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A preserving property of a the generalized Bernardi integral operator

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Abstract

In this paper we prove that the logarithmically n-spirallike of type α and order γ functions are preserved by a generalized Bernardi integral operator.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U and $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$.

Let consider the integral operator $I_a : A \to A$ defined as:

$$(1) \qquad f(z) = I_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt \ , \quad a \in \mathbb{C} \ , \quad Re\ a \geq 0.$$

In the case $a = 1, 2, 3, ...$ this operator was introduced by S. D. Bernardi and it was studied by many authors in different general cases.

Let D^n be the Sălăgean differential operator (see [7]) defined as:

$$D^n : A \to A \ , \quad n \in \mathbb{N} \ \text{ and } \ D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z) \ , \quad D^n f(z) = D(D^{n-1} f(z)).$$

2 Preliminary results

Definition 2.1. *Let $f \in A$ and $n \in \mathbb{N}$. We say that f is a n-starlike function if:*

$$Re \frac{D^{n+1} f(z)}{D^n f(z)} > 0 \ , \ z \in U.$$

We denote this class with ${S^*}_n$.

Definition 2.2. *Let $f \in A$ and $n \in \mathbb{N}$. We say that f is logarithmically n-spirallike of type $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and order $\gamma \in [0, 1)$ if $D^n f(z) \neq 0$, $z \in U$ and*

$$Re \left[e^{i\alpha} \frac{D^{n+1} f(z)}{D^n f(z)} \right] > \gamma cos \, \alpha \, , \, z \in U \, .$$

We denote this class with $S_{\alpha,n}(\gamma)$.

In the case $\gamma = 0$ we obtain the class $S_{\alpha,n}$ of the logarithmically n-spirallike of type α functions.

Remark 2.1. *If we consider $\alpha = \gamma = 0$ we obtain the concept of n-starlike functions and for $n = 0$ we obtain the class $S_\alpha(\gamma)$ of the spirallike functions of type α and order γ.*

The next theorem is result of the so called "admissible functions method" introduced by P. T. Mocanu and S. S. Miller (see [3], [4], [5]).

Theorem 2.1. *Let h convex in U and $Re\,[\beta h(z)+\delta]>0\,,\ z\in U$. If $q\in\mathcal{H}(U)$ with $q(0)=h(0)$ and q satisfied $q(z)+\dfrac{zq'(z)}{\beta q(z)+\delta}\prec h(z)$, then $q(z)\prec h(z)$.*

3 Main results

Theorem 3.1. *If $F(z)\in S_{\alpha,n}(\gamma)$ then $f(z)=I_aF(z)\in S_{\alpha,n}(\gamma)$.*

Proof. By differentiating (1) we obtain

$$(1+a)F(z)=af(z)+zf'(z)\,.$$

By means of the applications of the linear operator D^{n+1} we obtain:

$$(1+a)D^{n+1}F(z)=aD^{n+1}f(z)+D^{n+1}(zf'(z))$$

or

$$(1+a)D^{n+1}F(z)=aD^{n+1}f(z)+D^{n+2}f(z)\,.$$

It is easy to see that in the conditions of the hypothesis we have $D^nf(z)\neq 0\,,\ z\in U$.

With notation $\dfrac{D^{n+1}f(z)}{D^nf(z)}=p(z)$, where $p(z)=1+p_1z+....$, by simple calculations we obtain

$$\frac{D^{n+1}F(z)}{D^nF(z)}=p(z)+\frac{1}{p(z)+a}\cdot zp'(z)\,.$$

From here we have

$$e^{i\alpha}\frac{D^{n+1}F(z)}{D^nF(z)}=e^{i\alpha}p(z)+\frac{e^{i\alpha}}{p(z)+a}\cdot zp'(z)\,.$$

If we denote $e^{i\alpha}p(z)=q(z)$ we obtain

$$e^{i\alpha}\frac{D^{n+1}F(z)}{D^nF(z)}=q(z)+\frac{1}{e^{-i\alpha}q(z)+a}\cdot zq'(z)\,. \tag{2}$$

If we consider $h(z)$ a convex function, with $h(0)=e^{i\alpha}$, which maps the unit disc into the half plane $Re\, z > b$, where $b=\gamma cos\,\alpha \in [0,1)$, we have from (2):

$$q(z)+\frac{1}{e^{-i\alpha}q(z)+a}\cdot zq'(z)\prec h(z)\,.$$

In this conditions, using $Re\,a\geq 0$, we obtain $Re\left[e^{-i\alpha}h(z)+a\right]>0$. From Theorem (2.1), with $\beta=e^{-i\alpha}$ and $\delta=a$, we have $q(z)\prec h(z)$ or

$$e^{i\alpha}p(z)=e^{i\alpha}\frac{D^{n+1}f(z)}{D^nf(z)}\prec h(z)\,.$$

Thus we obtain $Re\left[e^{i\alpha}\dfrac{D^{n+1}f(z)}{D^nf(z)}\right]>\gamma cos\,\alpha\,,\ z\in U$ or $f(z)=I_aF(z)\in S_{\alpha,n}(\gamma)$.

If we take $\alpha=\gamma=0$ in Theorem (3.1) we obtain

Corollary 3.1. *If $F(z)\in {S_n}^*$ then $f(z)=I_aF(z)\in {S_n}^*$.*

Remark 3.1. *If we consider $\gamma=0$ in Theorem (3.1) we obtain the main result from [1].*

Remark 3.2. *In the case $n=0$ from Theorem (3.1) we obtain:*

If $F(z)\in S_\alpha(\gamma)$ then $f(z)=I_aF(z)\in S_\alpha(\gamma)$.

This result is a particular case of the more general results given by P.T. Mocanu and S.S. Miller in [6].

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