Computer Calculation of the Degree of Maps into the Poincaré Homology Sphere

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CONTENTS

Let M and P be Seifert 3-manifolds. Is there a degree one map $f : M \to P$? The problem was completely solved by Hayat-Legrand, Wang, and Zieschang for all cases except when P is the Poincaré homology sphere. We investigate the remaining case by elaborating and implementing a computer algorithm that calculates the degree. As a result, we get an explicit experimental expression for the degree through numerical invariants of the induced homomorphism $f_{\#} : \pi_1(M) \to \pi_1(P)$.

1. INTRODUCTION

Let M, P be closed oriented 3-manifolds such that the fundamental group $\pi_1(P)$ is finite of order n. Let $\varphi : \pi_1(M) \to \pi_1(P)$ be a homomorphism. Using elementary facts from obstruction theory, one can easily show that

(1) φ is geometrically realizable; that is, there exists a map $f : M \to P$ such that $f_* = \varphi$;
(2) $\deg(f)$ mod n depends only on φ.

The paper is devoted to the elaboration of a computer algorithm for calculating the degree. We apply the algorithm to maps into the Poincaré homology sphere P and under certain restrictions give an experimental explicit formula for $\deg(f)$ through numerical invariants of φ. The formula reduces the problem of finding out degree one maps onto P to purely number-theoretical questions. For additional background see [Hayat-Legrand et al. 1997; Hayat-Legrand and Zieschang 2000].

The calculation of the degree seems difficult to us, even for concrete examples. For instance, Plotnick [1982] constructed a self mapping of the Poincaré homology sphere which has degree 49 and induces an automorphism of the fundamental group. Our approach requires vast manipulations with group presentations and calculations in group rings.

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2. A BIT OF THEORY

Let $f : M \to P$ be a map between closed oriented 3-manifolds such that the fundamental group $\pi_1(P)$ is finite of order n. Let both manifolds be equipped with CW structures such that f is cellular. We also assume that both CW complexes have exactly one vertex, and that P has only one 3-cell. Let $p : \tilde{P} \to P$, $p_1 : \tilde{M} \to M$ be the universal coverings and $\tilde{f} : \tilde{M} \to \tilde{P}$ a lift of f to the universal coverings. We describe three items needed for calculating the degree.

The Boundary Cycle

Let $B_1, B_2, \dots, B_k$ be the 3-cells of M and $\beta_M \in C_3(M;\mathbb{Z})$ the corresponding 3-chain composed from the 3-cells taken with the orientation inherited from the orientation of M. For every 3-cell B_i we take a lift $\tilde{B}_i$. Then the *boundary cycle* $\partial\tilde{\beta}_M = \sum_{i=1}^k \partial\tilde{B}_i$ will be the first item needed for the calculation of the degree.

Remark 2.1. The group $C_2(\tilde{M};\mathbb{Z})$ considered as a $\pi_1(M)$-module with respect to the covering translations can be identified with the free $\pi_1(M)$-module $C_2(M;\mathbb{Z}[\pi_1(M)])$. To specify the identification, one should fix the orientations and the base points for the attaching curves of all 2-cells, as well as a base point for $\tilde{M}$ over the vertex of M.

The Characteristic Cochain

Choose a point x_0 in the interior of the unique 3-cell B^3 of P. The set $X = p^{-1}(x_0)$ can be considered as a 0-dimensional cycle in P with coefficients in $\mathbb{Z}_n$. Since $\tilde{P}$ is path connected and the coefficients are in $\mathbb{Z}_n$, every 0-chain of $\tilde{P}$ with coefficient sum divisible by n is a boundary. Hence X, consisting of n points, bounds a 1-dimensional chain Y in $\tilde{P}$ with coefficients in $\mathbb{Z}_n$. Note that X and Y are actually elements of the corresponding chain groups of $\tilde{D}$, where $\tilde{D}$ is the decomposition of $\tilde{P}$ dual to the one induced by the cell decomposition of P. Alternatively, one can consider X and Y as singular chains with the additional requirement that Y should be transverse to the 2-skeleton of $\tilde{P}$. Hence, for any 2-chain $\sigma \in C_2(\tilde{P};\mathbb{Z})$, the intersection number $\sigma \cap Y \in \mathbb{Z}_n$ is well-defined. Therefore we get a homomorphism $\xi_P : C_2(\tilde{P};\mathbb{Z}) \to \mathbb{Z}_n$, that is, a 2-cochain in $C^2(\tilde{P};\mathbb{Z}_n)$; this is the second item needed.

The Induced Chain Map

The third item we need for the calculation is the module homomorphism $\tilde{f}_* : C_2(M;\mathbb{Z}[\pi_1(M)]) \to C_2(P;\mathbb{Z}[\pi_1(P)])$. It can be described as the chain map induced by f and takes

$$C_2(\tilde{M};\mathbb{Z}) = C_2(M;\mathbb{Z}[\pi_1(M)])$$

to $C_2(\tilde{P};\mathbb{Z}) = C_2(P;\mathbb{Z}[\pi_1(P)])$. It is easy to see that $\tilde{f}_*$ preserves the module structures in the sense that for all $g \in \pi_1(M)$, $\sigma \in C_2(M;\mathbb{Z}[\pi_1(M)])$ we have $\tilde{f}_*(g\sigma) = f_\#(g)\tilde{f}_*(\sigma)$, where $f_\# : \pi_1(M) \to \pi_1(P)$ is the induced homomorphism.

Theorem 2.1. $\deg(f) \equiv -\xi_P(\tilde{f}_*(\partial\tilde{\beta}_M)) \bmod n$.

Proof. Let $\sigma^1 \in C_1(Q;\mathbb{Z}_n)$, $\sigma^3 \in C_3(Q;\mathbb{Z})$ be two (say, singular) chains in an orientable 3-manifold Q in general position; in particular, their boundary cycles are disjoint. Then the linking number $\mathrm{lk}(\partial\sigma^3, \partial\sigma^1) \in \mathbb{Z}_n$ is well-defined. It can be calculated as the intersection number $\sigma^3 \cap \partial\sigma^1$ as well as the intersection number $\partial\sigma^3 \cap \sigma^1$ multiplied by (-1), see [Seifert and Threlfall 1934, Chap. 10], for details. Thus $\sigma^3 \cap \partial\sigma^1 = -\partial\sigma^3 \cap \sigma^1$. Taking $Q = \tilde{P}$, $\sigma^1 = Y$ and $\sigma^3 = \tilde{f}_*(\tilde{\beta}_M)$, we get $\tilde{f}_*(\tilde{\beta}_M) \cap X = -\partial\tilde{f}_*(\tilde{\beta}_M) \cap Y$. The right side of the equality is just $-\xi_P(\tilde{f}_*(\partial\tilde{\beta}_M))$, by the definition of ξ_P. It remains to show that the left part equals $\deg(f)$.

Let B_i be a closed 3-cell of M. Denote by k_i the degree of the restriction map $f_| : (B_i, \partial B_i) \to (B, \partial B)$. Then we have

$$\begin{array}{ccc}
\mathbb{Z} \cong H_3(B_i, \partial B_i) & \xrightarrow{\times k_i} & H_3(B, \partial B) \cong \mathbb{Z} \\
\Big\downarrow{\scriptstyle i_*} & & \Big\| \\
\mathbb{Z}^k \cong H_3(M, M^{(2)}) & \xrightarrow[f_*]{} & H_3(P, P^{(2)}) \cong \mathbb{Z},
\end{array}$$

where $M^{(2)}$ and $P^{(2)}$ denote the 2-skeletons of M and P, respectively. Here i_* sends the generator of $H_3(B_i, \partial B_i)$ to the i-th generator of $H_3(M, M^{(2)})$, considered as the 3-dimensional cellular chain group of the CW-complex M. Denote this cellular chain by $\{B_i\}$; thus $f_*(\{B_i\}) = k_i\{B\}$. For the fundamental cycle of M we have $\{M\} = \sum_{i=1}^k \{B_i\}$. Clearly, the algebraic intersection number $\{B\} \cap \{x_0\}$ is 1. It follows that k_i coincides with the algebraic intersection number $f_*(\{B_i\}) \cap \{x_0\}$; after a deformation of f in B_i which does not alter f on $M^{(2)}$, we may assume that this number is also the geometric intersection number, which is to say that there are exactly k_i points in B_i which are mapped to x_0. To calculate it, note that for any lifting $\tilde{B}_i$ of B_i we have

$$\begin{aligned}\tilde{f}_*(\{\tilde{B}_i\}) \cap X &= \tilde{f}_*(\{\tilde{B}_i\}) \cap p^{-1}(x_0) \\ &= f_*(\{B_i\}) \cap x_0 = k_i.\end{aligned}$$

Summing up, $\tilde{f}_*(\tilde{\beta}_M) \cap X = \deg(f)$. □

Remark 2.2. It is important to have in mind that ξ_P and $\partial\tilde{\beta}_M$ depend only on P and M, respectively, but not on f.

3. HOW TO CALCULATE THE BOUNDARY CYCLE

3.1. The General Case

Let M be an oriented CW 3-manifold such that its 2-skeleton K_M^2 has exactly one vertex. We say that the presentation $\langle a_1, \dots, a_r \mid R_1, \dots R_q \rangle$ of $\pi_1(M)$ corresponds to K_M^2 if

(1) all edges of K_M^2 are oriented and correspond bijectively to the generators $a_1, \dots, a_r$;

(2) all 2-cells of K_M^2 are oriented and correspond bijectively to the relations $R_1, \dots, R_q$;

(3) the boundary curve of each 2-cell is equipped with a base point such that, starting from this point, the curve follows the edges just so as they are written in the corresponding relation.

Let $\langle a_1, \dots a_r \mid R_1, \dots R_q \rangle$ corresponds to K_M^2, and let B be a 3-cell of M. The simplest way to calculate the contribution $\partial\tilde{\beta}_B$ to the boundary cycle made by the boundary of B is to construct a *spherical diagram* for the attaching map $h_B : S_B^2 \to K_M^2$ of B, that is, a cellular decomposition θ_B of S_B^2 such that:

(1) every edge of θ_B is oriented and labeled with a generator a_i;

(2) every 2-cell of θ_B is oriented and labeled with a relation R_j;

(3) the boundary curve of each 2-cell is equipped with a base point such that, starting from this point, the curve follows the edges just so as they are written in the corresponding relation;

(4) the attaching map h_B is cellular and preserves the labels, base points, and orientations of the edges and 2-cells.

Fix a vertex v of S_B^2 as base point for S_B^2, and let us assign to every 2-cell c of S_B^2 the following data:

(1) the sign $\varepsilon(c) = \pm 1$ that shows whether or not the orientation of c agrees with the orientation of S_B^2 induced by the fixed orientation of $B \subset M$;

(2) the element $g(c) = [h_B(\gamma(c))]$ of $\pi_1(M)$, where $\gamma(c)$ is a path in S_B^2 joining v to the base point of c and $[h_B(\gamma(c))]$ denotes the element of $\pi_1(M)$ that corresponds to the loop $h_B(\gamma(c))$ in K_M^2;

(3) the relation $R_{i(c)}$ which labels c.

It is convenient to regard the chain group $C_2(\tilde{M}, \mathbb{Z})$ as the $\pi_1(M)$-module freely generated by the relations $R_1, \dots, R_q$. The proof of the following statement is evident.

Lemma 3.1.1. *The contribution $\partial\tilde{\beta}_B$ made by a 3-cell B of M to $\partial\tilde{\beta}_M$ equals $\sum_c \varepsilon(c) g(c) R_{i(c)}$, where the sum is taken over all 2-cells c in S_B^2.*

3.2. A Useful Example

We consider an informative example. Present the torus $T^2 = S^1 \times S^1$ as a CW complex with one vertex, two edges $a = S^1 \times \{*\}$, $t = \{*\} \times S^1$, and one 2-cell r_1 that corresponds to the relation $R_1 = at^{-1}a^{-1}t$ of $\pi_1(T^2) = \langle a, t \mid R_1 \rangle$. Choose a pair α, β of coprime integers, $\alpha \geq 0$. We extend the cell decomposition of T^2 to a cell decomposition of a solid torus M with $\partial M = T^2$ by attaching two new cells: a 2-cell r_2 and a 3-cell B. Note that the boundary curve m of r_2 lies in $a \cup t$ and thus can be written as a word in generators a, t. To make the situation interesting, we require that m wraps totally α times around a and β times around t. In other words, the corresponding word (denote it by $w_{\alpha,\beta}(a,t)$) should determine the element $\alpha a + \beta t$ in the homology group $H_1(a \cup t; \mathbb{Z})$. In general, we cannot take $w_{\alpha,\beta}(a,t) = a^\alpha t^\beta$ since the curve $a^\alpha t^\beta$ in $a \cup t$ bounds a meridional disc in M with embedded interior if and only if $|\alpha| = 1$ or $|\beta| = 1$.

We describe a simple geometric procedure for finding out $w_{\alpha,\beta}(a,t)$. Present a regular neighborhood N of $a \cup t$ in T^2 as the union of a disc with two disjoint strips (handles of index one). The key observation is that m, being the boundary of a meridional disc of the solid torus, can be shifted in N to a simple closed curve m_1 which is normal (in the sense of [Haken 1961]) with respect to the handle decomposition of N. The normality of m_1 means that it can be constructed as follows:

1. Take α parallel copies of a and $|\beta|$ parallel copies of t in the strips such that the end points of them lie on the boundary of the disc around the vertex;
2. Join the copies inside the disc by disjoint arcs such that no arc has both end points at the same end of the same strip. This can be done in two ways, and the right choice depends on the sign of β.

To get $w_{\alpha,\beta}$, it only remains to read it off by traversing m_1. See Figure 1 for the case $\alpha = 5$, $\beta = 2$, when we get $w_{\alpha,\beta} = a^3ta^2t$.

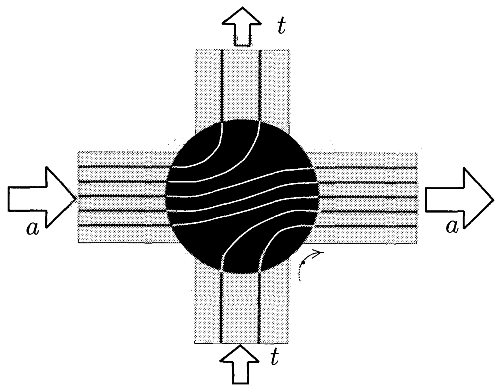

FIGURE 1. Simple closed curve of the type (α, β).

Lemma 3.2.1. *Let K be a CW complex realizing the presentation $\langle a, t \mid R_1, R_2\rangle$, where $R_1 = at^{-1}a^{-1}t$ and $R_2 = w_{\alpha,\beta}(a,t)$ for a pair α, β of coprime integers, $\alpha \geq 0$. Suppose that a solid torus M is obtained from K by attaching a 3-cell B. Then $\partial\tilde{\beta}_M = -R_1 + (1 - a^xt^y)R_2$, where $\alpha y - \beta x = 1$.*

Proof. To compute the boundary cycle $\partial\tilde{\beta}_M$ (which in our case coincides with the contribution $\partial\tilde{\beta}_B$ made by B), we construct a spherical diagram for B. It contains only three 2-cells c_1, c_2, c_3 labelled by R_1, R_2, R_2 and having signs $-1, 1, -1$, respectively. We choose the common base point of c_1, c_2 as a global vertex v; see Figure 2.

By Lemma 3.1.1, we get $\partial\tilde{\beta}_B = -R_1 + R_2 - g(c_3)R_2$, where $g(c_3) \in \pi_1(M)$ corresponds to a path $\gamma(c_3)$ in S^2_B joining v to the base point of c_3. Note that if we push slightly the loop $h_B(\gamma(c_3))$ into the

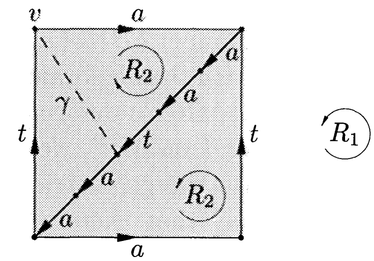

FIGURE 2. Spherical diagram for the 3-cell of M.

interior of M, we get a circle that intersects the meridional disc r_2 of the solid torus M positively in exactly one point. It follows that $g(c_3)$ can be presented as $a^xt^y \in \pi_1(M)$, where the integers x, y satisfy the equation $\alpha y - \beta x = 1$ and thus serve as the coordinates of a positively oriented longitude of the torus. □

Remark 3.2.1. The following simple rules can be used for recursive calculating of the word $w_{\alpha,\beta}$:

$$w_{1,0}(a,t) = a,\ w_{0,\pm1}(a,t) = t^{\pm1};$$
$$w_{\alpha+\beta,\beta}(a,t) = w_{\alpha,\beta}(a,at);$$
$$w_{\alpha,\alpha+\beta}(a,t) = w_{\alpha,\beta}(at,t).$$

An alternative approach to the $w_{\alpha,\beta}$ can be found in [Osborne and Zieschang 1981; Lustig et al. 1995]; compare also [Gonzálvez-Acuña and Ramírez 1999].

3.3. Boundary Cycles of Seifert Manifolds

We restrict ourselves to Seifert manifolds fibered over the 2-sphere with three exceptional fibers. Let $M = M\big((\alpha_1,\beta_1);(\alpha_2,\beta_2);(\alpha_3,\beta_3)\big)$ be an oriented Seifert manifold, where the α_i and β_i are non-normalized parameters of the exceptional fibers, with $\alpha_i \geq 0$, $1 \leq i \leq 3$. The orientation of M is induced by the standard orientation of S^2 and an orientation of a regular fiber. Then $\pi_1(M)$ can be presented by

$$\langle a_1, a_2, a_3, t \mid a_ita_i^{-1}t^{-1},\ a_i^{\alpha_i}t^{\beta_i}, i = 1,2,3,\ a_1a_2a_3\rangle$$

(t corresponds to the oriented regular fiber), but this presentation does not correspond to the 2-skeleton of a CW structure of M. To ameliorate this shortcoming, we will use another presentation of the same group with the same generators, namely,

$$\langle a_1, a_2, a_3, t \mid R_j, 1 \leq j \leq 7\rangle,$$

where $R_{2i-1} = a_it^{-1}a_i^{-1}t, R_{2i} = w_{\alpha_i\beta_i}(a_i,t)$ for $i = 1,2,3$, $R_7 = a_1a_2a_3$. The words $w_{\alpha_i\beta_i}$ have been described above. The CW complex that realizes this presentation embeds in M such that the complement consists of four 3-balls $B_i, 1 \leq i \leq 4$. The first three of them correspond to solid tori containing exceptional fibers, and the last corresponds to a regular fiber.

Theorem 3.3.1. *If* $M = M\big((\alpha_1,\beta_1);(\alpha_2,\beta_2);(\alpha_3,\beta_3)\big)$ *then*

$$\partial\tilde{\beta}_M = \sum_{i=1}^{3}\left(-R_{2i-1} + (1 - a_i^{x_i}t^{y_i})R_{2i}\right) + \big(R_1 + a_1R_3 + a_1a_2R_5 - (1-t)R_7\big),$$

where $\alpha_i y_i - \beta_i x_i = 1$.

Proof. The first three summands can be obtained by Lemma 3.2.1. The last summand can be obtained in a similar way. The only difference from the proof of Lemma 3.2.1 is that the corresponding spherical diagram, instead of one 2-cell with one odd R-label, contains three positively oriented 2-cells with labels R_1, R_3, R_5. □

4. HOW TO CALCULATE THE CHARACTERISTIC COCHAIN

Let P be a closed oriented 3-manifold with the finite fundamental group $\pi_1(P)$ of order n. We assume that P is equipped with a CW structure such that there is only one vertex v, only one 3-cell B, and the 2-dimensional skeleton K_P^2 of P defines a presentation $\langle b_1,\dots,b_s \mid Q_1,\dots,Q_s\rangle$ of $\pi_1(P) = \pi_1(P;v)$. We will identify generators and relations of $\pi_1(P)$ with edges and 2-cells of K_P^2, respectively. Fix a base point x_0 in the interior of B and a base point $\tilde{x}_0 \in p^{-1}(x_0)$ in the universal cover $\tilde{P}$ of P. For each $i = 1,\dots,s$ choose a loop u_i in P with end points in x_0 such that $K_P^2 \cap u_i$ is a point in Q_i and the intersection is transverse and positive. Clearly, the loops $u_i, 1 \le i \le s$, generate the group $\pi_1(P;x_0)$ isomorphic to $\pi_1(P;v)$; we will call them *dual generators*.

Let w be a loop in P written as word in the generators u_i. Denote by $\tilde{w}$ the lifting of w to $\tilde{P}$ with the initial point $\tilde{x}_0$. Then the formula $\xi_w(c) = c \cap \tilde{w}$, where $c \in C_2(\tilde{P};\mathbb{Z})$ and $c\cap\tilde{w}$ is the intersection number, determines a homomorphism $\xi_w : C_2(\tilde{P};\mathbb{Z}) \to \mathbb{Z}$ and hence a cochain $\xi_w \in C^2(\tilde{P};\mathbb{Z})$. It follows from the construction that

(1) $\xi_{u_i}(Q_j) = \delta_i^j$ where δ_i^j is the Kronecker symbol;
(2) $\xi_{w_1w_2} = \xi_{w_1} + \bar{w}_1\xi_{w_2}$ where $\bar{w}_1 \in \pi_1(P)$ corresponds to w_1 and the action of $\pi_1(P)$ on cochains is given by $g(\xi)(c) = \xi(g(c))$.

These rules describe actually nothing more than a sort of *Fox calculus* [Crowell and Fox 1963, Chap. VII; Burde and Zieschang 1985, Chap. 9]. They are sufficient for finding ξ_w for any word w in the generators u_i.

Proposition 4.1. *Let the words* $w_0,\dots,w_{n-1}$ *in generators* u_i *present* (*without repetitions*) *all elements of* $\pi_1(P;x_0)$. *Then*

$$\xi_P = \sum_{i=0}^{n-1}\xi_{w_i} \bmod n$$

is a characteristic cochain for P.

Proof. Evident, since the union of the paths given by w_i for $0 \le i \le n-1$ presents a 1-chain Y such that ∂Y is the union of all points in $p^{-1}(x_0)$. □

We may conclude that all what we need for calculation is a sort of *normal form* for elements of $\pi_1(P)$, that is, a list of words in the dual generators that present without repetitions all elements of $\pi_1(P)$.

5. HOW TO CALCULATE THE INDUCED CHAIN MAP

5.1. Taking Logs

Recall that M and P are closed oriented CW three-manifolds such that $\pi_1(P)$ is finite of order n. Let $\langle a_1,\dots a_r \mid R_1,\dots R_q\rangle$ and $\langle b_1,\dots,b_s \mid Q_1,\dots,Q_s\rangle$ be presentations of their fundamental groups, corresponding to the 2-skeletons of M and P, respectively. Suppose that the homomorphism $\varphi = f_\# : \pi_1(M) \to \pi_1(P)$ is given by a set of words h_i in the generators b_j that represent the elements $\varphi(a_i)$ of $\pi_1(P), 1 \le i \le r$. We consider the chain group $C_2(M,\mathbb{Z}[\pi_1(M)])$ as a free $\mathbb{Z}[\pi_1(M)]$-module generated by the set $R_1,\dots,R_q$; similarly, $C_2(P,\mathbb{Z}[\pi_1(P)])$ is a free module generated by $Q_1,\dots,Q_s$. Denote by K the kernel of the quotient map $F(b_1,\dots,b_s) \to \pi_1(P)$, where $F = F(b_1,\dots,b_s)$ is the free group generated by $b_1,\dots,b_s$.

Let R be one of the relations R_i. To calculate the image $f_*(R) \in C_2(P,\mathbb{Z}[\pi_1(P)])$ of the corresponding 2-cell, one may do the following:

(1) Replace each generator $a_i^{\pm1}$ in R by the corresponding word $h_i^{\pm1}$. We get a word $w \in K$.

(2) Present w as a product of conjugates of the defining relations, that is, in the form

$$w = \prod_k v_k Q_{i_k}^{\varepsilon_k} v_k^{-1},$$

where $\varepsilon_k = \pm 1$ and $v_k \in F$.

(3) Then the image $f_*(R)$ is obtained by "taking logs": $f_*(R) = \log w$, where $\log w = \sum_k \varepsilon_k \bar{v}_k Q_{i_k}$ and $\bar{v}_k$ is the image of v_k in $\pi_1(P)$.

Note that $\log w$ is a multivalued function since w can be presented as a product of conjugated relations in many different ways. This corresponds to the fact that φ can be realized by many different f; this arbitrariness does not affect the degree. One should point out that finding $\log w$ (actually, step (2) above) is a nontrivial procedure. The problem is to realize it algorithmically. We solve the problem for the case when P is the Poincaré homology sphere.

5.2. On the Fundamental Group of the Poincaré Sphere

Henceforth we denote by P the Poincaré sphere. It is homeomorphic to the Seifert manifold

$$M\big((2,1),\,(3,1),\,(5,-4)\big).$$

Its fundamental group $\pi_1(P)$ consists of 120 elements and is isomorphic to $\mathrm{SL}_2(\mathbb{Z}_5)$ and to the binary icosahedral group I^*. We will denote it by π.

We will use the presentation $\langle a, c \,|\, Q_1, Q_2\rangle$, where $Q_1 = c^5a^{-2}$ and $Q_2 = aca^{-1}ca^{-1}c$. This presentation corresponds to the Heegaard diagram of P shown in Figure 3. Therefore it corresponds to the 2-skeleton of the natural CW structure of M.

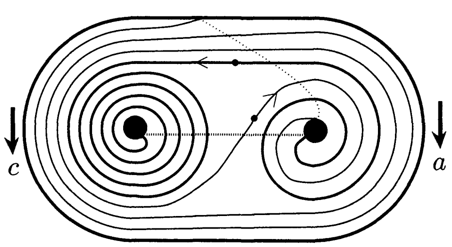

FIGURE 3. Heegaard diagram of the Poincaré homology sphere.

Remark 5.2.1. Looking at Figure 3, one can easily find the dual generators (see Section 4): $u_1 = c, u_2 = c^{-1}a$.

We now work a little with the relations $c^5a^{-2} = 1$, $aca^{-1}ca^{-1}c = 1$. Our goal is to get new relations appropriate for later use. We will use essentially one basic transformation:

($*$) If $w_1 = w_2$ is a relation, then $uw_1v = uw_2v$ is a relation for any words u, v.

Lemma 5.2.1. *In π, the following relations are true*:

(1) $aca = c^4ac^{-1}, \quad ac^{-1}a = cac$;
(2) $ac^{-2}a = c^{-4}ac^2ac$;
(3) $ac^2ac^2a = c^4ac^2ac^{-1}$;
(4) $ac^2ac^{-2}ac^{-1} = cac^2ac^{-2}a$;
(5) $c^{10} = 1$.

Proof. The first three relations are easily obtained, so we concentrate on the last two.

(a) Multiplying $Q_2 = 1$ by $c^{-1}ac^{-1}$, we get $aca^{-1} = c^{-1}ac^{-1}$, which implies (1).

(b) Using (a), we obtain

$$ac^{-2}a = c\,c^{-1}ac^{-1}\,c^{-1}ac^{-1}\,c = c\,aca^{-1}aca^{-1}c = cac^2a^{-1}c.$$

(c) Let $w = ac^2ac^{-2}a$. A consequence of (b) and $a^2 = c^5$ is

$$wc^{-1} = ac^2\,ac^{-2}a\,c^{-1} = ac^3ac^2a^{-1} = ac^{-2}a\,c^2a = c\,ac^2a^{-1}c^3a = cw.$$

(d) It follows from (c) that

$$c^{10} = c^5ww^{-1}c^5 = wc^{-5}w^{-1}c^5.$$

On the other hand, the relation $a^2 = c^5$ allows one to permute c^{-5} with any other word. Thus $c^{10} = wc^{-5}w^{-1}c^5 = ww^{-1}c^{-5}c^5 = 1$. □

The following list L presents the 120 elements of π: c^i, c^iac^j, $c^iac^2ac^j$, $c^iac^2ac^{-2}a$, where $0 \le i \le 9$ and $0 \le j \le 4$. The words from L will be called *normal forms.*

Lemma 5.2.2. *There exists an effective algorithm that transforms any word in the generators a, c to a normal form presenting the same element of π.*

Proof. The normalizing algorithm works with reduced words. Thus, before and after each step, one should reduce the word we are working with. By the *a-size* of a word in generators a, c we mean the total number of occurrences of $a^{\pm 1}$ in it.

Let w be a word in generators a, c. Steps 1–5 below are based on the corresponding relations 1–5 from Lemma 5.2.1.

Step 0. Using the relation $a^2 = c^5$, we transform w to the form $w = c^{k_1}ac^{k_2}a\dots c^{k_m}ac^{k_{m+1}}$, where $m \geq 0$ and $k_i = \pm 1, \pm 2$ for $1 < i < m+1$ and $|k_{m+1}| \leq 2$.

Step 1. If w contains the subwords aca or $ac^{-1}a$, we replace them by c^4ac^{-1} or cac, respectively.

Step 2. If the initial segment of w has the form $c^{k_1}ac^{-2}a$ (that is, if $k_2 = -2$ and $m \geq 2$), we replace it by $c^{k_1-4}ac^2ac$.

Step 3. If the initial segment of w has the form $c^{k_1}ac^2ac^2a$, we replace it by $c^{k_1+4}ac^2ac^{-1}$.

Step 4. If the initial segment of w has the form $c^{k_1}ac^2ac^{-2}ac^{k_4}$, we replace it by $c^{k_1-k_4}ac^2ac^{-2}a$.

Step 5. We reduce the power k_1 of the first term c^{k_1} of w modulo 10.

Now we are ready to describe the algorithm. We apply Steps 0–5 as long as possible to the given word w. Since Steps 1–4 strictly decrease the a-size, the process terminates after a finite number of steps. It is easy to verify that the resulting word is in normal form. □

5.3. Logs in the Case of the Poincaré Sphere

The algorithm for calculating logs is similar to the one described in Lemma 5.2.2. The only difference is that instead of operating with words in the free group $F = F(a, c)$ we will operate with their *shadows*.

Let $\mathcal{M}$ be the free π-module generated by Q_1, Q_2. We define the *shadow group* $\mathcal{S}(F)$ of F to be the semidirect product $\mathcal{M} \rtimes F$, where the operation of F on $\mathcal{M}$ is induced by the action of π (recall that π is the quotient of F). In other words, $\mathcal{S}(F)$ consists of pairs (μ, w), where $\mu \in \mathcal{M}$, $w \in F$. The multiplication is given by the rule $(\nu, u)(\mu, w) = (\nu + \bar{u}\mu,\, uw)$, where $\bar{u}$ is the image of u in π. Note that the unit of the group $\mathcal{S}(F)$ is $(0, 1)$, and the inverse element of (μ, w) is $(-\bar{w}^{-1}\mu,\, w^{-1})$.

Remark 5.3.1. The above formula $(\nu, u)(\mu, w) = (\nu + \bar{u}\mu, uw)$ is a formalization of the algebraic identity $\nu u \cdot \mu w = \nu\mu^u \cdot uw$, where $\mu^u = u\mu u^{-1}$ and ν, μ are products of the relations $Q_1^{\pm 1}, Q_2^{\pm 1}$ and their conjugates.

If $w \in F$, any pair $(\mu, w) \in \mathcal{S}(F)$ is called a *shadow* of w. The shadow $(0, w)$ is called *pure*. Similarly, the pair $(\mu, 1)$ is the *pure shadow* of $\mu \in \mathcal{M}$. Note that any product of shadows can be replaced by a product of pure shadows:

$$\textstyle\prod_{i=1}^{k}(\lambda_i, w_i) = (\mu, 1)\prod_{i=1}^{k}(0, w_i) = (\mu, 1)\big(0, \prod_{i=1}^{k} w_i\big),$$

where

$$\mu = \lambda_1 + \bar{w}_1\lambda_2 + \dots + \bar{w}_1\bar{w}_2\dots\bar{w}_{k-1}\lambda_k.$$

In other words, one can *purify* the factors. Recall that the kernel K of the quotient map $F \to \pi$ is the normal subgroup of F generated by Q_1, Q_2. Let $\mathcal{S}(K)$ denote the normal subgroup of $\mathcal{S}(F)$ generated by elements $(-Q_i, Q_i)$, for $i = 1, 2$. Note that the left Q_i in the above expression is considered as a generator of $\mathcal{M}$ while the right Q_i is the word c^5a^{-2} or $aca^{-1}ca^{-1}c$. We define the *shadow group* of π as

$$\mathcal{S}(\pi) = \mathcal{S}(F)/\mathcal{S}(K).$$

Another way to get $\mathcal{S}(\pi)$ is to take the quotient of $\mathcal{S}(F)$ by the relations $(0, Q_i) = (Q_i, 1)$, for $i = 1, 2$.

Lemma 5.3.1. $(-\mu, w) \in \mathcal{S}(K) \iff w \in K$ *and* $\mu = \log w$.

Proof. Recall that an element of a group lies in the normal subgroup generated by some elements if and only if it is a product of conjugates of the generators and their inverses. Thus for some $(\lambda_k, v_k) \in \mathcal{S}(F)$, $\varepsilon_k = \pm 1$, and $i_k \in \{1, 2\}$, the condition

$$(-\mu, w) \in \mathcal{S}(K)$$

is equivalent to

$$\begin{aligned}(-\mu, w) &= \prod_k (\lambda_k, v_k)(-\varepsilon_k Q_{i_k}, Q_{i_k}^{\varepsilon_k})(-\bar{v}_k^{-1}\lambda_k, v_k^{-1}) \\ &= \prod_k (-\varepsilon_k \bar{v}_k Q_{i_k}, v_k Q_{i_k}^{\varepsilon_k}\bar{v}_k^{-1}) \\ &= \Big(-\sum_k \varepsilon_k \bar{v}_k Q_{i_k}, \prod_k v_k Q_{i_k}^{\varepsilon_k} v_k^{-1}\Big) \\ &= (-\log w,\, w)\end{aligned}$$

by the definition of log (see Section 5.1). □

To construct an algorithm for calculating logs, we need a shadow counterpart of Lemma 5.2.1.

Lemma 5.3.2. *One can calculate* $\lambda_0, \dots, \lambda_5 \in \mathcal{M}$ *such that in* $\mathcal{S}(\pi)$ *the following relations are true*:

$\mathcal{S}(1)$ $(0, aca) = (\lambda_0, c^4ac^{-1})$, $(0, ac^{-1}a) = (\lambda_1, cac)$;

$\mathcal{S}(2)$ $(0, ac^{-2}a) = (\lambda_2, c^{-4}ac^2ac)$;

$\mathcal{S}(3)$ $(0, ac^2ac^2a) = (\lambda_3, c^4ac^2ac^{-1})$;

$\mathcal{S}(4)$ $(0, ac^2ac^{-2}ac^{-1}) = (\lambda_4, cac^2ac^{-2}a)$;

$\mathcal{S}(5)$ $(0, c^{10}) = (\lambda_5, 1)$.

Proof. The existence of λ_i is evident, since both sides of each relation are shadows of the same element of π. The problem consists in calculating λ_i. To solve it, we repeat the proof of Lemma 5.2.1 in terms of shadows starting with the relations $(0, Q_i) = (Q_i, 1)$ instead of $Q_i = 1$, $i = 1, 2$. In particular, we apply the following shadow version of the basic transformation $(*)$ of Section 5.2:

$\mathcal{S}(*)$ If $(0, w_1) = (\lambda, w_2)$ is a relation, so is $(0, uw_1v) = (\bar{u}\lambda, uw_2v)$ for any words u, v.

For example, the shadow versions of the items (a) and (b) in the proof of Lemma 5.2.1 look as follows:

(a) Multiplying $(0, Q_2) = (Q_2, 1)$ by $(0, c^{-1}ac^{-1})$, we get $(0, aca^{-1}) = (Q_2, c^{-1}ac^{-1})$ or, equivalently, $(0, c^{-1}ac^{-1}) = (-Q_2, aca^{-1})$;

(b) Using (a), we get

$$\begin{aligned}(0, ac^{-2}a) &= (0, c\, c^{-1}ac^{-1}\; c^{-1}ac^{-1}\; c)\\ &= (0, c)(-Q_2, aca^{-1})(-Q_2, aca^{-1})(0, c)\\ &= (-(\bar{c} + \bar{c}\bar{a}\bar{c}\bar{a}^{-1})Q_2, cac^2a^{-1}c).\end{aligned}$$

Recall that each λ_i is an element of the free π-module $\mathcal{M}$ with two generators Q_1, Q_2 and thus can be described by an array of 240 integers. We do not present here the values of λ_i since they are large (especially λ_5) in the sense that many of the 240 integers presenting each λ_i are not zeros. Nevertheless, the authors calculated them, and in the sequel we will think of them as known. $\square$

Proposition 5.3.1. *There exists an effective algorithm that, given $w \in K$, calculates $\log w \in \mathcal{M}$.*

Proof. The algorithm is a shadow twin of the one described in Lemma 5.2.2. Starting with the shadow $(0, w)$ of w, we apply shadow versions of Steps 0–5 as long as possible. It means that we use the shadow relations $\mathcal{S}(1)$–$\mathcal{S}(5)$ from Lemma 5.3.2 instead of the relations (1)–(5) from Lemma 5.2.1. After each step we purify the words by taking nonzero lambdas to the beginning of the word. We terminate with a shadow $(\mu, 1)$ of 1. By Lemma 5.3.1, $\log w = \mu$ since $(0, w) = (\mu, 1)$ in $\mathcal{S}(\pi)$ implies $(-\mu, w) \in \mathcal{S}(K)$. $\square$

6. COMPUTER IMPLEMENTATION

6.1. Description and Verification of the Program

Recall that calculation of the degree of a map $f : M \to P$ requires knowledge of the three items: the boundary cycle $\partial\tilde{\beta}_M$, the characteristic cochain ξ_P, and the induced chain map $\tilde{f}_*$, see Section 2. For $M = M\big((\alpha_1, \beta_1); (\alpha_2, \beta_2); (\alpha_3, \beta_3)\big)$ an explicit expression for $\partial\tilde{\beta}_M$ was obtained in Theorem 3.3.1. Proposition 4.1 and the information on $\pi = \pi_1(P)$ obtained in Section 5.2 show how to calculate ξ_P. The authors did this by hand, without computers. It is the calculation of $\tilde{f}_*$ that requires a computer.

We assume that f is given by images $\tau, x_1, x_3 \in F = F(a, c)$ of the generators t, a_1, a_3 of $\pi_1(M)$, respectively; the image x_2 of the generator a_2 can be found from the relation $a_1a_2a_3 = 1$. We describe the main steps of the computer program.

(1) For each relation $R_i, 1 \le i \le 7$, of the presentation of $\pi_1(M)$ (see Section 3.3), the computer determines its image w_i in F.

(2) Then the computer works according to the algorithm described in Proposition 5.3.1 and finds the logs of all w_i. This is sufficient to obtain $\tilde{f}_*$ since $\tilde{f}_*(R_i) = \log w_i$.

(3) To get $\tilde{f}_*(\partial\tilde{\beta}_M)$, the computer substitutes all relations R_i in the expression for $\partial\tilde{\beta}_M$ by the corresponding $\tilde{f}_*(R_i)$.

(4) The computer calculates the degree by evaluating ξ_P on $\tilde{f}_*(\partial\tilde{\beta}_M)$.

An extended version of the program calculates the degree for all possible homomorphisms $\pi_1(M) \to \pi$ by letting each one of τ, x_1, x_3 run over all 120 elements of π and casting off the assignments that do not determine homomorphisms.

The program is written in PASCAL and occupies about 1000 lines (not including commentaries). It works sufficiently fast: the extended version requires a few seconds to run over all 120^3 cases. The maximal range of α_i, β_i is about 1000. The cause of the restriction is that for large α_i, β_i the words $w_{\alpha_i\beta_i}(a_i, t)$ can be too long, especially after substituting the generators by their images.

The program has passed several tests. In particular:

- It gives correct answers for obvious cases, in particular, for the identity homomorphism $\pi \to \pi$.

- It gives the same list of degrees for maps into P for many cases of differently presented homeomorphic Seifert manifolds.
- It gives the same degree for maps into P that differ by an inner automorphism of π. The multiplication of a degree d map $M \to P$ by a degree 49 map $P \to P$ inducing the unique nontrivial element of $\mathrm{Out}(\pi)$ produces a map of degree $49d$.
- The results of a vast computer experiment completely agree with all known facts about the degree of maps into P. In particular, the computer rediscovered the set of Seifert homology spheres that admit degree one maps onto P. By this we mean homology spheres $M\big((\alpha_1,\beta_1);(\alpha_2,\beta_2);(\alpha_3,\beta_3)\big)$ such that $\alpha_1/2$, $\alpha_2/3$, $\alpha_3/5$ are integers and $\alpha_1\alpha_2\alpha_3/30 \equiv \pm1, \pm49 \bmod 120$. They are the only known Seifert homology spheres with three exceptional fibers that admit degree one maps onto P; see [Hayat-Legrand et al. 1997].

6.2. Results

Let $M = M\big((\alpha_1,\beta_1);(\alpha_2,\beta_2);(\alpha_3,\beta_3)\big)$ be an oriented Seifert manifold and

$$\langle a_1,a_2,a_3,t \mid a_ita_i^{-1}t^{-1},\ a_i^{\alpha_i}t^{\beta_i},\ 1\le i\le 3,\ a_1a_2a_3\rangle$$

the standard presentation of $\pi_1(M)$. Let d be an integer modulo 120. Denote by $N(d)$ the number of all homomorphisms $\pi_1(M) \to \pi$ induced by degree d maps $M \to P$. We present a few examples of computations.

Example 1. Suppose that $(\alpha_1,\alpha_2,\alpha_3) = (2,3,5)$ and $(\beta_1,\beta_2,\beta_3) = (-1,1,k)$, with $1 \le k \le 4$. Then the possible values of d and corresponding numbers $N(d)$ are:

k	1	2	3	4
d	0 1 49	0 7 103	0 13 37	0 19 91
$N(d)$	1 60 60	1 60 60	1 60 60	1 60 60

The case $k=1$ corresponds to maps $P \to P$. There exist exactly 60 inner automorphisms of π. They determine degree 1 maps. Multiplying them by an exterior automorphism of π (which exists by [Plotnick 1982]), we get 60 automorphisms that induce maps of degree 49. In the cases $k = 2,3,4$ we have similar situations: there are two nonzero degrees related by multiplication by 49. For each $k = 1,2,3,4$ all 120 nontrivial homomorphisms $\pi(M) \to \pi$ take the generator t to the unique nontrivial element c^5 of the center of π. Nevertheless, the next example shows that the situation may be quite different.

Example 2. Suppose that $(\alpha_1,\alpha_2,\alpha_3) = (3,6,30)$ and $(\beta_1,\beta_2,\beta_3) = (2,1,-1)$. Then the possible values of d, corresponding numbers $N(d)$, and corresponding images of t are:

d	0	0	4	40	60	76	80
$N(d)$	30	42	120	120	240	120	120
$t \mapsto$	1	c^5	c^5	c^5	c^5	c^5	1

The main goal of the computer experiment was to investigate the following question:

Problem 1. Let $\alpha_1,\alpha_2,\alpha_3$ be positive integers such that

(1) $\gcd(\alpha_i,\alpha_j) = 1$ for $i \ne j$ with $1 \le i,j \le 3$, and
(2) $2 \mid \alpha_1$, $3 \mid \alpha_2$, $5 \mid \alpha_3$.

Does there exist a degree one map of a Seifert manifold $M = M\big((\alpha_1,\beta_1);(\alpha_2,\beta_2);(\alpha_3,\beta_3)\big)$ onto P?

The conditions $2 \mid \alpha_1$, $3 \mid \alpha_2$, $5 \mid \alpha_3$, and $\gcd(\alpha_1,15) = 1$ are necessary for the existence of a degree one map $M \to P$, see [Hayat-Legrand et al. 1997, Corollary 9.3]. Also, for the most interesting case when M is a homology sphere we have

$$\alpha_1\alpha_2\beta_3 + \alpha_1\alpha_3\beta_2 + \alpha_2\alpha_3\beta_1 = \pm1,$$

whence $\gcd(\alpha_i,\alpha_j) = 1$ for $i \ne j$.

Remark 6.2.1. It is known that every homomorphism $\varphi : \pi_1(M) \to \pi$ induced by a degree one map $f : M \to P$ must be surjective. Moreover, every map $f : M \to P$ can be lifted to a map $\hat f : M \to \hat P$, where $\hat P$ is the covering of P corresponding to the subgroup $G = f_\#(\pi_1(M)) \subset \pi$. Note that $\deg(f) = [\pi{:}\,G]\deg(\hat f)$ where $[\pi{:}\,G]$ is the index of G in π. For $[\pi{:}\,G] > 1$ this reduces the calculation of the degree for f to that for $\hat f$ which is simpler.

Let $M = M\big((\alpha_1,\beta_1);(\alpha_2,\beta_2);(\alpha_3,\beta_3)\big)$ be a manifold such that

(1) $\gcd(\alpha_i,\alpha_j) = 1$ for $i \ne j, 1 \le i,j \le 3$;
(2) $2 \mid \alpha_1, 3 \mid \alpha_2, 5 \mid \alpha_3$;
(3) all β_i are odd.

Under these assumptions the rules

$t \mapsto a^2$, $a_1 \mapsto a$, $a_2 \mapsto a^{-1}c$, $a_3 \mapsto c^{-1}$ if α_1 is not divisible by 4,

$t \mapsto 1$, $a_1 \mapsto a$, $a_2 \mapsto a^{-1}c^{-4}$, $a_3 \mapsto c^4$ if α_1 is divisible by 4,

yield a surjective homomorphism $\varphi_0 : \pi_1(M) \to \pi$, which we call *standard*. Denote by ext_a the external automorphism of π that takes a to a and c to cac^2ac^{-1}. It is induced by a map $P \to P$ of degree 49; see [Plotnick 1982].

Remark 6.2.2. Assumption (5) can be easily achieved by transformations $\beta_i \mapsto \beta_i + \alpha_i$, $\beta_j \mapsto \beta_j - \alpha_j$ for $i \neq j$, which preserve the manifold.

Lemma 6.2.1. *Let α_i and β_i satisfy the above assumptions. Then for any homomorphism $\varphi : \pi_1(M) \to \pi$ the following conditions are equivalent*:

(1) *φ is surjective.*
(2) *φ has the form $\varphi = \psi\varphi_0$, where $\psi : \pi \to \pi$ is either an inner automorphism of π or the product of ext_a and an inner automorphism.*
(3) *$\varphi(t) = a^2$ if α_1 is not divisible by 4, and $\varphi(t) = 1$ if it is.*

Proof. Let $\alpha_1 = 2p_1$, $\alpha_2 = 3p_2$, $\alpha_3 = 5p_3$. Then p_1 is not divisible by 3 or 5. We assume that p_1 is odd; the case p_1 is even (that is, α_1 is divisible by 4) is similar. Denote $\varphi(t)$ by τ, $\varphi(a_i)$ by x_i, and the order of x_i by k_i, $1 \leq i \leq 3$. If τ is in the center $\{1, a^2\}$ of π, then it follows from $x_1^{2p_1}\tau^{\beta_1} = 1$ and $\tau^{2\beta_1} = 1$ that k_1 divides $4p_1$. Note that all possible orders of elements of π are contained in the following list: 1, 2, 3, 4, 5, 6, 10. Recall that p_1 is not divisible by 3 or 5. It follows that k_1 divides 4. Similar arguments show that k_2 divides 6 and k_3 divides 10 and that $\gcd(p_2, 2\cdot 5) = \gcd(p_3, 3\cdot 3) = 1$.

(1) $\Rightarrow$ (2). Step 1. Since φ is surjective, τ is in the center of π and, as shown above, k_1 divides 4. It follows that $k_1 = 4$. Indeed, the relation $x_1x_2x_3 = 1$ shows that for $k_1 = 1, 2$ the image of φ would be generated by τ, x_2 and possibly a^2, the unique element of π having order 2. In this case the image would be abelian, contradicting the surjectivity.

Note that all order 2 elements of the icosahedral group $I = \pi/\langle t\rangle$ are conjugate. It follows that every element of order 4 in π is conjugate to a or a^{-1}. Since for $x = c^3ac^2ac^{-2}$ we have $xax^{-1} = a^{-1}$, we can conclude that all elements of order 4 in π are conjugates of a (we learned this first by means of a computer program). Thus, after a multiplication of φ by an inner automorphism of π, we may assume that $x_1 = a$. Since p_1 and β_1 are odd, the relation $x_1^{2p_1}\tau^{\beta_1} = 1$ implies that $\tau = a^2$.

Step 2. Recall that k_2 divides 6. Just as above, we cannot have $k_2 = 1, 2$ because of the surjectivity of φ. Since $x_2^{3p_2}\tau^{\beta_2} = 1$, $\tau = a^2$, $\gcd(p_2, 2) = 1$, and β_2 is odd, we have $k_2 \neq 3$. The only remaining case is $k_2 = 6$. Similarly, $k_3 = 10$.

Step 3. There are only four elements x of π such that x has order 10 and $a^{-1}x^{-1}$ has order 6: c^{-1}, $ac^{-1}a^{-1}$, and their images under ext_a. Certainly, this fact could be theoretically obtained, but the authors got it by letting a simple computer program run over all elements of π. It implies easily (2).

(2) $\Rightarrow$ (3). Since the center $\{1, a^2\}$ is fixed under all automorphisms of π, this implication is evident.

(3) $\Rightarrow$ (1). Since $\tau = a^2$ is in the center, k_1, k_2, and k_3 divide 4,6, and 10, respectively (see above). The equations $x_1^{2p_1}a^{2\beta 1} = x_1^{2p_1}a^2 = 1$ imply that $k_1 = 1, 2$ is impossible. Thus $k_1 = 4$. We cannot have $k_2 = 1, 2$, since then $x_3 = x_2^{-1}x_1^{-1}$ would have order 4, which is impossible. Thus k_2 is divisible by 3. Similarly, k_3 is divisible by 5. It follows that the order of the subgroup $G \subset \pi$ generated by x_1, x_2, x_3 is divisible by 4, 3 and 5. Since π contains no subgroup of order 60, $G = \pi$. □

To a great extent, Lemma 6.2.1 facilitates the computer search for new degree one maps of Seifert manifolds onto P: under above conditions on α_i, β_i, it suffices to check only *standard* maps $M \to P$, that is, those that correspond to the standard homomorphisms $\pi_1(M) \to \pi$. The result of the corresponding computer experiment was negative: no new examples of degree one maps. Nevertheless, a manual analysis of the output had shown that the degrees of the standard maps are periodic with respect to any of the parameters $p_1 = \alpha_1/2$, $p_2 = \alpha_2/3$, $p_3 = \alpha_3/5$, and β_i. This observation allows one to suggest explicit artificial formulas for the degrees of standard maps. Since we do not have a theoretical proof of the periodicity, we present the formulas in a form of a conjecture. By $[x]_k$ we denote the residue of x modulo k. In other words, $[x]_k$ is the integer satisfying the conditions $x - [x]_k$ is divisible by k and $0 \leq [x]_k < k$.

Conjecture 6.2.1. *Let*

$$f_0 : M\big((2p_1, \beta_1); (3p_2, \beta_2); (5p_3, \beta_3)\big) \to P$$

be the standard map, where all β_i are odd.

(a) *If p_1 is even then*

$$\deg(f_0) \equiv 30[\tfrac{1}{2}p_1\beta_1]_4 + 40[p_2\beta_2]_3 + 96[p_3\beta_3^3]_5.$$

(b) *if p_1 is odd then* $\deg(f_0) \equiv A_1 + A_2 + A_3 + 39$ mod 120, *where*

$$A_1 = 30\Big(\big[\tfrac{p_1+\beta_1}{2}\big]_2\big[\tfrac{\beta_1-p_1-2}{2}\big]_4 + \big[\tfrac{p_1+\beta_1+2}{2}\big]_2\big[\tfrac{\beta_1+p_1+4}{2}\big]_4\Big),$$
$$A_2 = 10\Big(\big[\tfrac{1+\beta_2}{2}\big]_3\big[\beta_2+p_2-1\big]_{12} + \big[\tfrac{1-\beta_2}{2}\big]_3\big[\beta_2-p_2+1\big]_{12}\Big),$$
$$A_3 = 12\Big(\big[\tfrac{1+\beta_3^2}{2}\big]_5\big[\tfrac{1+p_3\beta_3}{2}\big]_{10} + \big[\tfrac{1-\beta_3^2}{2}\big]_5\big[\tfrac{11-p_3\beta_3}{2}\big]_{10}\Big).$$

6.3. Degree One Maps of Homology Spheres

In this section we characterize (modulo Conjecture 6.2.1) all homology spheres of the type

$$M = M\big((2p_1, \beta_1); (3p_2, \beta_2); (5p_3, \beta_3)\big)$$

that admit a degree one map onto P. The question was posed in [Hayat-Legrand et al. 1997].

Lemma 6.3.1. *Let a, b be integers such that $0 < b < 1001$ and $\gcd(a, b)$ is not divisible by 2, 3, or 5. Then for any m, $0 \leq m \leq 6$ the set $\{a + 120i \mid 0 \leq i \leq m\}$ contains at most two numbers which are not relatively prime to b.*

Proof. Denote $a + 120i$ by a_i and $\gcd(a_i, b)$ by d_i, for $0 \leq i \leq m$. Let d be a positive common divisor of $d_i, d_j, i \neq j$. Then d divides a_i, a_j and $a_i - a_j = 120(i - j)$. Since $|i - j| \leq 6$ and $\gcd(a_i, b)$ is not divisible by 2,3, or 5, we may conclude that $d = 1$. Thus all d_i are relatively prime. It follows that not more than two of them differ from 1, since the product of the three smallest values 7, 11, 13 of $d_i \neq 1$ is equal to 1001 and cannot divide $b < 1001$. □

Lemma 6.3.2. *Let p_1, p_2, p_3 be integers such that, for $1 \leq i < j \leq 3$, $\gcd(p_i, p_j)$ is not divisible by 2, 3, 5. Then there exist positive integers $\bar{p}_1, \bar{p}_2, \bar{p}_3$ such that*

(1) $\bar{p}_i \equiv p_i$ mod 120;
(2) $\bar{p}_1 < 600, \bar{p}_2 < 360, \bar{p}_3 < 120$;
(3) $\gcd(\bar{p}_i, \bar{p}_j) = 1$ *for* $i \neq j$.

Proof. Reducing p_i modulo 120, we get q_i such that $q_i \equiv p_i$ mod 120 and $0 \leq q_i < 120, 1 \leq i \leq 3$. Set $\bar{p}_3 = q_3$. Applying Lemma 6.3.1 to $a = q_2$, $b = \bar{p}_3$, and $m = 2$, we see that at most two of the numbers $q_2, q_2 + 120, q_2 + 240$ are not relatively prime to $\bar{p}_3$. Hence at least one of them (denote it by $\bar{p}_2$) is relatively prime to $\bar{p}_3$. Consider now five numbers $q_1 + 120i$, where $0 \leq i \leq 4$. Applying Lemma 6.3.1 twice to $a = q_1, m = 4$ and $b = \bar{p}_2, \bar{p}_3$, we find among them at least one number $\bar{p}_1$ relatively prime to $\bar{p}_2$ and $\bar{p}_3$. By construction, $\bar{p}_1, \bar{p}_2, \bar{p}_3$ satisfy the conditions (1)–(3) of Lemma 6.3.2. □

Remark 6.3.1. Any Seifert manifold fibered over S^2 with three exceptional fibers of orders $\alpha_1, \alpha_2, \alpha_3$ with $\alpha_2 \equiv \alpha_3 \equiv 1$ mod 2 can be presented in the form $M = M\big((\alpha_1, \beta_1); (\alpha_2, \beta_2); (\alpha_3, \beta_3)\big)$ with $\beta_2 \equiv \beta_3 \equiv 1$ mod 4. Indeed, any presentation can be transformed to one with $\beta_2 \equiv \beta_3 \equiv 1$ mod 4 by transformations $\beta_1 \mapsto \beta_1 + \alpha_1$, $\beta_2 \mapsto \beta_2 - \alpha_2$ and $\beta_1 \mapsto \beta_1 + \alpha_1$, $\beta_3 \mapsto \beta_3 - \alpha_3$ that preserve M.

Proposition 6.3.1. *For any homology sphere $M_1 = M\big((2p_1, \beta_1); (3p_2, \beta_2); (5p_3, \beta_3)\big)$ with $\beta_2 \equiv \beta_3 \equiv 1$ mod 4 there exists a homology sphere*

$$M_2 = M\big((2\bar{p}_1, \bar{\beta}_1); (3\bar{p}_2, \bar{\beta}_2); (5\bar{p}_3, \bar{\beta}_3)\big)$$

such that

(1) $\bar{p}_i \equiv p_i$ mod 120;
(2) $\bar{p}_1 < 600, \bar{p}_2 < 360, \bar{p}_3 < 120$;
(3) $\bar{\beta}_1 \equiv \beta_1$ mod 8, $\bar{\beta}_2 \equiv \beta_2$ mod 12, *and* $\bar{\beta}_3 \equiv \beta_3$ mod 20.

Proof. Since the group $H_1(M; \mathbb{Z})$ is trivial, its order

$$|6p_1p_2\beta_3 + 10p_1p_3\beta_2 + 15p_2p_3\beta_1|$$

is equal to 1. It follows that p_1, p_2, p_3 satisfy the assumption of Lemma 6.3.2. Therefore, one can find $\bar{p}_1, \bar{p}_2, \bar{p}_3$ satisfying conditions (1)–(3) of Lemma 6.3.2. Since $\gcd(\bar{p}_i, \bar{p}_j) = 1$ for $i \neq j$, there exist $\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3$ such that $6\bar{p}_1\bar{p}_2\bar{\beta}_3 + 10\bar{p}_1\bar{p}_3\bar{\beta}_2 + 15\bar{p}_2\bar{p}_3\bar{\beta}_1 = 6p_1p_2\beta_3 + 10p_1p_3\beta_2 + 15p_2p_3\beta_1$. Using transformations $\bar{\beta}_1 \mapsto \bar{\beta}_1 + 2\bar{p}_1$, $\bar{\beta}_2 \mapsto \bar{\beta}_2 - 3\bar{p}_2$ and $\bar{\beta}_1 \mapsto \bar{\beta}_1 + 2\bar{p}_1$, $\bar{\beta}_3 \mapsto \bar{\beta}_3 - 5\bar{p}_3$, one can achieve $\bar{\beta}_2 \equiv \bar{\beta}_3 \equiv 1$ mod 4. Note that the conditions (1), (2) of Lemma 6.3.2 are fulfilled by the construction of $\bar{p}_i$. To prove (3), consider the expression

$$\begin{aligned} S = 6&\big((p_1p_2 - \bar{p}_1\bar{p}_2)\beta_3 + \bar{p}_1\bar{p}_2(\beta_3 - \bar{\beta}_3)\big) \\ &+ 10\big((p_1p_3 - \bar{p}_1\bar{p}_3)\beta_2 + \bar{p}_1\bar{p}_3(\beta_2 - \bar{\beta}_2)\big) \\ &+ 15\big((p_2p_3 - \bar{p}_2\bar{p}_3)\beta_1 + \bar{p}_2\bar{p}_3(\beta_1 - \bar{\beta}_1)\big). \end{aligned}$$

The definition of $\bar\beta_i$ and a simple calculation show that $S = 0$. Reducing the expression S modulo 8 and taking into account that $p_i - \bar p_i \equiv 0 \bmod 120$ for $1 \le i \le 3$ and $\beta_j - \bar\beta_j \equiv 0 \bmod 4$ for $2 \le j \le 3$, we get $\bar p_2 \bar p_3(\beta_1 - \bar\beta_1) \equiv 0 \bmod 8$. It follows that $\bar\beta_1 \equiv \beta_1 \bmod 8$. Reducing S modulo 3, we get

$$\bar p_1 \bar p_3(\beta_2 - \bar\beta_2) \equiv 0 \bmod 3,$$

which, together with $\beta_2 - \bar\beta_2 \equiv 0 \bmod 4$, gives us that $\bar\beta_2 \equiv \beta_2 \bmod 12$. Similarly, reducing S modulo 5, we get $\bar\beta_3 \equiv \beta_3 \bmod 20$. □

Corollary 6.3.1. *If Conjecture* 6.2.1 *is true, a Seifert homology sphere* $M\big((2p_1, \beta_1); (3p_2, \beta_2); (5p_3, \beta_3)\big)$ *admits degree one map onto* P *if and only if* $p_1p_2p_3 \equiv \pm 1, \pm 49 \bmod 120$. (Compare [Hayat-Legrand et al. 1997].)

Proof. Assuming that Conjecture 6.2.1 is true and taking into account that the first formula from Conjecture 6.2.1 always gives an even number, we can reduce the problem of degree one maps onto P to the question when the second formula gives 1. Note that $\deg(f_0)$ given by the formula is preserved under the replacements $p_i \mapsto p_i \pm 120$, $\beta_1 \mapsto \beta_1 \pm 8$, $\beta_2 \mapsto \beta_2 \pm 12$, $\beta_3 \mapsto \beta_3 \pm 20$.

Thus Proposition 6.3.1 reduces the question to checking a finite number of possibilities. For each triple $p_1 < 600$, $p_2 < 360$, $p_3 < 120$ one should check whether $\gcd(p_i, p_j) = 1$ for $i \neq j$, find some $\beta_1, \beta_2, \beta_3$ such that $\beta_2 \equiv \beta_3 \equiv 1 \bmod 4$ and

$$6p_1p_2\beta_3 + 10p_1p_3\beta_2 + 15p_2p_3\beta_1 = \pm 1,$$

and calculate $\deg(f_0)$. It turned out that in all cases $\deg(f_0) = \pm 1$ if and only if $p_1p_2p_3 \equiv \pm 1, \pm 49 \bmod 120$. □

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