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Spatial wave solutions for generalized atmospheric Ekman equations

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Abstract. In this paper, we use Prandtl mixing-length theory and semiempirical theory to extend the classical problem of the wind in the steady atmospheric Ekman layer with constant eddy viscosity. New generalized atmospheric Ekman equations are established and qualitative properties of the corresponding ODEs are studied. Spatial wave solutions results for the nonlinear and implicit equations with different nonlinearities are presented.

Keywords: generalized atmospheric Ekman equations, nonlinear and implicit equations, spatial wave solutions.

2020 Mathematics Subject Classification: 34A09, 86A10.

1 Introduction

Lamina sublayer, surface layer and Ekman layer are three important parts for the atmospheric boundary layer [24,26]. In particular, the Ekman layer covers ninety percent of the atmospheric boundary layer, which is driven by a three-way balance among frictional effects, pressure gradient and the influence of the Coriolis force in non-equatorial regions [13,24,33]. However, this balance breaks down in equatorial regions, where the Coriolis effect due to the Earths rotation vanishes, the Coriolis force changes sign across the Equator, so the nonlinear effects have to be accounted for [4–8,11,23,25].

Ekman was the first to formula and analyse a mathematical model which describes the behavior of wind-generated steady surface currents [13], the theory is the basis for our understanding of wind-driven currents, and is also relevant for the air flow in the atmospheric boundary layer.

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We consider a rotating framework with the origin at a point on the Earth's surface, with the x axis chosen horizontally due east, the y axis horizontally due north, and the z axis upward, it is known that the standard Ekman equations are given by

$$\begin{cases} f(v - v_g) = -\frac{\partial}{\partial z}(k\frac{\partial u}{\partial z}), \\ f(u - u_g) = \frac{\partial}{\partial z}(k\frac{\partial v}{\partial z}), \end{cases} \tag{1.1}$$

where $u = u(t,x,y,z)$, $v = v(t,x,y,z)$ are the components of the wind in the x and y directions respectively, P is the atmospheric pressure, ρ is the reference density, $f = 2\Omega \sin\phi$ is the Coriolis parameter at the fixed latitude ϕ, u_g and v_g are the corresponding geostrophic wind components, k denotes the eddy viscosity [22].

Ekman derived the flow from this model and obtained three characteristics, two of which have been shown to hold in the general case of depth-dependent eddy viscosity. However, regarding of the value of the deflection angle of the surface flow from the wind direction, some data in non-equatorial regions predicted significant differences [9,17,34]. It is natural to attribute this difference to the assumption of constant vorticity. Some results have been made on the explicit formula of the solution to (1.1) with the hight-dependent eddy viscosity and the classic boundary conditions $u = v = 0$ at $z = 0$ and $u \to u_g$, $v \to v_g$ for $z \to \infty$ for the atmospheric Ekman equations [9,10,16,19,20,32]. With respect to wind-driven surface current, one can refer to [1–3,12,30,31] for the depth-dependent eddy viscosity and the corresponding boundary conditions.

Noting that (1.1) is formulated by omitting the turbulent fluxes, which has obvious limitations. Recently, Guan et al. [18] introduced a new nonhomogeneous model containing turbulent flux terms, which improved the classical model proposed in [24]. Further, in this paper, we propose the following generalized model

$$\begin{cases} f(v - v_g) = -\frac{\partial}{\partial z}(k\frac{\partial u}{\partial z}) + 2l^2 u\frac{\partial u}{\partial x}, \\ f(u - u_g) = \frac{\partial}{\partial z}(k\frac{\partial v}{\partial z}) + 2l^2 v\frac{\partial v}{\partial y}, \end{cases} \tag{1.2}$$

where l is a constant number. We emphasize that (1.2) is a generalization of the standard Ekman equations since the turbulent flux term is considered. Comparing with the previous extension model in [18], (1.2) has a totally different and specific turbulent flux terms. In [18], the turbulent flux is assumed to be a function of height, but here we use semi-empirical method and assume turbulent flux to be a function of u, v and their partial derivatives, which are more reasonable than the turbulent fluxes only depending on the high z in [18] and also makes the current model more complex.

Note that explicit solution and dynamical properties of atmospheric Ekman flows with boundary conditions have been presented extensively. There are still very few contributions on the modified Ekman equation. In particular, periodic solutions and Hyers–Ulam stability are reported in a modified model in [18] by using the theory of ordinary differential equations and hyperbolic matrix theory. In this paper, we consider spatial wave solutions of (1.2), which satisfy certain ODEs, and we study qualitative properties of this corresponding ODEs. This is a novelty of this paper.

The rest of the paper is organized as follows. New generalized atmospheric Ekman equations are derived in Section 2. Section 3 deals with spatial wave solutions of (1.2). We study qualitative properties of the corresponding ODEs determining these solutions. Involving also other terms not just linear ones into (2.5), we continue our analysis in Section 4 with more general ODEs. Finally, (2.4) is investigated in Section 5. The obtained spatial wave ODEs are nonlinear and implicit, so their study is difficult. There are still many open challenging problems for further research. These aspects are presented in Section 6.

2 Model description

In the local Cartesian coordinate system, the earth's surface is approximately regarded as a plane, and the curvature term can be omitted, so the Ekman layer is governed by the following equations, see [24,26]

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\Omega\sin\phi v - 2\Omega\cos\phi w + F_{rx}, \\ \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - 2\Omega\sin\phi u + F_{ry}, \\ \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g + 2\Omega\cos\phi u + F_{rz}, \end{cases} \tag{2.1}$$

where

$$\begin{cases} F_{rx} = v[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}], \\ F_{ry} = v[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}], \\ F_{rz} = v[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}], \end{cases}$$

and $v = \frac{\mu}{\rho}$ is the kinematic viscosity coefficient [24], $u = u(t,x,y,z)$, $v = v(t,x,y,z)$ and $w = w(t,x,y,z)$ are the components of the wind in the x,y and z directions respectively. Besides, $\overrightarrow{U} = (u,v,w)$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \overrightarrow{U}) = 0. \tag{2.2}$$

For a wide range of air movements, $w \ll u,v$ [33], so we assume $w = 0$, kinematic viscosity coefficient is negligible in the Ekman layer, so $F_{rx} = 0$, $F_{ry} = 0$, then (2.1) reduces to

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\Omega\sin\phi v, \\ \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - 2\Omega\sin\phi u. \end{cases} \tag{2.3}$$

Note that the Boussinesq approximation is an important simplifications in (2.2) and (2.3) for application in the boundary layer, in this approximation, density ρ in (2.2) and (2.3) are replaced by a constant mean value (everywhere except in the buoyancy term in the vertical momentum equation, see [24]). Clearly, (2.2) becomes to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

We assume that the variable consists of the mean value and the turbulence value, for example, $u = \overline{u} + u'$, the corresponding mean values are indicated by overbars and the fluctuating component by primes.

Under the Boussinesq approximation, the mean velocity fields satisfy the following continuity equations [24]

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0,$$

we separate each dependent variable into mean and fluctuating parts, and substitute into the chain rule of the differentiation, then we obtain

$$\overline{\frac{Du}{Dt}} = \frac{\partial \overline{u}}{\partial t} + \frac{\partial}{\partial x}(\overline{u'u'}) + \frac{\partial}{\partial y}(\overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u'w'}),$$

where

$$\frac{\overline{D}}{Dt} = \frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x} + \overline{v}\frac{\partial}{\partial y} + \overline{w}\frac{\partial}{\partial z}$$

is the rate of change following the mean motion.

Using the above relationships and (2.3), the mean equations thus have the following form:

$$\begin{cases} \frac{\overline{D}\overline{u}}{Dt} = -\frac{1}{\rho}\frac{\partial\overline{P}}{\partial x} + f\overline{v} - [\frac{\partial\overline{u'u'}}{\partial x} + \frac{\partial\overline{u'v'}}{\partial y} + \frac{\partial\overline{u'w'}}{\partial z}], \\ \frac{\overline{D}\overline{v}}{Dt} = -\frac{1}{\rho}\frac{\partial\overline{P}}{\partial y} - f\overline{u} - [\frac{\partial\overline{u'v'}}{\partial x} + \frac{\partial\overline{v'v'}}{\partial y} + \frac{\partial\overline{v'w'}}{\partial z}]. \end{cases}$$

We omit the inertial acceleration terms because they are much smaller that the Cariolis force and pressure gradient force terms for midlatitude synoptic-scale motions [24], using the geostrophic balance, we obtain

$$\begin{cases} f(\overline{v} - \overline{v_g}) - [\frac{\partial\overline{u'u'}}{\partial x} + \frac{\partial\overline{u'v'}}{\partial y} + \frac{\partial\overline{u'w'}}{\partial z}] = 0, \\ -f(\overline{u} - \overline{u_g}) - [\frac{\partial\overline{u'v'}}{\partial x} + \frac{\partial\overline{v'v'}}{\partial y} + \frac{\partial\overline{v'w'}}{\partial z}] = 0. \end{cases}$$

By the Flux-Gradient theory [24], we get

$$\begin{cases} \overline{u'w'} = -k\frac{\partial u}{\partial z}, \\ \overline{v'w'} = -k\frac{\partial v}{\partial z}, \end{cases}$$

where k is the eddy viscosity coefficient, then we obtain

$$\begin{cases} f(\overline{v} - \overline{v_g}) = -\frac{\partial}{\partial z}(k\frac{\partial\overline{u}}{\partial z}) + \frac{\partial\overline{u'u'}}{\partial x} + \frac{\partial\overline{u'v'}}{\partial y}, \\ f(\overline{u} - \overline{u_g}) = \frac{\partial}{\partial z}(k\frac{\partial\overline{v}}{\partial z}) + \frac{\partial\overline{u'v'}}{\partial x} + \frac{\partial\overline{v'v'}}{\partial y}, \end{cases}$$

usually we omit the terms $\frac{\partial\overline{u'u'}}{\partial x}$, $\frac{\partial\overline{u'v'}}{\partial y}$, $\frac{\partial\overline{u'v'}}{\partial x}$ and $\frac{\partial\overline{v'v'}}{\partial y}$ because they are small in comparison to the terms $\frac{\partial\overline{u'w'}}{\partial z}$, $\frac{\partial\overline{v'w'}}{\partial z}$, but here we retain $\frac{\partial\overline{u'u'}}{\partial x}$ and $\frac{\partial\overline{v'v'}}{\partial y}$ and obtain

$$\begin{cases} f(\overline{v} - \overline{v_g}) = -\frac{\partial}{\partial z}(k\frac{\partial\overline{u}}{\partial z}) + \frac{\partial\overline{u'u'}}{\partial x}, \\ f(\overline{u} - \overline{u_g}) = \frac{\partial}{\partial z}(k\frac{\partial\overline{v}}{\partial z}) + \frac{\partial\overline{v'v'}}{\partial y}. \end{cases}$$

By the Prandtl mixing-length theory [24], we have $u' = -l'\frac{\partial\overline{u}}{\partial z}$, $v' = -l'\frac{\partial\overline{v}}{\partial z}$, so

$$\begin{cases} f(\overline{v} - \overline{v_g}) = -\frac{\partial}{\partial z}(k\frac{\partial\overline{u}}{\partial z}) + l^2\frac{\partial}{\partial x}(\frac{\partial\overline{u}}{\partial z})^2, \\ f(\overline{u} - \overline{u_g}) = \frac{\partial}{\partial z}(k\frac{\partial\overline{v}}{\partial z}) + l^2\frac{\partial}{\partial y}(\frac{\partial\overline{v}}{\partial z})^2. \end{cases} \tag{2.4}$$

where $l = \overline{l'}$ is the mean mixing-length.

Now replacing $\overline{u}$, $\overline{v}$, $\overline{u_g}$ and $\overline{v_g}$ by u, v, u_g and v_g, respectively, and we assume that

$$\frac{\partial u}{\partial z} \approx u, \quad \frac{\partial v}{\partial z} \approx v, \tag{2.5}$$

by semiempirical theory, one can obtain

$$\begin{cases} f(v - v_g) = -\frac{\partial}{\partial z}(k\frac{\partial u}{\partial z}) + 2l^2 u\frac{\partial u}{\partial x}, \\ f(u - u_g) = \frac{\partial}{\partial z}(k\frac{\partial v}{\partial z}) + 2l^2 v\frac{\partial v}{\partial y}. \end{cases}$$

3 Spatial wave solutions for (1.2)

Assuming that k is a nonzero constant, (1.2) becomes

$$\begin{cases} f(v-v_g) = -k\frac{\partial^2 u}{\partial z^2} + 2l^2 u\frac{\partial u}{\partial x}, \\ f(u-u_g) = k\frac{\partial^2 v}{\partial z^2} + 2l^2 v\frac{\partial v}{\partial y}. \end{cases} \tag{3.1}$$

We are looking for spatial wave solutions of (3.1) as follows

$$\begin{aligned} u(x,y,z) &= U(\alpha x + \beta y + z), \\ v(x,y,z) &= V(\alpha x + \beta y + z), \end{aligned} \tag{3.2}$$

where α and β are parameters. Then we get

$$\begin{aligned} f(V-v_g) &= -kU'' + 2\alpha l^2 UU', \\ f(U-u_g) &= kV'' + 2\beta l^2 VV'. \end{aligned} \tag{3.3}$$

For $\alpha = 0$ and $\beta = 0$, we get the standard Ekman equations. Taking

$$X = \begin{bmatrix} U \\ V \\ U' \\ V' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

(3.3) becomes

$$X' = F(X) = \begin{bmatrix} x_3 \\ x_4 \\ \frac{f}{k}(v_g - x_2) + \frac{2\alpha l^2}{k}x_1x_3 \\ \frac{f}{k}(x_1 - u_g) - \frac{2\beta l^2}{k}x_2x_4 \end{bmatrix}. \tag{3.4}$$

Note (3.4) has a unique equilibrium

$$X_0 = \begin{bmatrix} u_g \\ v_g \\ 0 \\ 0 \end{bmatrix}$$

and its Jacobian matrix is

$$DF(X_0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{f}{k} & \frac{2\alpha l^2 u_g}{k} & 0 \\ \frac{f}{k} & 0 & 0 & -\frac{2\beta l^2 v_g}{k} \end{bmatrix} \tag{3.5}$$

with the characteristic polynomial

$$\chi(\lambda) = \lambda^4 + \lambda^3\frac{2\beta l^2 v_g - 2\alpha l^2 u_g}{k} - \lambda^2\frac{4\alpha\beta l^4 u_g v_g}{k^2} + \frac{f^2}{k^2}. \tag{3.6}$$

Lemma 3.1. *χ defined in (3.6) has no pure imaginary roots.*

Proof. Suppose $\lambda = \imath\omega$, $\omega \in \mathbb{R}$ is a root of χ, then we get

$$0 = \chi(\imath\omega) = \omega^4 - \imath\omega^3 \frac{2\beta l^2 v_g - 2\alpha l^2 u_g}{k} + \omega^2 \frac{4\alpha\beta l^4 u_g v_g}{k^2} + \frac{f^2}{k^2}.$$

So

$$\omega^4 + \omega^2 \frac{4\alpha\beta l^4 u_g v_g}{k^2} + \frac{f^2}{k^2} = 0,$$
$$\omega^3 \frac{2\beta l^2 v_g - 2\alpha l^2 u_g}{k} = 0.$$

Clearly $\omega \neq 0$, then $\beta v_g - \alpha u_g = 0$, so

$$\omega^4 + \omega^2 \frac{4\alpha^2\beta^2 l^4 u_g v_g}{k^2} + \frac{f^2}{k^2} = 0,$$

which is not possible. The proof is finished. □

Consequently, $DF(X_0)$ is hyperbolic. When $\alpha = \beta = 0$, we get

$$DF(X_0) = A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{f}{k} & 0 & 0 \\ \frac{f}{k} & 0 & 0 & 0 \end{bmatrix}$$

with (3.6) of the form

$$\lambda^4 + \frac{f^2}{k^2} = 0$$

and possessing four eigenvalues

$$\sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}}\imath, \quad \sqrt{\frac{f}{2k}} - \sqrt{\frac{f}{2k}}\imath, \quad -\sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}}\imath, \quad -\sqrt{\frac{f}{2k}} - \sqrt{\frac{f}{2k}}\imath.$$

Thus there are two eigenvalues of A on both sides of the imaginary axis. By Lemma 3.1 this property remains for any $DF(X_0)$ with arbitrary α and β. Consequently, X_0 has a 2-dimensional stable manifold $W^s_{X_0}$. So we have a 4-parameterized family of functions

$$X(\alpha, \beta, s_1, s_2; t)$$

such that

$$X(t) = X(\alpha, \beta, s_1, s_2; t)$$

is a solution of (3.4) with $X(0) \in W^s_{X_0}$. Then $X(t) \to X_0$ exponentially fast as $t \to \infty$. Summarizing, we arrive at the following result.

Theorem 3.2. *Functions*

$$\begin{aligned} u_{\alpha,\beta,s_1,s_2}(x,y,z) &= U(\alpha, \beta, s_1, s_2; \alpha x + \beta y + z), \\ v_{\alpha,\beta,s_1,s_2}(x,y,z) &= V(\alpha, \beta, s_1, s_2; \alpha x + \beta y + z) \end{aligned} \tag{3.7}$$

give a 4*-parameterized family of solutions for* (3.1) *with*

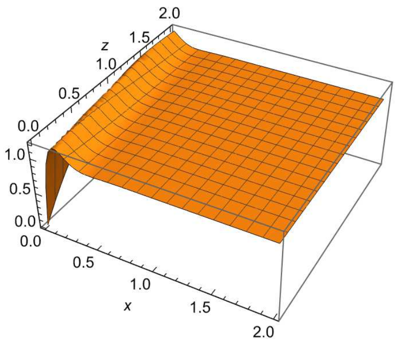
$$u_{\alpha,\beta,s_1,s_2}(x,y,z) \to u_g,$$
$$v_{\alpha,\beta,s_1,s_2}(x,y,z) \to v_g,$$

as $x + y + z \to \infty$, $x \geq 0$, $y \geq 0$, $z \geq 0$, $\alpha > 0$ *and* $\beta > 0$. *In general, the above asymptotic properties hold for* $\alpha x + \beta y + z \to \infty$.

For $l = 0$, $u_g = v_g = 1, \frac{f}{2k} = 1$, we have an implicit solution [20]

$$\begin{aligned} u_{10,0,1,-1}(x,y,z) &= e^{-(10x+z)}\sin(10x+z) - e^{-(10x+z)}\cos(10x+z) + 1, \\ v_{10,0,1,-1}(x,y,z) &= -e^{-(10x+z)}\sin(10x+z) - e^{-(10x+z)}\cos(10x+z) + 1, \end{aligned} \tag{3.8}$$

visualizing their spatial wave forms on Figure 3.1.



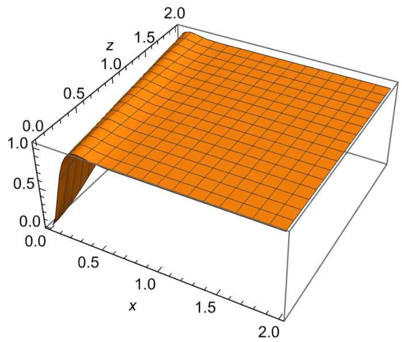


Figure 3.1: Solutions of (3.8): left $u_{10,0,1,-1}(x,y,z)$, right $v_{10,0,1,-1}(x,y,z)$

We need the next observation.

Lemma 3.3. *If $\alpha > 0$ and $\beta > 0$, then*

$$L(X) = \frac{k}{2}(x_4^2 - x_3^2) - f x_1 x_2 + f v_g x_1 + f u_g x_2$$

is a Lyapunov function of (3.4) *on the set*

$$\Pi = \{x_1 \geq 0, x_2 \geq 0\} \subset \mathbb{R}^4.$$

Proof. For any solution $X(t) \in \Pi$ of (3.4), we compute

$$\begin{aligned} L(X(t))' &= k(x_4(t)x_4'(t) - x_3(t)x_3'(t)) - f x_1'(t)x_2(t) - f x_1(t)x_2'(t) + f v_g x_1'(t) + f u_g x_2'(t) \\ &= x_4(t)(f(x_1(t) - u_g) - 2\beta l^2 x_2(t)x_4(t)) - x_3(t)(f(v_g - x_2(t)) \\ &\quad +2\alpha l^2 x_1(t)x_3(t)) - f x_3(t)x_2(t) - f x_1(t)x_4(t) \\ &\quad +f v_g x_3(t) + f u_g x_4(t) = -2\beta l^2 x_2(t)x_4(t)^2 - 2\alpha l^2 x_1(t)x_3(t)^2 \leq 0. \end{aligned}$$

The proof is finished. □

Now we present a uniqueness result for nonnegative solutions in Theorem 3.2.

Theorem 3.4. *If $\alpha > 0$ and $\beta > 0$, then any bounded solution $X(t) \in \Pi$, $\forall t \geq 0$ of* (3.4) *tends to X_0 as $t \to \infty$, i.e., $X(t) \in W^s_{X_0}$, $\forall t \geq 0$.*

Proof. Set

$$\dot{L}(X) = -2\beta l^2 x_2 x_4^2 - 2\alpha l^2 x_1 x_3^2.$$

The ω-limit of set of $X(0)$ is denoted by $\omega(X(0))$. The largest invariant subset of the set

$$\{X \in \Pi \mid \dot{L}(X) = 0\}$$

is denoted by M. A simple analysis shows that $M = \{X_0\}$. Next, by Lemma 3.3 and [21, Theorem 9.22], we know $\omega(X(0)) = \{X_0\}$. The proof is finished. □

Next, using (3.2), we consider that a solution depends on z. Now we study opposite, that is we take

$$\begin{aligned} u(x,y,z) &= U(\alpha x + \beta y), \\ v(x,y,z) &= V(\alpha x + \beta y). \end{aligned} \tag{3.9}$$

Then (3.3) is transformed to

$$\begin{aligned} f(V - v_g) &= 2\alpha l^2 U U', \\ f(U - u_g) &= 2\beta l^2 V V', \end{aligned} \tag{3.10}$$

which is an implicit ODE (see [15,29]). Now (3.10) gives

$$\begin{aligned} 0 &= \alpha U U'(U - u_g) - \beta V V'(V - v_g) \\ &= \frac{d}{dt}\left[\alpha\left(\frac{U^3}{3} - \frac{U^2}{2}u_g\right) - \beta\left(\frac{V^3}{3} - \frac{V^2}{2}v_g\right)\right], \end{aligned}$$

thus implicit solutions are given by

$$H(U,V) = \alpha\left(\frac{U^3}{3} - \frac{U^2}{2}u_g\right) - \beta\left(\frac{V^3}{3} - \frac{V^2}{2}v_g\right) = c \in \mathbb{R}. \tag{3.11}$$

Theorem 3.5. *There is a family of periodic spatial solutions* (3.9) *of* (3.1) *given by the equation* (3.11) *under the following condition*

$$\alpha\beta u_g v_g < 0. \tag{3.12}$$

Proof. The gradient of $H(U,V)$ is

$$\nabla H(U,V) = \begin{bmatrix} \alpha U(U - u_g) \\ -\beta V(V - v_g) \end{bmatrix},$$

so

$$\begin{bmatrix} u_g \\ v_g \end{bmatrix}, \tag{3.13}$$

is a critical point of $H(U,V)$ with the Hessian

$$\mathrm{Hess}\, H(u_g, v_g) = \begin{bmatrix} \alpha u_g & 0 \\ 0 & -\beta v_g \end{bmatrix}.$$

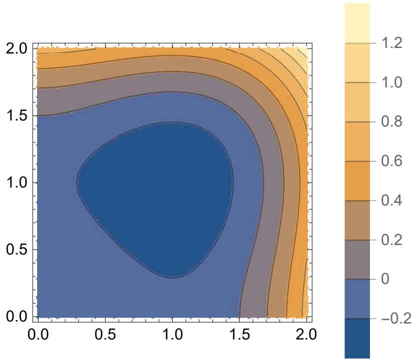
Clearly, if (3.12) holds then (3.13) is a strong local extreme of $H(U,V)$ and it is a center for (3.10). If $\alpha\beta u_g v_g > 0$, then (3.13) is a non-degenerate saddle point of $H(U,V)$ and it is hyperbolic. Consequently, (3.11) are periodic for suitable $c \approx \frac{\beta v_g^3 - \alpha u_g^3}{6}$ under (3.12). The proof is finished. □

Implicit ODE (3.10) has the same phase portrait as the following ODE

$$\begin{aligned} a' &= \beta b(b - v_g), \\ b' &= \alpha a(a - u_g) \end{aligned} \tag{3.14}$$

when $a \neq 0$ and $b \neq 0$. (3.14) has 4 equilibria $(0,0)$, $(u_g, 0)$, $(0, v_g)$ and (u_g, v_g) which are either centers or hyperbolic. Thus, implicit ODE (3.10) has impasse solutions, so solutions terminating in singularities $U = 0$ or $V = 0$ in finite time [15,29], which are impasse spatial solutions of (3.1). This is demonstrated on Figure 3.2.

We end this section with the following notes.



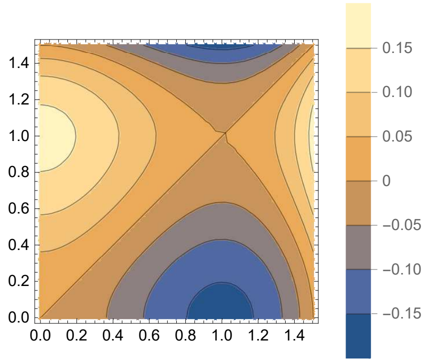


Figure 3.2: Periodic and impasse solutions of (3.10): left $\alpha = -\beta = u_g = v_g = 1$, right $\alpha = \beta = u_g = v_g = 1$

1. We can reduce parameters in (3.3) by taking

$$\begin{aligned} U(t) &= u_g + \frac{\sqrt{fk}}{2l^2} U_1\left(\sqrt{\frac{f}{k}}t\right), \\ V(t) &= v_g + \frac{\sqrt{fk}}{2l^2} V_1\left(\sqrt{\frac{f}{k}}t\right) \end{aligned} \tag{3.15}$$

to get

$$\begin{aligned} V_1 &= -U_1'' + \alpha U_1 U_1', \\ U_1 &= V_1'' + \beta V_1 V_1'. \end{aligned} \tag{3.16}$$

We do not consider (3.16) until instead of (3.3) to keep the role of other parameters in the above results.

2. Let (3.16) have a T-periodic solution. Then integrating (3.16) we have

$$\begin{aligned} \int_0^T V_1(t)dt &= \int_0^T (-U_1''(t) + \alpha U_1(t) U_1'(t))dt = \left[-U'(t) + \alpha \frac{U_1(t)^2}{2}\right]_{t=0}^{t=T} = 0, \\ \int_0^T U_1(t)dt &= \int_0^T (V_1''(t) + \beta V_1(t) V_1'(t))dt = \left[-V'(t) + \beta \frac{V_1(t)^2}{2}\right]_{t=0}^{t=T} = 0. \end{aligned}$$

So we can use Wirtinger inequality [27, p. 9] to derive

$$\begin{aligned} \|U_1''\|_2 &\le \|V_1\|_2 + |\alpha| \|U_1 U_1'\|_2 \le \frac{T^2}{4\pi^2}\|V_1''\|_2 + \frac{T}{2\pi}|\alpha| \|U_1\|_\infty \|U_1''\|_2, \\ \|V_1''\|_2 &\le \|U_1\|_2 + |\beta| \|V_1 V_1'\|_2 \le \frac{T^2}{4\pi^2}\|U_1''\|_2 + \frac{T}{2\pi}|\beta| \|V_1\|_\infty \|V_1''\|_2, \end{aligned} \tag{3.17}$$

where

$$\|U\|_2 = \sqrt{\int_0^T U(t)^2 dt}, \quad \|U\|_\infty = \max_{t\in[0,T]} |U(t)|.$$

Adding the two equations of (3.17), we arrive at

$$\|U_1''\|_2 + \|V_1''\|_2 \le \left(\frac{T^2}{4\pi^2} + \frac{T}{2\pi}\max\{|\alpha| \|U_1\|_\infty, |\beta| \|V_1\|_\infty\}\right) \|U_1''\|_2 + \|V_1''\|_2. \tag{3.18}$$

So if

$$\|U_1''\|_2 + \|V_1''\|_2 \neq 0,$$

then (3.18) implies

$$1 \leq \frac{T^2}{4\pi^2} + \frac{T}{2\pi}\max\{|\alpha|\|U_1\|_\infty, |\beta|\|V_1\|_\infty\},$$

which leads to

$$\left(\sqrt{4 + (\max\{|\alpha|\|U_1\|_\infty, |\beta|\|V_1\|_\infty\})^2} - \max\{|\alpha|\|U_1\|_\infty, |\beta|\|V_1\|_\infty\}\right)\pi \leq T. \tag{3.19}$$

Using (3.15) and (3.19), we obtain

Theorem 3.6. *A period T of any nonconstant T-periodic solution of* (3.3) *with*

$$\max_{t\in[0,T]} |U(t) - u_g| \leq M, \quad \max_{t\in[0,T]} |V(t) - v_g| \leq N$$

satisfying

$$\pi\sqrt{\frac{k}{f}}\left(\sqrt{4 + \frac{4l^2}{fk}(\max\{|\alpha|M, |\beta|N\})^2} - \frac{2l^2}{\sqrt{fk}}\max\{|\alpha|M, |\beta|N\}\right) \leq T.$$

Results similar to Theorem 3.6 are presented in [14].

3. We are focusing in this paper on the case for fixed $f \neq 0$. This leads to a hyperbolic-like dynamics. On the other hand, if $f = 0$, then (3.4) has a form

$$\begin{aligned} x_1' &= x_3, \\ x_2' &= x_4, \\ x_3' &= \frac{2\alpha l^2}{k}x_1x_3, \\ x_4' &= -\frac{2\beta l^2}{k}x_2x_4. \end{aligned} \tag{3.20}$$

Clearly

$$\Sigma = \{x_3 = x_4 = 0\}$$

is a fixed point set of (3.20) with Jacobian matrices

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{2\alpha l^2 x_1}{k} & 0 \\ 0 & 0 & 0 & -\frac{2\beta l^2 x_2}{k} \end{bmatrix} \tag{3.21}$$

possessing eigenvalues

$$0, \quad 0, \quad \frac{2\alpha l^2 x_1}{k}, \quad -2\beta l^2 x_2$$

and the corresponding eigenvectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ \frac{2\alpha l^2 x_1}{k} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\frac{2\beta l^2 x_2}{k} \end{bmatrix}$$

for $x_1 \neq 0$ and $x_2 \neq 0$. (3.20) is decoupling to

$$\begin{aligned} x_1'' &= \frac{2\alpha l^2}{k} x_1 x_1', \\ x_2'' &= -\frac{2\beta l^2}{k} x_2 x_2'. \end{aligned} \tag{3.22}$$

Integrating (3.22), we derive

$$\begin{aligned} x_1' &= \frac{\alpha l^2}{k} x_1^2 + c_1, \\ x_2' &= -\frac{\beta l^2}{k} x_2^2 + c_2 \end{aligned} \tag{3.23}$$

(3.23) is solvable and leading to these cases [28]:

i) $c_1 = 0$:

$$\begin{aligned} x_1(t) &= \frac{k x_1(0)}{k - \alpha l^2 x_1(0) t}, \\ x_3(t) &= \frac{\alpha k l^2 x_1(0)^2}{(k - \alpha l^2 x_1(0) t)^2} \end{aligned}$$

is a blow-up solution.

ii) $\frac{\alpha l^2 c_1}{k} < 0$:

$$\begin{aligned} x_1(t) &= \frac{x_1(0)\sqrt{-\frac{\alpha l^2 c_1}{k}} + c_1 \tanh\left(\sqrt{-\frac{\alpha l^2 c_1}{k}}t\right)}{\sqrt{-\frac{\alpha l^2 x c_1}{k}} - \frac{\alpha l^2 x_1(0)}{k} \tanh\left(\sqrt{-\frac{\alpha l^2 c_1}{k}}t\right)}, \\ x_3(t) &= \frac{-\frac{\alpha l^2 c_1}{k}\left(c_1 + \frac{\alpha l^2 x_1(0)^2}{k}\right)}{\left(\sqrt{-\frac{\alpha l^2 c_1}{k}} \cosh\left(\sqrt{-\frac{\alpha l^2 c_1}{k}}t\right) - \frac{\alpha l^2 x_1(0)}{k} \sinh\left(\sqrt{-\frac{\alpha l^2 c_1}{k}}t\right)\right)^2} \end{aligned}$$

is an asymptotic solution for $|x_1(0)| < \sqrt{-\frac{\alpha l^2}{k c_1}}$ connecting two points on Σ:

$$\begin{aligned} \lim_{t \to -\infty} x_1(t) = \pm\sqrt{-\frac{c_1 k}{\alpha l^2}}, &\quad \lim_{t \to \infty} x_1(t) = \mp\sqrt{-\frac{c_1 k}{\alpha l^2}}, \\ \lim_{t \to -\infty} x_3(t) = 0, &\quad \lim_{t \to \infty} x_3(t) = 0 \end{aligned}$$

is a blow-up solution for $|x_1(0)| \geq \sqrt{-\frac{\alpha l^2}{k c_1}}$.

iii) $\frac{\alpha l^2 c_1}{k} > 0$:

$$\begin{aligned} x_1(t) &= \frac{x_1(0)\sqrt{\frac{\alpha l^2 c_1}{k}} + c_1 \tan\left(\sqrt{\frac{\alpha l^2 c_1}{k}}t\right)}{\sqrt{\frac{\alpha l^2 x c_1}{k}} - \frac{\alpha l^2 x_1(0)}{k} \tan\left(\sqrt{\frac{\alpha l^2 c_1}{k}}t\right)}, \\ x_3(t) &= \frac{\frac{\alpha l^2 c_1}{k}\left(c_1 + \frac{\alpha l^2 x_1(0)^2}{k}\right)}{\left(\sqrt{\frac{\alpha l^2 c_1}{k}} \cos\left(\sqrt{\frac{\alpha l^2 c_1}{k}}r\right) - \frac{\alpha l^2 x_1(0)}{k} \sin\left(\sqrt{\frac{\alpha l^2 c_1}{k}}t\right)\right)^2} \end{aligned}$$

is a blow-up solution.

iv) Similar formulas hold for $x_2(t)$ and $x_4(t)$ by exchanging (α, c_1) with $(-\beta, c_2)$.

v) Note

$$c_1 = x_3(0) - \frac{\alpha l^2}{k} x_1(0)^2, \quad c_2 = x_4(0) + \frac{\beta l^2}{k} x_2(0)^2.$$

Clearly blow up solutions persist in (3.4) for $f \neq 0$ small. It will be our next study the asymptotic solutions ii).

Finally, we note that (3.4) for l small has a hyperbolic structure on bounded sets due to the Hartman–Grobman theorem. On the other hand, when l large, say $l = \epsilon^{-1/2} > 0$ then (3.3) becomes

$$\begin{aligned} \epsilon f(V - v_g) &= -\epsilon k U'' + 2\alpha U U', \\ \epsilon f(U - u_g) &= \epsilon k V'' + 2\beta V V'. \end{aligned}$$

Scaling

$$U(t) = U_1(t/\epsilon), \quad V(t) = V_1(t/\epsilon),$$

we get

$$\begin{aligned} \epsilon^2 f(V_1 - v_g) &= -k U_1'' + 2\alpha U_1 U_1', \\ \epsilon^2 f(U_1 - u_g) &= k V_1'' + 2\beta V_1 V_1'. \end{aligned} \tag{3.24}$$

(3.24) has a form of (3.22) for $\epsilon = 0$, so we can apply above results and remarks. We see that (3.4) has different dynamics for l small and large.

4 General nonlinearities

Assuming that (2.5) involves also other terms not just linear ones, we suppose that

$$\left(\frac{\partial u}{\partial z}\right)^2 \approx p(u), \quad \left(\frac{\partial v}{\partial z}\right)^2 \approx q(v)$$

for $p, q \in C^2(\mathbb{R}, \mathbb{R})$. Then instead of (1.2), we obtain

$$\begin{cases} f(v - v_g) = -\frac{\partial}{\partial z}(k \frac{\partial u}{\partial z}) + l^2 p'(u) \frac{\partial u}{\partial x}, \\ f(u - u_g) = \frac{\partial}{\partial z}(k \frac{\partial v}{\partial z}) + l^2 q'(v) \frac{\partial v}{\partial y}. \end{cases} \tag{4.1}$$

Then (3.4) becomes

$$X' = F(X) = \begin{bmatrix} x_3 \\ x_4 \\ \frac{f}{k}(v_g - x_2) + \frac{\alpha l^2}{k} p'(x_1) x_3 \\ \frac{f}{k}(x_1 - u_g) - \frac{\beta l^2}{k} q'(x_2) x_4 \end{bmatrix}. \tag{4.2}$$

(4.2) still has a unique equilibrium X_0 with a Jacobian matrix

$$DF(X_0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{f}{k} & \frac{\alpha l^2 p''(u_g)}{k} & 0 \\ \frac{f}{k} & 0 & 0 & -\frac{\beta l^2 q''(v_g)}{k} \end{bmatrix}.$$

We see again that X_0 is hyperbolic with 2-dimensional stable and unstable manifolds. Note (4.2) has a form

$$X' = B(X)(X - X_0) \tag{4.3}$$

for

$$B(X) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{f}{k} & \frac{\alpha l^2 p'(x_1)}{k} & 0 \\ \frac{f}{k} & 0 & 0 & -\frac{\beta l^2 q'(x_2)}{k} \end{bmatrix}.$$

For any X, $B(X)$ is hyperbolic with 2-dimensional stable and unstable manifolds.

This motivates us to show the following results. Let W_s and W_u be stable and unstable subspaces of A defined in (3.5). Let $P_s : \mathbb{R}^4 \to W_s$ and $P_u : \mathbb{R}^4 \to W_u$ be projections with $P_s + P_u = I$. Then from [20] we have

$$e^{At}P_s = \frac{e^{-\tilde{k}t}}{2}\begin{bmatrix} \cos\tilde{k}t & -\sin\tilde{k}t & \frac{-\cos\tilde{k}t+\sin\tilde{k}t}{2\tilde{k}} & \frac{\cos\tilde{k}t+\sin\tilde{k}t}{2\tilde{k}} \\ \sin\tilde{k}t & \cos[\tilde{k}t] & -\frac{\cos\tilde{k}t+\sin\tilde{k}t}{2\tilde{k}} & \frac{-\cos\tilde{k}t+\sin\tilde{k}t}{2\tilde{k}} \\ -\tilde{k}(\cos\tilde{k}t+\sin\tilde{k}t) & \tilde{k}(-\cos\tilde{k}t+\sin\tilde{k}t) & \cos\tilde{k}t & -\sin\tilde{k}t \\ \tilde{k}(\cos\tilde{k}t-\sin\tilde{k}t) & -\tilde{k}(\cos\tilde{k}t+\sin\tilde{k}t) & \sin\tilde{k}t & \cos\tilde{k}t \end{bmatrix}$$

and

$$e^{At}P_u = \frac{e^{\tilde{k}t}}{2}\begin{bmatrix} \cos\tilde{k}t & \sin\tilde{k}t & \frac{\cos\tilde{k}t+\sin\tilde{k}t}{2\tilde{k}} & \frac{-\cos\tilde{k}t+\sin\tilde{k}t}{2\tilde{k}} \\ -\sin\tilde{k}t & \cos\tilde{k}t & \frac{\cos\tilde{k}t-\sin\tilde{k}t}{2\tilde{k}} & \frac{\cos\tilde{k}t+\sin\tilde{k}t}{2\tilde{k}} \\ \tilde{k}(\cos\tilde{k}t-\sin\tilde{k}t) & \tilde{k}(\cos\tilde{k}t+\sin\tilde{k}t) & \cos\tilde{k}t & \sin\tilde{k}t \\ -\tilde{k}(\cos\tilde{k}t+\sin\tilde{k}t) & \tilde{k}(\cos\tilde{k}t-\sin\tilde{k}t) & -\sin\tilde{k}t & \cos\tilde{k}t \end{bmatrix}$$

for

$$\tilde{k} = \sqrt{\frac{f}{2k}}.$$

By considering a norm

$$\|X\| = \max_{i=1,2,3,4} |x_i|$$

on $\mathbb{R}^4$, we compute

$$\begin{aligned} \|e^{At}P_{s,u}\| &\le Ke^{\tilde{k}t}, \quad K = \frac{1}{\sqrt{2}} + \max\left\{\frac{1}{2\tilde{k}}, \tilde{k}\right\}, \\ \|B(X) - A\| &= \frac{l^2}{k}\max\{|\alpha p'(x_1)|, |\beta q'(x_2)|\}. \end{aligned} \tag{4.4}$$

We are ready to prove the next theorem.

Theorem 4.1. *Let $M > 0$ and set*

$$S_{X_0}(M) = \{X \in \mathbb{R}^4 \mid |x_1| \le M, |x_2| \le M\}.$$

Suppose

$$\kappa = \frac{l^2}{k}\max_{X \in S_{X_0}(M)} \max\{|\alpha p'(x_1)|, |\beta q'(x_2)|\} < \frac{\tilde{k}}{2K}, \tag{4.5}$$

where K is given in (4.4). Then (4.2) has $X(t) = X_0$ as the only bounded solution on $\mathbb{R}$ with $X(t) \in S_{X_0}(M)$.

Proof. Rewriting (4.3) as

$$(X - X_0)' = A(X - X_0) + (B(X) - A)(X - X_0),$$

its bounded solution $X(t) \in S_{X_0}(M)$ on $\mathbb{R}$ is given by

$$X(t) - X_0 = \int_{-\infty}^{t} e^{A(t-s)} P_s(B(X(s)) - A)(X(s) - X_0)ds \\ - \int_{t}^{\infty} e^{A(t-s)} P_u(B(X(s)) - A)(X(s) - X_0)ds,$$

which by (4.4) implies

$$\|X(t) - X_0\| \leq K \int_{-\infty}^{t} e^{\tilde{k}(t-s)} \|B(X(s)) - A\| \|X(s) - X_0\| ds \\ + \int_{t}^{\infty} e^{\tilde{k}(t-s)} \|B(X(s)) - A\| \|X(s) - X_0\| ds \leq \frac{2K\kappa}{\tilde{k}} \sup_{t\in\mathbb{R}} \|X(t) - X_0\|.$$

This gives

$$\sup_{t\in\mathbb{R}} \|X(t) - X_0\| \leq \frac{2K\kappa}{\tilde{k}} \sup_{t\in\mathbb{R}} \|X(t) - X_0\|,$$

which by (4.5) implies $\sup_{t\in\mathbb{R}} \|X(t) - X_0\| = 0$, i.e., $X(t) = X_0$. The proof is finished. □

Theorem 4.1 leads to the following extension of Theorem 3.4.

Corollary 4.2. *If* (4.5) *holds then a bounded solution $X(t) \in S_{X_0}(M)$, $t \geq 0$ of* (4.2) *satisfies*

$$\lim_{t\to\infty} X(t) = X_0.$$

Proof. If $X(t) \in S_{X_0}(M)$, $t \geq 0$ is a bounded solution of (4.2), then its ω-limit set $\omega(X(0)) \subset S_{X_0}(M)$ is compact and invariant. Thus for any $\tilde{X}_0 \in \omega(X(0))$, the solution $\tilde{X}(t)$, $\tilde{X}(0) = \tilde{X}_0$, $t \in \mathbb{R}$ of (4.2) is bounded and it satisfies $X(t) \in S_{X_0}(M)$, since $\tilde{X}(t) \in \omega(X(0)) \subset S_{X_0}(M)$, $t \in \mathbb{R}$. Theorem 4.1 gives $\tilde{X}(t) = X_0$, so $\tilde{X}_0 = X_0$ and thus $\omega(X(0)) = \{X_0\}$. The proof is finished. □

Corollary 4.3. *If*

$$\Theta = \max\left\{ \sup_{x_1\in\mathbb{R}} |p'(x_1)|, \sup_{x_2\in\mathbb{R}} |q'(x_2)| \right\} < \infty,$$

then for any

$$\max\{|\alpha|, |\beta|\} < \frac{\tilde{k}k}{2Kl^2\Theta}, \tag{4.6}$$

all bounded solutions $X(t)$, $t \geq 0$ of (4.2) *satisfies*

$$\lim_{t\to\infty} X(t) = X_0.$$

Proof. Since condition (4.6) implies (4.5), the proof is finished by Corollary 4.2. □

We continue with utilizing a hyperbolic structure of $B(X)$ by considering a slowly variable system

$$X' = \begin{bmatrix} x_3 \\ x_4 \\ \frac{f}{k}(v_g - x_2) + \frac{\alpha l^2}{k} p'(\epsilon x_1) x_3 \\ \frac{f}{k}(x_1 - u_g) - \frac{\beta l^2}{k} q'(\epsilon x_2) x_4 \end{bmatrix} \tag{4.7}$$

for a small parameter $\epsilon \in \mathbb{R}$. Then (4.7) has a form

$$X' = B(\epsilon X)(X - X_0).$$

We have the following conclusion.

Theorem 4.4. *If $p'(0) = q'(0) = 0$, then for any $M > 0$ there is an $\epsilon_M > 0$ such that any bounded solution $X(t) \in S_{X_0}(M)$, $t \geq 0$ of (4.7) with $|\epsilon| < \epsilon_M$ satisfies*

$$\lim_{t\to\infty} X(t) = X_0.$$

Proof. Now (4.5) means

$$\kappa = \frac{l^2}{k} \max_{X\in S_{X_0}(M)} \max\{|\alpha p'(\epsilon x_1)|, |\beta q'(\epsilon x_2)|\} < \frac{\tilde{k}}{2K},$$

which clearly holds for any ϵ small due $p'(0) = q'(0) = 0$. The proof is finished. □

Results of this section lead to Theorem 3.2.

5 Spatial wave solutions for (2.4)

Motivated by the above method and results, we consider (2.4) for constant k

$$\begin{cases} f(v - v_g) = -k\frac{\partial^2 u}{\partial z^2} + 2l^2 \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z\partial x}, \\ f(u - u_g) = k\frac{\partial^2 v}{\partial z^2} + 2l^2 v\frac{\partial v}{\partial z}\frac{\partial^2}{\partial z\partial y}. \end{cases} \tag{5.1}$$

We are looking again for spatial wave solutions (3.2) of (5.1) to get

$$\begin{aligned} f(V - v_g) &= -kU'' + 2\alpha l^2 U'U'', \\ f(U - u_g) &= kV'' + 2\beta l^2 V'V''. \end{aligned} \tag{5.2}$$

We observe that (5.2) is more sophisticated than (3.3). Shifting

$$U \longleftrightarrow U - u_g, \quad V \longleftrightarrow V - v_g,$$

we study

$$\begin{aligned} fV &= -kU'' + 2\alpha l^2 U'U'', \\ fU &= kV'' + 2\beta l^2 V'V''. \end{aligned} \tag{5.3}$$

Integrating both equations of (5.3), we obtain

$$\begin{aligned} f\int V(t)dt &= -kU'(t) + \alpha l^2 U'^2(t), \\ f\int U(t)dt &= kV'(t) + \beta l^2 V'^2(t). \end{aligned} \tag{5.4}$$

By introducing

$$W_1 = \int U(t)dt, \quad W_2 = \int V(t)dt,$$

we get

$$\begin{aligned} fW_2 &= -kW_1'' + \alpha l^2 W_1''^2, \\ fW_1 &= kW_2'' + \beta l^2 W_2''^2. \end{aligned} \tag{5.5}$$

When $W_1(t) = 0$ then $U(t) = 0$ and (5.3) implies $V(t) = 0$, so $W_2(t) = 0$. Consequently $W_1(t) = 0 \Longrightarrow W_2(t) = 0$. Similarly $W_2(t) = 0 \Longrightarrow W_1(t) = 0$. Thus (5.5) gives

$$\begin{aligned} W_1'' &= \frac{k - \sqrt{k^2 + 4\alpha f l^2 W_2}}{2\alpha l^2}, \\ W_2'' &= \frac{-k + \sqrt{k^2 + 4\beta f l^2 W_1}}{2\beta l^2}. \end{aligned} \tag{5.6}$$

Next, we take in (5.6)

$$Y_1 = k^2 + 4\beta f l^2 W_1, \quad Y_2 = k^2 + 4\alpha f l^2 W_2$$

to get

$$\begin{aligned} Y_1'' &= \frac{2\beta f}{\alpha}(k - \sqrt{Y_2}), \\ Y_2'' &= \frac{2\alpha f}{\beta}(-k + \sqrt{Y_1}). \end{aligned} \tag{5.7}$$

Next, we set

$$Y_i(t) = k^2 Z_i\left(\sqrt{\frac{2f}{k}}t\right), \quad i = 1, 2$$

in (5.7) to obtain

$$\begin{aligned} Z_1'' &= \mu^{-1}(1 - \sqrt{Z_2}), \\ Z_2'' &= \mu(-1 + \sqrt{Z_1}). \end{aligned} \tag{5.8}$$

for

$$\mu = \frac{\alpha}{\beta}.$$

Taking

$$X = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_1' \\ Z_2' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

(5.8) becomes

$$X' = G(X) = \begin{bmatrix} x_3 \\ x_4 \\ \mu^{-1}(1 - \sqrt{x_2}) \\ \mu(-1 + \sqrt{x_1}) \end{bmatrix}. \tag{5.9}$$

Note (5.9) has a unique equilibrium

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and its Jacobian matrix is

$$DG(X_1) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{2\mu} & 0 & 0 \\ \frac{\mu}{2} & 0 & 0 & 0 \end{bmatrix}$$

with eigenvalues

$$\frac{-1-\imath}{2}, \quad \frac{-1+\imath}{2}, \quad \frac{1-\imath}{2}, \quad \frac{1+\imath}{2}$$

and the corresponding complex eigenvectors

$$\begin{bmatrix} -\frac{1+\imath}{\mu} \\ -1+\imath \\ \frac{\imath}{\mu} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -\frac{1-\imath}{\mu} \\ -1-\imath \\ -\frac{\imath}{\mu} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1-\imath}{\mu} \\ 1+\imath \\ -\frac{\imath}{\mu} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1+\imath}{\mu} \\ 1-\imath \\ \frac{\imath}{\mu} \\ 1 \end{bmatrix}.$$

Consequently, X_1 is a hyperbolic equilibrium. Thus we have the following result similar to the statement of Theorem 3.2.

Theorem 5.1. *There is a 4-parametrized family of spacial waves solutions of* (5.2) *asymptotic to the equilibrium.*

Furthermore, (5.9) has a first integral

$$I(x_1,x_2,x_3,x_4) = x_3x_4 - \mu^{-1}\left[x_2 - \frac{2}{3}x_2^{3/2}\right] - \mu\left[-x_1 + \frac{2}{3}x_1^{3/2}\right].$$

Its reduction on the level

$$I(x_1,x_2,x_3,x_4) = C \tag{5.10}$$

is given by

$$\begin{aligned} x_1' &= x_3, \\ x_3x_2' &= \mu^{-1}\left[x_2 - \frac{2}{3}x_2^{3/2}\right] + \mu\left[-x_1 + \frac{2}{3}x_1^{3/2}\right] + C, \\ x_3' &= \mu^{-1}(1-\sqrt{x_2}). \end{aligned} \tag{5.11}$$

(5.11) is an implicit ODE [15, 29] and its analysis seems to be difficult in general. Some numerical simulations should help. On the other hand, taking

$$y_i(t) = x_i(\mu t), \quad i = 1,2,3, \tag{5.12}$$

we get

$$\begin{aligned} y_1' &= \mu y_3, \\ y_3y_2' &= y_2 - \frac{2}{3}y_2^{3/2} + C\mu + \mu^2\left[-y_1 + \frac{2}{3}y_1^{3/2}\right], \\ y_3' &= 1-\sqrt{y_2}. \end{aligned} \tag{5.13}$$

(5.13) is reducing for $\mu = 0$ to

$$\begin{aligned} y_1' &= 0 \\ y_3y_2' &= y_2 - \frac{2}{3}y_2^{3/2} \\ y_3' &= 1-\sqrt{y_2}. \end{aligned} \tag{5.14}$$

The first equation of (5.14) gives $y_1(t) = y_1(0)$, and the second and third ones imply

$$\frac{dy_3}{y_3} = \frac{1-\sqrt{y_2}}{y_2 - \frac{2}{3}y_2^{3/2}}dy_2. \tag{5.15}$$

Integrating (5.15), we have

$$\ln y_3 = \ln(y_2(2\sqrt{y_2}-3)) + \tilde{C}$$

for a constant $\tilde{C}$, which implies

$$y_3 = C_0(3-2\sqrt{y_2})y_2 \tag{5.16}$$

for a constant C_0. Note y_1, y_2 and y_3 are depending on t, so differentiating (5.16) with respect to t, we get

$$y_3' = 3C_0(1-\sqrt{y_2})y_2',$$

which together with the third equation of (5.14) give

$$3C_0 y_2' = 1,$$

which possesses a solution

$$y_2(t) = \frac{t}{3C_0} + y_2(0)$$

and (5.16) leads to

$$y_3(t) = C_0 \left(3 - 2\sqrt{\frac{t}{3C_0} + y_2(0)}\right) \left(\frac{t}{3C_0} + y_2(0)\right).$$

Clearly

$$y_3(0) = C_0 \left(3 - 2\sqrt{y_2(0)}\right) y_2(0).$$

Consequently, (5.13) has a solution

$$\begin{aligned}
y_1(t) &= y_1(0) + O(\mu), \\
y_2(t) &= \frac{t}{3C_0} + y_2(0) + O(\mu), \\
y_3(t) &= C_0 \left(3 - 2\sqrt{\frac{t}{3C_0} + y_2(0)}\right) \left(\frac{t}{3C_0} + y_2(0)\right) + O(\mu), \\
C_0 &= \frac{y_3(0)}{(3 - 2\sqrt{y_2(0)})y_2(0)}.
\end{aligned} \tag{5.17}$$

Summarizing, (5.17), (5.12) and (5.10) give a first order approximate solution of (5.9) with respect to μ small. Higher orders can be computed similarly. But since the right hand side of (5.13) is not analytic, it is better instead of (5.13) to take

$$u_1^2 = y_1, \quad u_2^2 = y_2, \quad u_3 = y_3$$

and consider

$$\begin{aligned}
2u_1 u_1' &= \mu u_3, \\
2u_3 u_2 u_2' &= u_2^2 - \frac{2}{3}u_2^3 + C\mu + \mu^2 \left[-u_1^2 + \frac{2}{3}u_1^3\right], \\
u_3' &= 1 - u_2.
\end{aligned} \tag{5.18}$$

Then we expand

$$u_i(t) = \sum_{k=0}^{r} \mu^k u_{ik}(t), \quad i = 1,2,3, \quad u_{ik}(0) = 0, \quad k \geq 1 \tag{5.19}$$

and plugging (5.19) into (5.18), we derive other terms. By (5.17), we have

$$\begin{aligned}
u_{10}(t) &= u_1(0), \\
u_{20}(t) &= \sqrt{\frac{t}{3C_0} + u_2(0)^2}, \\
u_{30}(t) &= C_0 \left(3 - 2\sqrt{\frac{t}{3C_0} + u_2(0)^2}\right) \left(\frac{t}{3C_0} + u_2(0)^2\right), \\
C_0 &= \frac{u_3(0)}{(3 - 2u_2(0))u_2(0)^2}.
\end{aligned}$$

Note that (5.18) is not solvable at the surface $u_1u_2u_3 = 0$, so it is implicit in the terminology of [15,29]. But it is orbitally equivalent for $u_1u_2u_3 \neq 0$ to a standard ODE

$$\begin{aligned} \hat{u}_1' &= \mu \hat{u}_2 \hat{u}_3^2, \\ \hat{u}_2' &= \hat{u}_1 \hat{u}_2^2 - \frac{2}{3}\hat{u}_1 \hat{u}_2^3 + C\mu + \mu^2 \hat{u}_1 \left[-\hat{u}_1^2 + \frac{2}{3}\hat{u}_1^3 \right], \\ \hat{u}_3' &= \hat{u}_1 \hat{u}_2 \hat{u}_3 - \hat{u}_1 \hat{u}_2^2 \hat{u}_3. \end{aligned} \tag{5.20}$$

Hence expansion (5.19) really works for (5.18).

6 Conclusion

We use Prandtl mixing-length theory and semiempirical theory to extend the classical problem of the wind in the steady atmospheric Ekman layer with constant eddy viscosity. This establishes new generalized atmospheric Ekman equations. Then paper deals with the existence of spatial wave solutions for these generalized atmospheric Ekman equations. Such kind of solutions are determined by certain 4-dimensional autonomous ODEs with quadratic nonlinearities. We apply methods of dynamical systems for investigating qualitative properties of these ODEs. The existence of families of asymptotic and periodic spatial wave solutions is proved. Exact and approximative solutions of the corresponding ODEs are also derived. Two figures are presented for visualization of certain these solutions. The derived spatial wave ODEs are nonlinear and could be implicit, so their study is difficult in general. Consequently, there are still many open challenging problems for further research such as existence or nonexistence of quasiperiodic, homoclinic or even chaotic solutions.

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References

[1] A. Bressan, A. Constantin, The deflection angle of surface ocean currents from the wind direction, *J. Geophys. Res.* **124**(2019), 7412–7420. https://doi.org/10.1029/2019JC015454

[2] A. Constantin, Frictional effects in wind-driven ocean currents, *Geophys. Astrophys. Fluid Dyn.* **115**(2021), 1–14. https://doi.org/10.1080/03091929.2020.1748614; MR4205096

[3] A. Constantin, D. G. Dritschel, N. Paldor, The deflection angle between a wind-forced surface current and the overlying wind in an ocean with vertically varying Eddy viscosity, *Phys. Fluids* **32**(2020), 116604. https://doi.org/10.1063/5.0030473

[4] A. Constantin, R. I. Ivanov, Equatorial wave-current interactions, *Comm. Math. Phys.* **370**(2019), 1–48. https://doi.org/10.1007/s00220-019-03483-8; MR3982690

[5] A. Constantin, R. S. Johnson, The dynamics of waves interacting with the Equatorial Undercurrent, *Geophys. Astrophys. Fluid Dyn.* **109**(2015), 311–358. https://doi.org/10.1080/03091929.2015.1066785; MR3375654

[6] A. Constantin, R. S. Johnson, Steady large-scale ocean flows in spherical coordinates, *Oceanography* **31**(2018), 42–50. https://doi.org/10.5670/oceanog.2018.308

[7] A. Constantin, R. S. Johnson, Ekman-type solutions for shallow-water flows on a rotating sphere: A new perspective on a classical problem, *Phys. Fluids* **31**(2019), 021401. https://doi.org/10.1063/1.5083088

[8] A. Constantin, R. S. Johnson, On the nonlinear, three-dimensional structure of equatorial oceanic flows, *J. Phys. Oceanogr.* **49**(2019), 2029–2042. https://doi.org/10.1175/JPO-D-19-0079.1

[9] A. Constantin, R. S. Johnson, Atmospheric Ekman flows with variable eddy viscosity, *Bound.-Layer Meteorol.* **170**(2019), 395–414. https://doi.org/10.1007/s10546-018-0404-0

[10] I. K. Delia, Analytical atmospheric Ekman-type solutions with height-dependent e0ddy viscosities, *J. Math. Fluid Mech.* **23**(2021), Article 18. https://doi.org/10.1007/s00021-020-00543-1; MR4200934

[11] I. K. Delia, C. I. Martin, Periodic equatorial water flows from a Hamiltonian perspective, *J. Differential Equations* **262**(2017), 4451–4474. https://doi.org/10.1016/j.jde.2017.01.001; MR3603277

[12] D. G. Dritschel, N. Paldor, A. Constantin, The Ekman spiral for piecewise-uniform diffusivity, *Ocean Science* **16**(2020), 1089–1093. https://doi.org/10.5194/os-2020-31

[13] V. W. Ekman, On the influence of the Earth's rotation on ocean-currents, *Ark. Mat. Astr. Fys.* **2**(1905), 1–52. Zbl 36.1009.04

[14] M. Fečkan, Minimal periods of periodic solutions, *Miskolc Math. Notes*, **7**(2006) 121–139. https://doi.org/10.18514/MMN.2006.155

[15] M. Fečkan, A survey on the Melnikov theory for implicit ordinary differential equations with applications to RLC circuits, in: *F. Smith, H. Dutta, J. Mordeson, Eds., Mathematics Applied to Engineering, Modelling, and Social Issues, Studies in Systems, Decision and Control*, Vol 200, Springer-Verlag, Cham, 2019. https://doi.org/10.1007/978-3-030-12232-4_4; MR3931547

[16] M. Fečkan, Y. Guan, D. O'Regan, J. Wang, Existence and uniqueness and first order approximation of solutions to atmospheric Ekman flows, *Monatsh. Math.* **193**(2020), 623–636. https://doi.org/10.1007/s00605-020-01414-7; MR4153333

[17] B. Grisogono, The angle of the near-surface wind-turning in weakly stable boundary layers, *Q. J. R. Meteorol. Soc.* **137**(2011), 700–708. https://doi.org/10.1002/qj.789

[18] Y. Guan, M. Fečkan, J. Wang, Periodic solutions and Hyers–Ulam stablity of atmospheric Ekman flows, *Discrete Contin. Dyn. Syst.* **41**(2021), 1157–1176. https://doi.org/10.3934/dcds.2020313

[19] Y. Guan, M. Fečkan, J. Wang, Explicit solution of atmospheric Ekman flows with some types of Eddy viscosity, *Monatsh Math.* **197**(2022), 71–84. https://doi.org/10.1007/s00605-021-01551-7; MR4368631

[20] Y. Guan, J. Wang, M. Fečkan, Explicit solution and dynamical properties of atmospheric Ekman flows with boundary conditions, *Electron. J. Qual. Theory Differ. Equ.* **2021**, No. 30, 1–19. https://doi.org/10.14232/ejqtde.2021.1.30; MR4240284

[21] J. K. Hale, H. Koçak, *Dynamics and bifurcations*, Springer-Verlag, New York, 1991. MR1138981

[22] G. J. Haltinar, R. T. Williams, *Numercial prediction and dynamic meteorology*, Wiley Press, New York, 1980. https://doi.org/10.1017/S0022112084210938

[23] D. Henry, Nonlinear features of equatorial ocean flows, *Oceanography* **31**(2018), 22–27. https://doi.org/10.5670/oceanog.2018.305

[24] J. R. Holton, *An introduction to dynamic meteorology*, Academic Press, New York, 2004. https://doi.org/10.1016/C2009-0-63394-8; MR0111594

[25] G. C. Johnson, M. J. McPhaden, E. Firing, Equatorial Pacific Ocean horizontal velocity, divergence, and upwelling, *J. Phys. Oceanogr.* **31**(2001), 839–849. https://doi.org/10.1175/1520-0485(2001)031<0839:EPOHVD>2.0.CO;2

[26] J. Marshall, R. A. Plumb, *Atmosphere ocean and climate dynamic: an introduction text*, Academic Press, New York, 2016.

[27] J. Mawhin, M. Willem, *Critical point theory and Hamiltonian systems*, Springer-Verlag, New York, 1989. https://doi.org/10.1007/978-1-4757-2061-7; MR0982267

[28] K. Rektorys, *Survey of applicable mathematics*, Springer, Dordrecht, 1994. https://doi.org/10.1007/978-94-015-8308-4; MR1282494

[29] R. Riaza, *Differential-algebraic systems, analytical aspects and circuit applications*, World Scien. Publ. Co. Pte. Ltd., Singapore, 2008. https://doi.org/10.1142/9789812791818

[30] L. Roberti, Perturbation analysis for the surface deflection angle of Ekman-type flows with variable eddy viscosity, *J. Math. Fluid Mech.* **23**(2021), Article 57, 7 pp. https://doi.org/10.1007/s00021-021-00586-y; MR4254067

[31] L. Roberti, The Ekman spiral for piecewise-constant eddy viscosity, *Appl. Anal.* **101**(2022), No. 15, 5528–5536. https://doi.org/10.1080/00036811.2021.1896709

[32] J. Wang, M. Fečkan, Y. Guan, Local and global analysis for discontinuous atmospheric Ekman equations, *J. Dynam. Differential Equations*, July 2021, published online. https://doi.org/10.1007/s10884-021-10037-x

[33] W. Zdunkowski, A. Bott, *Dynamics of the atmosphere*, Cambridge University Press, Cambridge, 2003. https://doi.org/10.1029/2004EO160015

[34] Y. Yoshikawa, A. Masuda, Seasonal variations in the speed factor and deflection angle of the wind-driven surface flow in the Tsushima Strait, *J. Geophys. Res.* **114**(2009), C12022. https://doi.org/10.1029/2009JC005632