\in\mathbb{N}$. $[(2m+1)p]=m,\ \ m\le(2m+1)p<m+1,\ \ \dfrac{m}{2m+1}\le p<\dfrac{m+1}{2m+1}$.

Remark 2.2. *For $m\to\infty$ results $p=\dfrac{1}{2}$.*

Case 3. $[np]=nq-1$, $nq\in\mathbb{Z}$, $q\in(0,1)$, $p=1-q$.

From the fact that $nq\in\mathbb{Z}$, it follows that $np\in\mathbb{Z}$.

We have: $np=nq-1$, $np=n-np-1$, $2np=n-1$. Equality is valid if n is an odd number, that is $n=2m+1,\ m\in\mathbb{N}$.

Consequently

$$p=\frac{m}{2m+1}\ ,\quad q=\frac{m+1}{2m+1}. \tag{21}$$

Using Theorem 1, (17) and (21), we get the following regular transformation of the sequence of real numbers $(s_n)_{n\in\mathbb{N}}$:

$$T^{(4)}_{2m+1} = \frac{1}{(2m+1)^{2m+1}} \cdot \left\{ \sum_{k=0}^{m} \binom{2m+1}{k} m^k \cdot (m+1)^{2m+1-k} \cdot s_k + \right. \tag{22}$$
$$\left. + \sum_{k=m+1}^{2m+1} \binom{2m+1}{k} m^{2m+1-k} \cdot (m+1)^k \cdot s_k \right\}.$$

Case 4. $[np] < nq - 1$, $nq \in \mathbb{Z}$, $q \in (0,1)$, $p = 1 - q$. From the fact that $nq \in \mathbb{Z}$ it follows that $np \in \mathbb{Z}$.

We obtain: $np < n - np - 1$,

$$p < \frac{n-1}{2n}. \tag{23}$$

By using Theorem 1, (16) and (22), we get the following regular transformation of the sequence of real numbers $(s_n)_{n\in\mathbb{N}}$:

$$T^{(5)}_n(\alpha_n) = \sum_{k=0}^{n\alpha_n} c^{(5)}_{n,k}(\alpha) \cdot s_k + \sum_{k=n(1-\alpha_n)}^{n} c^{(5)}_{n,k}(\alpha) \cdot s_k, \text{ where} \tag{24}$$

$$c^{(5)}_{n,k}(\alpha) = \begin{cases} \binom{n}{k} \alpha_n^k (1-\alpha_n)^{n-k} & , \quad k = \overline{0, n\alpha_n} \\ 0 & , \quad k = \overline{n\alpha_n + 1, n(1-\alpha_n) - 1} \\ \binom{n}{k} \alpha_n^{n-k} (1-\alpha_n)^k & , \quad k = \overline{n(1-\alpha_n), n}, \end{cases}$$

$\alpha_n \in \left(0, \dfrac{n-1}{2n}\right)$ and $n \cdot \alpha_n \in \mathbb{Z}$.

From (9), for $t = 0$, we have

$$\left\{ \begin{array}{l} \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \dfrac{P_n^{(k)}(0)}{k!} \theta \left(\sum\limits_{i=1}^{n} p_i - k \right) = \dfrac{1}{2} \\ \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \dfrac{P_n^{(n-k)}(0)}{(n-k)!} \theta \left(\sum\limits_{i=1}^{n} p_i - n + k \right) = \dfrac{1}{2} \end{array} \right. \tag{25}$$

where $P_n(x) = \prod\limits_{i=1}^{n} (p_i x + q_i)$, $p_i \in (0,1)$, $q_i = 1 - p_i$ for $i = \overline{1,n}$ or

$$\left\{ \begin{array}{l} \lim\limits_{n\to\infty} \sum\limits_{k=0}^{\left[\sum_{i=1}^{n} p_i\right]} \dfrac{P_n^{(k)}(0)}{k!} = \dfrac{1}{2} \\ \lim\limits_{n\to\infty} \sum\limits_{k=\left[n-\sum_{i=1}^{n} p_i\right]+1}^{n} \dfrac{P_n^{(n-k)}(0)}{(n-k)!} = \dfrac{1}{2}\ , \quad \left[n - \sum\limits_{i=1}^{n} p_i \right] \notin \mathbb{Z} \\ \lim\limits_{n\to\infty} \sum\limits_{k=n-\sum_{i=1}^{n} p_i}^{n} \dfrac{P_n^{(n-k)}(0)}{(n-k)!} = \dfrac{1}{2}\ , \quad n - \sum\limits_{i=1}^{n} p_i \in \mathbb{Z} \end{array} \right. \tag{26}$$

We consider the following cases:

Case 5. $\left[\sum\limits_{i=1}^{n} p_i \right] < \left[n - \sum\limits_{i=1}^{n} p_i \right]$ where $n - \sum\limits_{i=1}^{n} p_i \notin \mathbb{Z}$, $p_i \in (0,1)$, $i = \overline{1,n}$.

From properties i) and ii), we get

$$\sum_{i=1}^{n} p_i < \frac{n-1}{2}. \tag{27}$$

By using Theorem 1, and (25) and (26), we get the following regular transformation of the sequence of real numbers $(s_n)_{n\in\mathbb{N}}$:

$$T_n^{(6)}(p) = \sum_{k=0}^{\left[\sum_{i=1}^{n} p_i\right]} c_{n,k}(p) s_k + \sum_{k=\left[n-\sum_{i=1}^{n} p_i\right]+1}^{n} c_{n,k}(p) s_k \tag{28}$$

$$\text{where } c_{n,k}(p)=\begin{cases}\dfrac{P_n^{(k)}(0)}{k!} & ,\quad k=\overline{0,\left[\sum_{i=1}^{n}p_i\right]}\\ 0 & ,\quad k=\overline{\left[\sum_{i=1}^{n}p_i\right]+1\ ,\quad \left[n-\sum_{i=1}^{n}p_i\right]}\ ,\\ \dfrac{P_n^{(n-k)}(0)}{(n-k)!} & ,\quad k=\overline{\left[n-\sum_{i=1}^{n}p_i\right]+1,n},\end{cases}$$

$0<\sum_{i=1}^{n}p_i<\dfrac{n-1}{2}$, $n-\sum_{i=1}^{n}p_i\notin\mathbb{Z}$, $p_i\in(0,1)$, $i=\overline{1,n}$.

Case 6. $\left[\sum_{i=1}^{n}p_i\right]=\left[n-\sum_{i=1}^{n}p_i\right]$, $n-\sum_{i=1}^{n}p_i\notin\mathbb{Z}$. We have $\left[\sum_{i=1}^{n}p_i\right]=\dfrac{n-1}{2}$. It follows that

$$\begin{cases}n=2m+1\ ,\quad m\in\mathbb{N}\\ \dfrac{n-1}{2}\le\sum_{i=1}^{n}p_i<\dfrac{n+1}{2}\ ,\quad m\le\sum_{i=1}^{2m+1}p_i<m+1.\end{cases}\tag{29}$$

Consequently, we consider the following regular transformation of the sequence $(s_n)_{n\in\mathbb{N}}$:

$$T_{2m+1}^{(7)}(p)=\sum_{k=0}^{\left[\sum_{i=1}^{2m+1}p_i\right]}c_{2m+1,k}(p)s_k+\sum_{k=\left[\sum_{i=1}^{2m+1}p_i\right]+1}^{2m+1}c_{2m+1,k}(p)s_k\tag{30}$$

where

$$c_{2m+1,k}(p)=\begin{cases}\dfrac{1}{k!}P_{2m+1}^{(k)}(0), & k=\left[0,\sum_{i=1}^{2m+1}p_i\right],m\in\mathbb{N}\\ \dfrac{1}{(2m+1-k)!}P_{2m+1}^{(2m+1-k)}(0), & k=\overline{\left[\sum_{i=1}^{2m+1}p_i\right]+1,2m+1}\end{cases}\ .$$

Case 7. $\left[\sum_{i=1}^{n} p_i\right] = n - \left(\sum_{i=1}^{n} p_i + 1\right)$, $n - \sum_{i=1}^{n} p_i \in \mathbb{Z}$.

We obtain $\sum_{i=1}^{n} p_i = \frac{n-1}{2}$, $n = 2m+1$, $m \in \mathbb{N}$ i.e.

$$\begin{cases} n = 2m+1 \\ \\ \sum_{i=1}^{2m+1} p_i = m \end{cases}, m \in \mathbb{N} \quad . \tag{31}$$

We consider the following regular transformation of the sequence $(s_n)_{n\in\mathbb{N}}$:

$$T_{2m+1}^{(8)}(p) = \sum_{k=0}^{m} c_{2m+1,k}(p) s_k + \sum_{k=m+1}^{2m+1} c_{2m+1,k}(p) s_k$$

where

$$c_{2m+1,k}(p) = \begin{cases} \dfrac{P_{2m+1}^{(k)}(0)}{k!} & , \ k = \overline{0,m} \\ \dfrac{P_{2m+1}^{(2m+1-k)}(0)}{(2m+1-k)!} & , \ k = \overline{m+1, 2m+1} \end{cases} , \sum_{i=1}^{2m+1} p_i \in \mathbb{Z}, m \in \mathbb{N}.$$

Case 8. $\left[\sum_{i=1}^{n} p_i\right] < n - \left(\sum_{i=1}^{n} p_i + 1\right)$, $\sum_{i=1}^{n} p_i \in \mathbb{Z}$. We obtain $\sum_{i=1}^{n} p_i < \frac{n-1}{2}$.

We consider the following regular transformation of the sequence $(s_n)_{n\in\mathbb{N}}$:

$$T_n^{(9)}(p) = \sum_{k=0}^{\sum_{i=1}^{n} p_i} c_{n,k}(p) s_k + \sum_{k=n-\sum_{i=1}^{n} p_i}^{n} c_{n,k}(p) s_k, \tag{32}$$

where $\begin{cases} \sum_{i=1}^{n} p_i \in \mathbb{Z} \ , \quad p_i \in (0,1) \ , \quad i = \overline{1,n} \\ \sum_{i=1}^{n} p_i < \dfrac{n-1}{2} \end{cases}$ and

$$c_{n,k}(p) = \begin{cases} \dfrac{P_n^{(k)}(0)}{k!} & , \ k = \overline{0, \sum_{i=1}^{n} p_i} \\ \dfrac{P_n^{(n-k)}(0)}{(n-k)!} & , \ k = \overline{\sum_{i=1}^{n} p_i + 1, n - \sum_{i=1}^{n} p_i - 1} \end{cases} .$$

Example 1. In (13) we consider $p = 1/2$; it follows that

$$T_n^{(1)}(0) = 2^{-n+1} \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{k} s_k.$$

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