Electronic Journal of Qualitative Theory of Differential Equations
2022, No. **23**, 1–18; https://doi.org/10.14232/ejqtde.2022.1.23 www.math.u-szeged.hu/ejqtde/

On a differential equation involving a new kind of variable exponents

Sami Aouaoui ✉

University of Kairouan, Department of Mathematics,
High Institute of Applied Mathematics and Informatics of Kairouan,
Avenue Assad Iben Fourat, 3100 Kairouan, Tunisia

Received 18 October 2021, appeared 17 May 2022
Communicated by Dimitri Mugnai

Abstract. In this paper, we are concerned with some new first order differential equation defined on the whole real axis $\mathbb{R}$. The principal part of the equation involves an operator with variable exponent p depending on the variable $x \in \mathbb{R}$ through the unknown solution while the nonlinear part involves the classical variable exponent $p(x)$. Such kind of situation is very related to the presence of the variable exponent and has not been treated before. Our existence result of nontrivial solution cannot be reached using standard variational or topological methods of nonlinear analysis and some sophisticated arguments have to be employed.

Keywords: $p(u)$-Laplacian, variable exponents, Schauder's fixed point theorem, approximation scheme, weighted Sobolev space, existence result.

2020 Mathematics Subject Classification: 26A24, 34A34, 47H05, 47H10.

1 Introduction and Statement of Main results

Nonlinear partial differential equations involving variable exponents have many applications in physics. In fact, such equations are used as models to describe many phenomena arising in applied sciences. For instance, we can mention the study of materials with strong inhomogeneities such as electrorheological fluids or thermo-rheological, image restoration, phenomenon of elasticity or the continuum mechanics. See [5,10,15,16,22].

Actually, the observation of the image restoration process through some numerical techniques has proved that considering the case of variable exponents depending on the solution u (or its derivatives) reduces the noise of the restored image u. See [8,9,17]. The same situation is observed when treating the problem of thermistor which describes the electric current in a conductor that may change its properties in dependence of temperature (see [4]).

When we try to deal with a problem involving an exponent depending on the solution, we are quickly faced with many obstacles which are essentially related to the theoretical well-posedness of the problem itself. Indeed, such a problem is not standard because its weak

✉ Email: sami.aouaoui@ismai.u-kairouan.tn

formulation cannot be written as an equation in terms of duality in a fixed Banach space. This is why, in the mathematical literature, one can find only few works devoted to the study of elliptic and parabolic equations involving an exponent of the type $p(u)$ with local and nonlocal dependence of p on u. The first one is due to B. Andreianov, M. Bendahmane and S. Ouaro who have considered in [1] the problem

$$\begin{cases} u - \operatorname{div}\left(|\nabla u|^{p(u)-2}\nabla u\right) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is some bounded domain of $\mathbb{R}^N$, $N \geq 2$, $f \in L^1(\Omega)$ and $p : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous such that $p^- = \inf_{s\in\mathbb{R}} p(s) > N$. Under the key restriction $p^- > N$, the authors proved that the problem (1.1) is well-posed in $L^1(\Omega)$. By this way, using some approximation method, they can establish the existence of so-called narrow and broad weak solution (definitions related to the fact that the source f is integrable). The version of the problem (1.1) with homogeneous Neumann boundary conditions has been treated in [14].

Recently, M. Chipot and H. B. de Oliveira proposed in [11] a new simple approach to deal with a problem very similar to (1.1). More precisely, M. Chipot and H. B. de Oliveira studied the problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(u)-2}\nabla u\right) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$ with smooth boundary, $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $p^- > N$, and $f \in W^{-1,(p^-)'}(\Omega)$. The approach in [11] is mainly based on a perturbation of the problem (1.2) and the use of Schauder's fixed point theorem to solve the approximated problem. Finally, a process of passage to the limit in the spirit of [23] is carried out to prove the existence of a weak solution u of the problem (1.2) in the sense that $u \in W_0^{1,p(u)}(\Omega)$ and satisfies

$$\int_\Omega |\nabla u|^{p(u)-2}\nabla u \nabla v dx = \langle f, v\rangle, \qquad \forall\, v \in W_0^{1,p(u)}(\Omega).$$

The nonlocal version of (1.2) has been also considered in [11]. More precisely, the authors studied the problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(b(u))-2}\nabla u\right) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where p is merely bounded continuous and satisfies that $1 < p^- < p(s)$, $\forall\, s \in \mathbb{R}$, and $b : W_0^{1,p^-}(\Omega) \to \mathbb{R}$ sends bounded sets of $W_0^{1,p^-}(\Omega)$ into bounded sets of $\mathbb{R}$. Using the Brower's fixed point theorem applied to some compact interval of $\mathbb{R}$, M. Chipot and H. B. de Oliveira proved that (1.3) has at least one weak solution u in the sense that $u \in W_0^{1,p(b(u))}(\Omega)$ and satisfies

$$\int_\Omega |\nabla u|^{p(b(u))-2}\nabla u \nabla v dx = \langle f, v\rangle, \qquad \forall\, v \in W_0^{1,p(b(u))}(\Omega).$$

This work has been completed in [20] where the authors treated the case when $f \in L^1(\Omega)$ for which they prove the existence of an entropy solution. The work of M. Chipot and H. B. de Oliveira has given a new impulse to the study of problems involving exponents depending on the unknown solution. In [2], S. Antontsev and S. Shmarev studied the homogeneous Dirichlet problem for the parabolic equation

$$u_t - \operatorname{div}\left(|\nabla u|^{p[u]-2} \nabla u\right) = f, \quad \text{in } Q_T = \Omega \times]0, T[,$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a smooth domain, $p[u] = p(l(u))$, p is a given differentiable function such that $\frac{2N}{N+2} < p^- \leq p^+ < 2$, and $\sup_{s\in\mathbb{R}} |p'(s)| < +\infty$; $l(u) = \int_\Omega |u(x,t)|^\alpha \, dx$, $\alpha \in [1,2]$, and $f \in L^{(p^-)'}(Q_T)$. A result of existence and uniqueness of a solution $u \in C^0\left([0,T]; L^2(\Omega)\right)$, $|\nabla u|^{p[u]} \in L^\infty\left(0,T; L^1(\Omega)\right)$, $u_t \in L^2(Q_T)$ has been proved. This result has been extended in [3] to the case when the source f is replaced by the nonlinear term $f((x,t), u, l(u))$, and in [4] where the authors (together with I. Kuznetsov) treated the case when the exponent p is depending on the gradient of u, i.e. when $p[u]$ is replaced by $p[\nabla u] = p(l(|\nabla u|))$. The case of unbounded domain has been considered in [7] where S. Aouaoui and A. E. Bahrouni studied the equation

$$-\operatorname{div}(w_1(x) |\nabla u|^{p(u)-2} \nabla u) + w_0(x) |u|^{p(u)-2} u = f(x,u), \qquad x \in \mathbb{R}^N, \ N \geq 2,$$

where $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $N < p^- < p^+ < +\infty$; $w_0, w_1 \in L^1(\mathbb{R}^N)$ and f is a Carathéodory function having a polynomial growth with exponent lower than $p^- - 1$. A result of the existence of a nontrivial solution has been established.

The present work is a contribution in the same direction. Indeed, in this paper, we are concerned with the following nonlinear differential equation:

$$-\left(w_1(x) |u'|^{p(u)-2} u'\right)' + w_0(x) |u|^{p(u)-2} u = g(x) |u|^{p(x)-2} u, \qquad x \in \mathbb{R}, \tag{1.4}$$

where $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that

$$1 < p^- = \inf_{s\in\mathbb{R}} p(s) < p^+ = \sup_{s\in\mathbb{R}} p(s) < +\infty.$$

The equation (1.4) is taken under the following assumptions:

(H_1) We assume that there exists $0 < \alpha < 1$ such that $p(\alpha) = p^+$. Moreover, we assume that the function $x \longmapsto x^{p(x)-1}$ is increasing on the interval $[0, \alpha]$.

(H_2) $w_0, w_1 \in L^1(\mathbb{R})$ are such that

$$0 < \sup_{|x|\leq R} w_0(x) < +\infty, \qquad 0 < \inf_{|x|\leq R} w_1(x) < +\infty, \qquad \forall\, R > 0.$$

We also assume that there exists a positive constant $C_0 > 0$ such that

$$w_1(x) \leq C_0 w_0(x), \qquad \forall\, x \in \mathbb{R}.$$

(H_3) $g \in L^1(\mathbb{R})$, $g(x) > 0$, $\forall\, x \in \mathbb{R}$. We assume that

$$g(x) \leq w_0(x) \leq g(x)\alpha^{p(x)-p^+}, \qquad \forall\, x \in \mathbb{R},$$

where α is defined in (H_1).

A similar differential equation to (1.4) has been treated in [6] where the author dealt with the nonlinear equation

$$-\left(\left|u'\right|^{p(x)-2}u'\right)+|u|^{p(x)-2}u=\lambda\varphi(x)\,|u|^{p(u)-2}u,\qquad x\in\mathbb{R},\tag{1.5}$$

where $p\in C^1(\mathbb{R})$ is such that $2<p^-<p^+<+\infty$, λ is a positive parameter and $\varphi\in C(\mathbb{R})\cap L^\infty(\mathbb{R})$, $\varphi(x)>0$, $\forall\ x\in\mathbb{R}$. Under some suitable additional conditions on p and φ, the author used a variational method to prove the existence of a nontrivial solution to (1.5). Comparing to (1.4), the problem (1.5) is easier because the exponent appearing in the principal part depends directly on the variable $x\in\mathbb{R}$ and by consequence the solution has been searched in the fixed classical variable exponent Sobolev space $W^{1,p(x)}(\mathbb{R})$.

Definition 1.1. A function $u:\mathbb{R}\to\mathbb{R}$ is said to be a weak solution to the equation (1.4) if it satisfies that $u\in L^1_{loc}(\mathbb{R})$,

$$\int_{\mathbb{R}}w_0(x)\,|u|^{p(u)}\,dx<+\infty,\qquad \int_{\mathbb{R}}w_1(x)\left|u'\right|^{p(u)}dx<+\infty,$$

and

$$\int_{\mathbb{R}}w_1(x)\left|u'\right|^{p(u)-2}u'v'dx+\int_{\mathbb{R}}w_0(x)\,|u|^{p(u)-2}uvdx=\int_{\mathbb{R}}g(x)\,|u|^{p(x)-2}uvdx,\qquad \forall\ v\in E_u,$$

where

$$E_u=\left\{v\in L^1_{loc}(\mathbb{R}),\ \int_{\mathbb{R}}w_0(x)\,|v|^{p(u)}\,dx<+\infty,\ \int_{\mathbb{R}}w_1(x)\left|v'\right|^{p(u)}dx<+\infty\right\}.$$

The main result in this work is given by the following theorem.

Theorem 1.2. *Assume that (H_1), (H_2) and (H_3) hold. Then, there exists at least one weak nontrivial and positive solution to the equation (1.4) in the sense of Definition 1.1.*

Example 1.3. As an example of functions p, w_0, w_1 and g satisfying the hypotheses of Theorem 1.2, one can choose

$$p(x)=k+e^{-(x-\frac{1}{2})^2},\quad k\geq 2,\quad w_0(x)=w_1(x)=g(x)=e^{-x^2},\quad x\in\mathbb{R}.$$

2 Preliminaries

In this section, we study the functional space E_u. For $u:\mathbb{R}\to\mathbb{R}$ a fixed measurable function, set $q=p(u)$. In view of this notation, one can easily see that E_u is the weighted Sobolev space with variable exponent

$$E_u=\left\{v\in L^1_{loc}(\mathbb{R}),\ \int_{\mathbb{R}}w_0(x)\,|v|^{q(x)}\,dx<+\infty,\ \int_{\mathbb{R}}w_1(x)\left|v'\right|^{q(x)}dx<+\infty\right\}.$$

This space is equipped with the well known Luxemburg norm

$$\|u\|_{E_u}=\inf\left\{\lambda>0,\ \int_{\mathbb{R}}\left(\frac{w_1(x)\,|u'|^{q(x)}+w_0(x)\,|u|^{q(x)}}{\lambda^{q(x)}}\right)dx\leq 1\right\}.$$

Since $w_0, w_1 \in L^1_{loc}(\mathbb{R})$ and $w_0^{-\frac{1}{q(x)-1}}, w_1^{-\frac{1}{q(x)-1}} \in L^1_{loc}(\mathbb{R})$, then $(E_u, \|\cdot\|_{E_u})$ is a Banach, reflexive and separable space (see [21]).

If $v \in E_u$, $(v_n)_n \subset E_u$, then the following relations hold true.

$$\|v\|_{E_u} < 1 \Rightarrow \|v\|_{E_u}^{q^+} \leq \int_{\mathbb{R}} \left(w_1(x) \left|v'\right|^{q(x)} + w_0(x) \left|v\right|^{q(x)} \right) dx \leq \|v\|_{E_u}^{q^-},$$

$$\|v\|_{E_u} > 1 \Rightarrow \|v\|_{E_u}^{q^-} \leq \int_{\mathbb{R}} \left(w_1(x) \left|v'\right|^{q(x)} + w_0(x) \left|v\right|^{q(x)} \right) dx \leq \|v\|_{E_u}^{q^+},$$

$$\|v_n - v\|_{E_u} \to 0 \Leftrightarrow \int_{\mathbb{R}} \left(w_1(x) \left|v_n' - v'\right|^{q(x)} + w_0(x) \left|v_n - v\right|^{q(x)} \right) dx \to 0, \qquad n \to +\infty.$$

One of the most important properties of the space E_u is the density of the space of smooth functions $C_0^\infty(\mathbb{R})$ in it with respect to the norm $\|\cdot\|_{E_u}$.

Proposition 2.1. *Assume that $u \in L^1_{loc}(\mathbb{R})$, and satisfies*

$$\int_{\mathbb{R}} w_0(x) \left|u\right|^{p^-} dx < +\infty, \quad \text{and} \quad \int_{\mathbb{R}} w_1(x) \left|u'\right|^{p^-} dx < +\infty.$$

Then, $C_0^\infty(\mathbb{R})$ is dense in E_u.

Proof. The proof relies essentially on a truncation procedure. Let $v \in E_u$, $\psi \in C_0^\infty(\mathbb{R})$ with $0 \leq \psi \leq 1$, $\psi(x) = 1$, if $|x| \leq 1$, $\psi(x) = 0$, if $|x| \geq 2$, and, for $n \geq 1$ an integer, set $\psi_n(x) = \psi(\frac{x}{n})$ and $v_n = v\psi_n$. We claim that $v_n \to v$ strongly in E_u. We have,

$$\int_{\mathbb{R}} w_0(x) \left|v_n - v\right|^{q(x)} dx = \int_{\mathbb{R}} w_0(x) \left|1 - \psi_n(x)\right|^{q(x)} \left|v\right|^{q(x)} dx.$$

Since $|1 - \psi_n(x)| \to 0$, $\forall\, x \in \mathbb{R}$ and $|1 - \psi_n(x)| \leq 2$, $\forall\, x \in \mathbb{R}$, $\forall\, n \geq 1$, then one can use the Lebesgue's dominated convergence theorem to deduce that

$$\lim_{n \to +\infty} \int_{\mathbb{R}} w_0(x) \left|v_n - v\right|^{q(x)} dx = 0. \tag{2.1}$$

On the other hand,

$$\begin{aligned} &\int_{\mathbb{R}} w_1(x) \left|v_n' - v'\right|^{q(x)} dx \\ &\quad = \int_{\mathbb{R}} w_1(x) \left|(1 - \psi_n) v' - v \psi_n'\right|^{q(x)} dx \\ &\quad \leq c_0 \int_{\mathbb{R}} w_1(x) \left|1 - \psi_n\right|^{q(x)} \left|v'\right|^{q(x)} dx + c_0 \int_{\mathbb{R}} w_1(x) \left|v\right|^{q(x)} \left(\frac{1}{n}\right)^{q(x)} \left|\psi'\left(\frac{x}{n}\right)\right|^{q(x)} dx \\ &\quad \leq c_0 \int_{\mathbb{R}} w_1(x) \left|1 - \psi_n\right|^{q(x)} \left|v'\right|^{q(x)} dx + c_0 C_0 \int_{\mathbb{R}} w_0(x) \left|v\right|^{q(x)} \left(\frac{1}{n}\right)^{q(x)} \left|\psi'\left(\frac{x}{n}\right)\right|^{q(x)} dx, \end{aligned} \tag{2.2}$$

where we used the fact that $w_1(x) \leq C_0 w_0(x)$, $\forall\, x \in \mathbb{R}$. Plainly,

$$\int_{\mathbb{R}} w_0(x) \left|v\right|^{q(x)} \left(\frac{1}{n}\right)^{q(x)} \left|\psi'\left(\frac{x}{n}\right)\right|^{q(x)} dx \to 0, \qquad n \to +\infty. \tag{2.3}$$

Again by the virtue of the Lebesgue's dominated convergence theorem, we get

$$\lim_{n \to +\infty} \int_{\mathbb{R}} w_1(x) \left|1 - \psi_n\right|^{q(x)} \left|v'\right|^{q(x)} dx = 0. \tag{2.4}$$

By (2.4) and (2.3), we infer

$$\lim_{n\to+\infty}\int_{\mathbb{R}} w_1(x)\left|v_n'-v'\right|^{q(x)}dx=0. \tag{2.5}$$

Combining (2.5) and (2.1), it follows that $v_n\to v$ strongly in E_u. Hence, for every $\epsilon>0$, there exists $n_0=n_0(\epsilon)\geq 1$ such that $\|v_{n_0}-v\|_{E_u}\leq\frac{\epsilon}{2}$. Now, taking into account that

$$0<\inf_{|x|<2n_0} w_0(x)\leq\sup_{|x|<2n_0} w_0(x)<+\infty,\quad\text{and}\quad 0<\inf_{|x|<2n_0} w_1(x)\leq\sup_{|x|<2n_0} w_1(x)<+\infty,$$

one can easily see that

$$\left\{w\in L^1(]-2n_0,2n_0[),\ \int_{-2n_0}^{2n_0} w_0(x)\,|w|^{q(x)}dx<+\infty,\ \int_{-2n_0}^{2n_0} w_1(x)\left|w'\right|^{q(x)}dx<+\infty\right\}$$
$$=W^{1,q(x)}(]-2n_0,2n_0[).$$

We also see that $u\in W^{1,p^-}(]-2n_0,2n_0[)$. Hence, $u\in\mathcal{C}([-2n_0,2n_0])$ and there exists a constant C depending on p and ϵ such that

$$|u(x)-u(y)|\leq C\,\|u\|_{W^{1,p^-}(]-2n_0,2n_0[)}\,|x-y|^{1-\frac{1}{p^-}},\qquad\forall\, x,y\in]-2n_0,2n_0[.$$

By hypothesis, there is a constant $L>0$ such that

$$|p(u(x))-p(u(y))|\leq L\,|u(x)-u(y)|,\qquad\forall\, x,y\in\mathbb{R},$$

which implies that

$$|q(x)-q(y)|\leq LC\,\|u\|_{W^{1,p^-}(]-2n_0,2n_0[)}\,|x-y|^{1-\frac{1}{p^-}},\qquad\forall\, x,y\in]-2n_0,2n_0[.$$

Therefore, q is log-Hölder continuous, that is, there is a constant $C'>0$ such that

$$|q(x)-q(y)|\leq\frac{-C'}{\log|x-y|},\qquad\forall\, x,y\in]-2n_0,2n_0[,\ |x-y|<\frac{1}{2}.$$

That result guarantees that $C_0^\infty(]-2n_0,2n_0[)$ is dense in $W^{1,q(x)}(]-2n_0,2n_0[)\cap W_0^{1,1}(]-2n_0,2n_0[)$ (see [13, 21]). Having in mind that $v_{n_0}\in W^{1,q(x)}(]-2n_0,2n_0[)\cap W_0^{1,1}(]-2n_0,2n_0[)$, we can conclude the proof of Proposition 2.1. □

3 Proof of Theorem 1.2

For $(x,s)\in\mathbb{R}^2$, set

$$f(x,s)=\begin{cases} g(x), & \text{if } s\geq 1,\\ g(x)s^{p(x)-1}, & \text{if } \alpha\leq s\leq 1,\\ g(x)\alpha^{p(x)-1}, & \text{if } s\leq\alpha.\end{cases}$$

Consider the weighted Sobolev space

$$W^{1,p^+}_{w_0,w_1}(\mathbb{R})=\left\{u\in L^1_{loc}(\mathbb{R}),\ \int_{\mathbb{R}} w_0(x)\,|u|^{p^+}dx<+\infty,\ \int_{\mathbb{R}} w_1(x)\left|u'\right|^{p^+}dx<+\infty\right\}.$$

This space is naturally equipped with the norm

$$\|u\|_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})}=\left(\int_{\mathbb{R}}\left(w_1(x)\left|u'\right|^{p^+}+w_0(x)\,|u|^{p^+}\right)dx\right)^{\frac{1}{p^+}}.$$

Lemma 3.1. *For each $\epsilon > 0$, there exists a function $u_\epsilon \in W^{1,p^+}_{w_0,w_1}(\mathbb{R})$ such that*

$$\begin{aligned}&\int_{\mathbb{R}} w_1(x) \left|u'_\epsilon\right|^{p(u_\epsilon)-2} u'_\epsilon v' dx + \int_{\mathbb{R}} w_0(x) \left|u_\epsilon\right|^{p(u_\epsilon)-2} u_\epsilon v dx \\ &\quad + \epsilon \left(\int_{\mathbb{R}} w_1(x) \left|u'_\epsilon\right|^{p^+-2} u'_\epsilon v' dx + \int_{\mathbb{R}} w_0(x) \left|u_\epsilon\right|^{p^+-2} u_\epsilon v dx \right) \\ &= \int_{\mathbb{R}} f(x, u_\epsilon) v dx, \ \forall\, v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}).\end{aligned}$$

Proof. Let $\epsilon > 0$ fixed. For $w : \mathbb{R} \to \mathbb{R}$ a measurable function, define the operator $A_w : W^{1,p^+}_{w_0,w_1}(\mathbb{R}) \to (W^{1,p^+}_{w_0,w_1}(\mathbb{R}))^*$ by

$$\begin{aligned}\langle A_w u, v\rangle = &\int_{\mathbb{R}} w_1 \left|u'\right|^{p(w)-2} u'v' dx + \int_{\mathbb{R}} w_0 \left|u\right|^{p(w)-2} uv dx \\ &+ \epsilon \left(\int_{\mathbb{R}} w_1 \left|u'\right|^{p^+-2} u'v' dx + \int_{\mathbb{R}} w_0 \left|u\right|^{p^+-2} uv dx \right), \qquad u, v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}).\end{aligned}$$

Observe that A_w is well defined. In fact, for $u, v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R})$, we have

$$\begin{aligned}&\left| \int_{\mathbb{R}} w_1 \left|u'\right|^{p(w)-2} u'v' dx + \int_{\mathbb{R}} w_0 \left|u\right|^{p(w)-2} uv dx \right| \\ &\quad \leq \int_{\mathbb{R}} w_1 \left|u'\right|^{p(w)-1} \left|v'\right| dx + \int_{\mathbb{R}} w_0 \left|u\right|^{p(w)-1} |v| \, dx \\ &\quad \leq \int_{\mathbb{R}} w_1 \left|v'\right| dx + \int_{\mathbb{R}} w_1 \left|u'\right|^{p^+-1} \left|v'\right| dx + \int_{\mathbb{R}} w_0 |v| \, dx + \int_{\mathbb{R}} w_0 |u|^{p^+-1} |v| \, dx \\ &\quad \leq |w_1|^{\frac{p^+-1}{p^+}}_{L^1(\mathbb{R})} \left(\int_{\mathbb{R}} w_1 \left|v'\right|^{p^+} dx \right)^{\frac{1}{p^+}} + \left(\int_{\mathbb{R}} w_1 \left|u'\right|^{p^+} dx \right)^{\frac{p^+-1}{p^+}} \left(\int_{\mathbb{R}} w_1 \left|v'\right|^{p^+} dx \right)^{\frac{1}{p^+}} \\ &\qquad + |w_0|^{\frac{p^+-1}{p^+}}_{L^1(\mathbb{R})} \left(\int_{\mathbb{R}} w_0 |v|^{p^+} dx \right)^{\frac{1}{p^+}} + \left(\int_{\mathbb{R}} w_0 |u|^{p^+} dx \right)^{\frac{p^+-1}{p^+}} \left(\int_{\mathbb{R}} w_0 |v|^{p^+} dx \right)^{\frac{1}{p^+}}.\end{aligned}$$

Hence, for u fixed in $W^{1,p^+}_{w_0,w_1}(\mathbb{R})$, the linear mapping $v \longmapsto \langle A_w u, v\rangle$ lies in the dual $(W^{1,p+}_{w_0,w_1}(\mathbb{R}))^*$. Clearly, A_w is coercive and continuous. Moreover, A_w is strictly monotone, i.e.

$$\langle A_w u_1 - A_w u_2, u_1 - u_2\rangle > 0, \qquad \forall\, u_1, u_2 \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}),\ u_1 \neq u_2.$$

On the other hand, for $w : \mathbb{R} \to \mathbb{R}$ measurable and $v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R})$, we have

$$\begin{aligned}\left| \int_{\mathbb{R}} f(x, w) v dx \right| &\leq \int_{w \geq 1} g(x) |v| \, dx + \int_{w \leq \alpha} g(x) \alpha^{p(x)-1} |v| \, dx + \int_{\alpha \leq w \leq 1} g(x) |w|^{p(x)-1} |v| \, dx \\ &\leq \int_{\mathbb{R}} g(x) |v| \, dx \\ &= \int_{\mathbb{R}} \frac{g(x)}{w_0^{\frac{1}{p^+}}} w_0^{\frac{1}{p^+}} |v| \, dx \\ &\leq \left(\int_{\mathbb{R}} \frac{(g(x))^{\frac{p^+}{p^+-1}}}{w_0^{\frac{1}{p^+-1}}} dx \right)^{\frac{p^+-1}{p^+}} \left(\int_{\mathbb{R}} w_0 |v|^{p^+} dx \right)^{\frac{1}{p^+}}.\end{aligned}$$

Thus, $(f(\cdot,w)) \in (W^{1,p^+}_{w_0,w_1}(\mathbb{R}))^*$. By the virtue of the Minty–Browder theorem (see [19, Theorem 26.A]), we deduce that there exists a unique element $u_w \in W^{1,p^+}_{w_0,w_1}(\mathbb{R})$ such that

$$A_w(u_w) = f(\cdot,w) \quad \text{in } (W^{1,p^+}_{w_0,w_1}(\mathbb{R}))^*.$$

That is

$$\langle A_w u_w, v\rangle = \int_{\mathbb{R}} f(x,w)v dx, \qquad \forall\, v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}). \tag{3.1}$$

Taking $v = u_w$ in (3.1), we infer

$$\int_{\mathbb{R}} w_1(x)\left|u_w'\right|^{p(w)} dx + \int_{\mathbb{R}} w_0(x)\left|u_w\right|^{p(w)} dx + \epsilon\left(\int_{\mathbb{R}} w_1(x)\left|u_w'\right|^{p^+} dx + \int_{\mathbb{R}} w_0(x)\left|u_w\right|^{p^+} dx\right)$$
$$= \int_{\mathbb{R}} f(x,w)u_w dx \leq \int_{\mathbb{R}} g(x)\left|u_w\right| dx.$$

Thus,

$$\epsilon\left\|u_w\right\|^{p^+}_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})} \leq \left(\int_{\mathbb{R}} \frac{g(x)^{\frac{p^+}{p^+-1}}}{w_0^{\frac{1}{p^+-1}}} dx\right)^{\frac{p^+-1}{p^+}} \left\|u_w\right\|_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})}.$$

Consequently, there exists a constant C_ϵ depending on ϵ but independent of w such that

$$\left\|u_w\right\|_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})} \leq C_\epsilon. \tag{3.2}$$

Now, we claim that $W^{1,p^+}_{w_0,w_1}(\mathbb{R})$ is compactly embedded in the weighted Lebesgue space

$$L^{p^-}_{w_0}(\mathbb{R}) = \left\{u : \mathbb{R} \to \mathbb{R} \text{ measurable}, \int_{\mathbb{R}} w_0(x)\left|u\right|^{p^-} dx < +\infty\right\}$$

equipped with the norm $u \longmapsto |u|_{L^{p^-}_{w_0}(\mathbb{R})} = \left(\int_{\mathbb{R}} w_0(x)\left|u\right|^{p^-} dx\right)^{\frac{1}{p^-}}$. For that aim, take a sequence $(u_n)_n \subset W^{1,p^+}_{w_0,w_1}(\mathbb{R})$ such that $u_n \rightharpoonup 0$ weakly in $W^{1,p^+}_{w_0,w_1}(\mathbb{R})$. We claim that, up to a subsequence $u_n \to 0$ strongly in $L^{p^-}_{w_0}(\mathbb{R})$. We have,

$$\int_{\mathbb{R}} w_0(x)\left|u_n\right|^{p^-} dx = \int_{\mathbb{R}} (w_0(x))^{1-\frac{p^-}{p^+}} (w_0(x))^{\frac{p^-}{p^+}} \left|u_n\right|^{p^-} dx. \tag{3.3}$$

Observing that the sequence $(w_0^{\frac{p^-}{p^+}}\left|u_n\right|^{p^-})_n$ is bounded in $L^{\frac{p^+}{p^-}}(\mathbb{R})$ and, up to a subsequence, is weakly convergent to 0 in $L^{\frac{p^+}{p^-}}(\mathbb{R})$, and that $w_0^{1-\frac{p^-}{p^+}} \in L^{\frac{p^+}{p^+-p^-}}(\mathbb{R})$, we can immediately see from (3.3) that

$$\int_{\mathbb{R}} w_0(x)\left|u_n\right|^{p^-} dx \to 0, \qquad n \to +\infty.$$

Let $C_1 > 0$ be a positive constant such that

$$|u|_{L^{p^-}_{w_0}(\mathbb{R})} \leq C_1\left\|u\right\|_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})}, \qquad \forall\, u \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}). \tag{3.4}$$

Set $\widetilde{C_\epsilon} = C_1 C_\epsilon$, and $\mathcal{K} = \{w \in L^{p^-}_{w_0}(\mathbb{R}),\ |w|_{L^{p^-}_{w_0}(\mathbb{R})} \leq \widetilde{C_\epsilon}\}$ the closed ball of $L^{p^-}_{w_0}(\mathbb{R})$ centered at the origin and of radius $\widetilde{C_\epsilon}$. Define the mapping $T : \mathcal{K} \to \mathcal{K}$ by $Tw = u_w$ given by (3.1). In view of (3.2) and (3.4), it yields that $T(\mathcal{K}) \subset \mathcal{K}$. Moreover, since $W^{1,p^-}_{w_0,w_1}(\mathbb{R})$ is compactly embedded in $L^{p^-}_{w_0}(\mathbb{R})$, we can easily show that $T(\mathcal{K})$ is relatively compact. Observing that T is continuous, then one can use the Schauder's fixed point Theorem (see [18, Theorem 2.A]) to deduce the existence of $\widetilde{w} \in \mathcal{K}$ such that $u_{\widetilde{w}} = \widetilde{w}$. Consequently,

$$\begin{aligned} &\int_{\mathbb{R}} w_1(x) \left|u'_{\widetilde{w}}\right|^{p(u_{\widetilde{w}})-2} u'_{\widetilde{w}} v' dx + \int_{\mathbb{R}} w_0(x) \left|u_{\widetilde{w}}\right|^{p(u_{\widetilde{w}})-2} u_{\widetilde{w}} v dx \\ &\quad + \epsilon \left(\int_{\mathbb{R}} w_1(x) \left|u'_{\widetilde{w}}\right|^{p^+-2} u'_{\widetilde{w}} v' dx + \int_{\mathbb{R}} w_0(x) \left|u_{\widetilde{w}}\right|^{p^+-2} u_{\widetilde{w}} v dx \right) \\ &= \int_{\mathbb{R}} f(x, u_{\widetilde{w}}) v dx, \qquad \forall\, v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}). \end{aligned}$$

This concludes the proof of Lemma 3.1. □

The completion of the proof of Theorem 1.2

Choosing $\epsilon = \frac{1}{n}$, $n \geq 1$, in Lemma 3.1, we deduce that there exists $u_n \in W^{1,p^+}_{w_0,w_1}(\mathbb{R})$ such that

$$\begin{aligned} &\int_{\mathbb{R}} w_1(x) \left|u'_n\right|^{p(u_n)-2} u'_n v' dx + \int_{\mathbb{R}} w_0(x) \left|u_n\right|^{p(u_n)-2} u_n v dx \\ &\quad + \frac{1}{n} \left(\int_{\mathbb{R}} w_1(x) \left|u'_n\right|^{p^+-2} u'_n v' dx + \int_{\mathbb{R}} w_0(x) \left|u_n\right|^{p^+-2} u_n v dx \right) \\ &= \int_{\mathbb{R}} f(x, u_n) v dx, \qquad \forall\, v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}). \end{aligned} \tag{3.5}$$

Taking $v = u_n$ as test function in (3.5), we get

$$\begin{aligned} &\int_{\mathbb{R}} w_1(x) \left|u'_n\right|^{p(u_n)} dx + \int_{\mathbb{R}} w_0(x) \left|u_n\right|^{p(u_n)} dx + \frac{1}{n} \|u_n\|^{p^+}_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})} \\ &= \int_{\mathbb{R}} f(x, u_n) u_n dx, \qquad \forall\, n \geq 1. \end{aligned} \tag{3.6}$$

We have

$$\begin{aligned} \left| \int_{\mathbb{R}} f(x, u_n) u_n dx \right| &\leq \int_{\mathbb{R}^N} g(x) \left|u_n\right| dx \\ &\leq \left(\int_{\mathbb{R}} \left(\frac{g(x)}{(w_0(x))^{\frac{1}{p^-}}} \right)^{\frac{p^-}{p^- -1}} dx \right)^{\frac{p^- -1}{p^-}} \left(\int_{\mathbb{R}} w_0(x) \left|u_n\right|^{p^-} dx \right)^{\frac{1}{p^-}} \\ &\leq c_2 \left(\int_{|u_n| \leq 1} w_0(x) \left|u_n\right|^{p^-} dx + \int_{|u_n| \geq 1} w_0(x) \left|u_n\right|^{p^-} dx \right)^{\frac{1}{p^-}} \\ &\leq c_2 \left(\int_{\mathbb{R}} w_0(x) \left|u_n\right|^{p(u_n)} dx + |w_0|_{L^1(\mathbb{R})} \right)^{\frac{1}{p^-}}. \end{aligned}$$

By (3.6), we infer

$$\int_{\mathbb{R}} w_1(x) \left|u'_n\right|^{p(u_n)} dx + \int_{\mathbb{R}} w_0(x) \left|u_n\right|^{p(u_n)} dx + \frac{1}{n} \|u_n\|^{p^+}_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})} \leq c_3, \qquad \forall\, n \geq 1. \tag{3.7}$$

In particular, there exists a positive constant $c_4 > 0$ (independent of n) such that

$$\int_{\mathbb{R}} w_1(x) \left|u'_n\right|^{p^-} dx + \int_{\mathbb{R}} w_0(x) \left|u_n\right|^{p^-} dx \leq c_4, \qquad \forall\, n \geq 1. \tag{3.8}$$

Indeed,

$$\begin{aligned}\int_{\mathbb{R}} w_1(x)\left|u_n'\right|^{p^-}dx &= \int_{|u_n'|\geq 1} w_1(x)\left|u_n'\right|^{p^-}dx + \int_{|u_n'|<1} w_1(x)\left|u_n'\right|^{p^-}dx\\ &\leq \int_{|u_n'|\geq 1} w_1(x)\left|u_n'\right|^{p(u_n)}dx + |w_1|_{L^1(\mathbb{R})}\\ &\leq \int_{\mathbb{R}} w_1(x)\left|u_n'\right|^{p(u_n)}dx + |w_1|_{L^1(\mathbb{R})}, \qquad \forall\, n\geq 1.\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}} w_0(x)\left|u_n\right|^{p^-}dx \leq \int_{\mathbb{R}} w_0(x)\left|u_n\right|^{p(u_n)}dx + |w_0|_{L^1(\mathbb{R})}, \qquad \forall\, n\geq 1.$$

Hence, (3.8) immediately follows from (3.7). By the reflexivity of the weighted Sobolev space $W^{1,p^-}_{w_0,w_1}(\mathbb{R})$, there exists $u \in W^{1,p^-}_{w_0,w_1}(\mathbb{R})$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $W^{1,p^-}_{w_0,w_1}(\mathbb{R})$ and $u_n(x) \to u(x)$ a.e. $x\in\mathbb{R}$. Now, we claim that

$$\int_{\mathbb{R}} w_1(x)\left|u'\right|^{p(u)}dx + \int_{\mathbb{R}} w_0(x)\left|u\right|^{p(u)}dx < +\infty. \tag{3.9}$$

For that aim, for $x\in\mathbb{R}$ and $n\geq 1$, set $q_n(x) = p(u_n(x))$ and $q(x) = p(u(x))$. For $k>0$, set

$$v_k = \begin{cases} qu\left|u\right|^{q-2}, & \text{if } |u|\leq k,\\ qk^{q-1}\frac{u}{|u|}, & \text{if } |u|>k.\end{cases}$$

By Young's inequality, it yields

$$u_n v_k \leq |u_n|^{q_n} + \frac{q_n-1}{q_n^{q_n'}}\left|v_k\right|^{q_n'}, \qquad \forall\, k>0,\ \forall\, n\geq 1,$$

where $q_n' = \frac{q_n}{q_n-1}$. Thus,

$$\int_{\mathbb{R}} w_0(x)u_n v_k dx \leq \int_{\mathbb{R}} w_0(x)\left|u_n\right|^{q_n}dx + \int_{\mathbb{R}} w_0(x)\frac{q_n-1}{q_n^{q_n'}}\left|v_k\right|^{q_n'}dx, \qquad \forall\, k>0,\ \forall\, n\geq 1.$$

Tending n to $+\infty$ and having (3.7) in mind, we get

$$\int_{\mathbb{R}} w_0(x)uv_k dx \leq c_3 + \int_{\mathbb{R}} w_0(x)\frac{q-1}{q^{q'}}\left|v_k\right|^{q'}dx.$$

Consequently,

$$\begin{aligned}&\int_{|u|\leq k} w_0(x)q\left|u\right|^{q}dx + \int_{|u|>k} w_0(x)qk^{q-1}\left|u\right|dx\\ &\qquad\leq c_3 + \int_{|u|\leq k} w_0(x)(q-1)\left|u\right|^{q}dx + \int_{|u|>k} w_0(x)(q-1)k^{q}dx.\end{aligned}$$

Thus,

$$\int_{|u|\leq k} w_0(x)\left|u\right|^{q}dx + \int_{|u|>k} w_0(x)k^{q}dx \leq c_3.$$

Passing to the limit as k tends to $+\infty$ in that last inequality, we obtain

$$\int_{\mathbb{R}} w_0(x)\left|u\right|^{q}dx \leq c_3.$$

Similarly, for $k > 0$, one can choose the function

$$\widetilde{v_k} = \begin{cases} qu' \left|u'\right|^{q-2}, & \text{if } |u'| \leq k, \\ qk^{q-1}\frac{u'}{|u'|}, & \text{if } |u'| > k, \end{cases}$$

Using Young's inequality and proceeding exactly as previously, we can easily show that

$$\int_{|u'|\leq k} w_1(x) \left|u'\right|^q dx \leq c_3, \qquad \forall\, k > 0,$$

and after passing to the limit as k tends to $+\infty$, we finally obtain

$$\int_{\mathbb{R}} w_1(x) \left|u'\right|^q dx \leq c_3.$$

Hence, the claim (3.9) holds. Let $v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R})$. We have

$$\begin{aligned}
&\int_{\mathbb{R}} w_1(x) \left(\left|u_n'\right|^{p(u_n)-2} u_n' - \left|v'\right|^{p(u_n)-2} v' \right) (u_n' - v')dx \\
&\qquad + \int_{\mathbb{R}} w_0(x) \left(|u_n|^{p(u_n)-2} u_n - |v|^{p(u_n)-2} v \right) (u_n - v)dx \\
&\qquad + \frac{1}{n} \int_{\mathbb{R}} w_1(x) \left(\left|u_n'\right|^{p^+-2} u_n' - \left|v'\right|^{p^+-2} v' \right) (u_n' - v')dx \\
&\qquad + \frac{1}{n} \int_{\mathbb{R}} w_0(x) \left(|u_n|^{p^+-2} u_n - |v|^{p^+-2} v \right) (u_n - v)dx \\
&\quad \geq 0, \qquad \forall\, n \geq 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}} w_1(x) \left(\left|u_n'\right|^{p(u_n)-2} u_n' + \frac{1}{n} \left|u_n'\right|^{p^+-2} u_n' \right) (u_n' - v')dx \\
&\qquad + \int_{\mathbb{R}} w_0(x) \left(|u_n|^{p(u_n)-2} u_n + \frac{1}{n} |u_n|^{p^+-2} u_n \right) (u_n - v)dx \\
&\quad \geq \int_{\mathbb{R}} w_1(x) \left(\left|v'\right|^{p(u_n)-2} v' + \frac{1}{n} \left|v'\right|^{p^+-2} v' \right) (u_n' - v')dx \\
&\qquad + \int_{\mathbb{R}} w_0(x) \left(|v|^{p(u_n)-2} v + \frac{1}{n} |v|^{p^+-2} v \right) (u_n - v)dx.
\end{aligned}$$

By (3.5), it follows

$$\begin{aligned}
\int_{\mathbb{R}} f(x,u_n)(u_n - v)dx \geq & \int_{\mathbb{R}} w_1(x) \left(\left|v'\right|^{p(u_n)-2} v' + \frac{1}{n} \left|v'\right|^{p^+-2} v' \right) (u_n' - v')dx \\
& + \int_{\mathbb{R}} w_0(x) \left(|v|^{p(u_n)-2} v + \frac{1}{n} |v|^{p^+-2} v \right) (u_n - v)dx.
\end{aligned} \tag{3.10}$$

We have

$$\begin{aligned}
&\left|\frac{1}{n}\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p^+-2} v'(u_n'-v')dx\right| \\
&\quad\le \frac{1}{n}\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p^+-1}\left|(u_n'-v')\right|dx \\
&\quad\le \frac{1}{n}\left(\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p^+}dx\right)^{\frac{p^+-1}{p^+}}\left(\int_{\mathbb{R}} w_1(x)\left|(u_n'-v')\right|^{p^+}dx\right)^{\frac{1}{p^+}} \\
&\quad= \left(\frac{1}{n}\right)^{\frac{p^+-1}{p^+}}\left(\frac{1}{n}\right)^{\frac{1}{p^+}}\left(\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p^+}dx\right)^{\frac{p^+-1}{p^+}}\left(\int_{\mathbb{R}} w_1(x)\left|(u_n'-v')\right|^{p^+}dx\right)^{\frac{1}{p^+}} \\
&\quad\le \left(\frac{1}{n}\right)^{\frac{p^+-1}{p^+}}\left(\frac{1}{n}\left\|u_n-v\right\|^{p^+}_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})}\right)^{\frac{1}{p^+}}\left\|v\right\|^{p^+-1}_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})}.
\end{aligned} \tag{3.11}$$

By (3.7), we know that

$$\sup_{n\ge 1}\left(\frac{1}{n}\left\|u_n\right\|^{p^+}_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})}\right) < +\infty.$$

Then, from (3.11), we deduce that

$$\frac{1}{n}\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p^+-2} v'(u_n'-v')dx \to 0, \qquad n\to+\infty. \tag{3.12}$$

Similarly,

$$\frac{1}{n}\int_{\mathbb{R}} w_0(x)\left|v\right|^{p^+-2} v(u_n-v)dx \to 0, \qquad n\to+\infty. \tag{3.13}$$

We claim that

$$\int_{\mathbb{R}} f(x,u_n)(u_n-v)dx \to \int_{\mathbb{R}} f(x,u)(u-v)dx, \qquad n\to+\infty. \tag{3.14}$$

First, note that $f(x,u_n(x))(u_n(x)-v(x)) \to f(x,u(x))(u(x)-v(x))$, a.e. $x\in\mathbb{R}$. Second, by (H_2), it yields

$$\left|f(x,u_n)(u_n-v)\right| \le g(x)\left|u_n-v\right| = \frac{g(x)}{(w_0(x))^{\frac{1}{p^-}}}(w_0(x))^{\frac{1}{p^-}}\left|u_n-v\right|, \qquad \forall\, n\ge 1.$$

Having in mind that $(w_0(x))^{\frac{1}{p^-}}\left|u_n-v\right| \rightharpoonup (w_0(x))^{\frac{1}{p^-}}\left|u-v\right|$ weakly in $L^{p^-}(\mathbb{R})$ and that $g/w_0^{\frac{1}{p^-}}$ belongs to the dual of $L^{p^-}(\mathbb{R})$, it follows that

$$\int_{\mathbb{R}} g(x)\left|u_n-v\right|dx \to \int_{\mathbb{R}} g(x)\left|u-v\right|dx, \qquad n\to+\infty,$$

which implies that $g(x)\left|u_n-v\right| \to g(x)\left|u-v\right|$ strongly in $L^1(\mathbb{R})$ and by consequence, there exists $g_1\in L^1(\mathbb{R})$ such that, up to a subsequence,

$$g(x)\left|u_n-v\right| \le g_1(x), \quad \text{a.e. } x\in\mathbb{R},\ \forall\, n\ge 1. \tag{3.15}$$

Using (3.15), we can easily apply the Lebesgue's dominated convergence theorem to deduce (3.14). In view of (3.12), (3.13), and (3.14), from (3.10) we get that

$$\begin{aligned}
\int_{\mathbb{R}} f(x,u)(u-v)dx \ge{}& \lim_{n\to+\infty}\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u_n)-2} v'(u_n'-v')dx \\
&+ \lim_{n\to+\infty}\int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u_n)-2} v(u_n-v)dx.
\end{aligned} \tag{3.16}$$

Now, observe that

$$\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u_n)-2} v'(u_n'-v')dx = \int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-2} v'(u_n'-v')dx + \int_{\mathbb{R}} w_1(x)\left(\left|v'\right|^{p(u_n)-2} v' - \left|v'\right|^{p(u)-2} v'\right)(u_n'-v')dx, \tag{3.17}$$

and

$$\int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u_n)-2} v(u_n-v)dx = \int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u)-2} v(u_n-v)dx + \int_{\mathbb{R}} w_0(x)\left(\left|v\right|^{p(u_n)-2} v - \left|v\right|^{p(u)-2} v\right)(u_n-v)dx. \tag{3.18}$$

Next, we introduce the functional subspace Z of $W^{1,p^+}_{w_0,w_1}(\mathbb{R})$ defined by

$$Z = \left\{ v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}),\ \int_{\mathbb{R}} w_1(x)\left|v'\right|^{\frac{p^-(p^+-1)}{p^--1}} dx < +\infty,\ \int_{\mathbb{R}} w_0(x)\left|v\right|^{\frac{p^-(p^+-1)}{p^--1}} dx < +\infty \right\}.$$

For $v \in Z$ and $w \in W^{1,p^-}_{w_0,w_1}(\mathbb{R})$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-2} v'w'dx\right| \\
&\le \int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-1}\left|w'\right| dx \\
&\le \left(\int_{\mathbb{R}} w_1(x)\left|v'\right|^{\frac{p^-(p(u)-1)}{p^--1}} dx\right)^{\frac{p^--1}{p^-}} \left(\int_{\mathbb{R}} w_1(x)\left|w'\right|^{p^-} dx\right)^{\frac{1}{p^-}} \\
&\le \left(|w_1|_{L^1(\mathbb{R})} + \int_{|v'|\ge 1} w_1(x)\left|v'\right|^{\frac{p^-(p^+-1)}{p^--1}} dx\right)^{\frac{p^--1}{p^-}} \left(\int_{\mathbb{R}} w_1(x)\left|w'\right|^{p^-} dx\right)^{\frac{1}{p^-}} \\
&\le \left(|w_1|_{L^1(\mathbb{R})} + \int_{\mathbb{R}} w_1(x)\left|v'\right|^{\frac{p^-(p^+-1)}{p^--1}} dx\right)^{\frac{p^--1}{p^-}} \|w\|_{W^{1,p^-}_{w_0,w_1}(\mathbb{R})},
\end{aligned}$$

where

$$W^{1,p^-}_{w_0,w_1}(\mathbb{R}) = \left\{ u \in L^1_{loc}(\mathbb{R}),\ \int_{\mathbb{R}} w_0(x)\left|u\right|^{p^-} dx < +\infty,\ \int_{\mathbb{R}} w_1(x)\left|u'\right|^{p^-} dx < +\infty \right\},$$

equipped with the norm

$$\|u\|_{W^{1,p^-}_{w_0,w_1}(\mathbb{R})} = \left(\int_{\mathbb{R}} \left(w_1(x)\left|u'\right|^{p^-} + w_0(x)\left|u\right|^{p^-}\right) dx\right)^{\frac{1}{p^-}}.$$

Thus, the functional

$$w \longmapsto \int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-2} v'w'dx$$

belongs to the dual of $W^{1,p^-}_{w_0,w_1}(\mathbb{R})$. The same result holds for the functional

$$w \longmapsto \int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u)-2} vwdx.$$

Since $(u_n - v) \rightharpoonup (u - v)$ weakly in $W^{1,p^-}_{w_0,w_1}(\mathbb{R})$, then

$$\begin{aligned}\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-2} v'(u_n' - v')dx + \int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u)-2} v(u_n - v)dx \\ \to \int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-2} v'(u' - v')dx + \int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u)-2} v(u - v)dx.\end{aligned} \tag{3.19}$$

Furthermore, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}} w_1(x)\left(\left|v'\right|^{p(u_n)-2} v' - \left|v'\right|^{p(u)-2} v'\right)(u_n' - v')dx\right| \\ &\leq \left(\int_{\mathbb{R}} w_1(x)\left|\,\left|v'\right|^{p(u_n)-2} v' - \left|v'\right|^{p(u)-2} v'\right|^{\frac{p^-}{p^- -1}} dx\right)^{\frac{p^- -1}{p^-}} \\ &\quad\times \left(\int_{\mathbb{R}} w_1(x)\left|(u_n' - v')\right|^{p^-} dx\right)^{\frac{1}{p^-}} \\ &\leq c_5 \left(\int_{\mathbb{R}} w_1(x)\left|\,\left|v'\right|^{p(u_n)-2} v' - \left|v'\right|^{p(u)-2} v'\right|^{\frac{p^-}{p^- -1}} dx\right)^{\frac{p^- -1}{p^-}}.\end{aligned} \tag{3.20}$$

Observe that

$$\begin{aligned}&w_1(x)\left|\,\left|v'\right|^{p(u_n)-2} v' - \left|v'\right|^{p(u)-2} v'\right|^{\frac{p^-}{p^- -1}} \\ &\leq w_1(x) 2^{\frac{p^-}{p^- -1}} \left|v'\right|^{\frac{p^-(p^+-1)}{p^- -1}} \mathbb{1}_{\{|v'|\geq 1\}} + w_1(x) 2^{\frac{p^-}{p^- -1}} \mathbb{1}_{\{|v'|\leq 1\}} \\ &\leq w_1(x) 2^{\frac{p^-}{p^- -1}} \left(1 + \left|v'\right|^{\frac{p^-(p^+-1)}{p^- -1}}\right).\end{aligned}$$

Taking into account that, for a.e. $x \in \mathbb{R}$, $p(u_n(x)) \to p(u(x))$ as $n \to +\infty$, then we can apply the Lebesgue's dominated convergence theorem to get

$$\int_{\mathbb{R}} w_1(x)\left|\,\left|v'\right|^{p(u_n)-2} v' - \left|v'\right|^{p(u)-2} v'\right|^{\frac{p^-}{p^- -1}} dx \to 0, \qquad n \to +\infty.$$

By (3.20), it follows

$$\int_{\mathbb{R}} w_1(x)\left(\left|v'\right|^{p(u_n)-2} v' - \left|v'\right|^{p(u)-2} v'\right)(u_n' - v')dx \to 0, \qquad n \to +\infty. \tag{3.21}$$

In a similar way, we have

$$\int_{\mathbb{R}} w_0(x)\left(\left|v\right|^{p(u_n)-2} v - \left|v\right|^{p(u)-2} v\right)(u_n - v)dx \to 0, \qquad n \to +\infty. \tag{3.22}$$

Combining (3.21), (3.22), (3.17) and (3.18), we deduce that

$$\begin{aligned}\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u_n)-2} v'(u_n' - v')dx + \int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u_n)-2} v(u_n - v)dx \\ \to \int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-2} v'(u' - v')dx + \int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u)-2} v(u - v)dx.\end{aligned} \tag{3.23}$$

Inserting (3.23) in (3.16), we infer

$$\begin{aligned}\int_{\mathbb{R}} f(x,u)(u - v)dx \geq \int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-2} v'(u' - v')dx \\ + \int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u)-2} v(u - v)dx, \qquad \forall\, v \in Z.\end{aligned} \tag{3.24}$$

In particular,

$$\begin{aligned}\int_{\mathbb{R}} f(x,u)(u-v)dx \geq \int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-2} v'(u-v)'dx \\ + \int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u)-2} v(u-v)dx, \qquad \forall\, v \in C_0^{\infty}(\mathbb{R}).\end{aligned} \tag{3.25}$$

Next, observe that the function $w \longmapsto \int_{\mathbb{R}} w_0(x)\,|u-w|^{p^-}$ is continuous on $(E_u,\ \|\cdot\|_{E_u})$. Taking into account that

$$\int_{\mathbb{R}} g(x)\,|u-w|\,dx \leq \left(\int_{\mathbb{R}}\left(\frac{g(x)}{(w_0(x))^{\frac{1}{p^-}}}\right)^{\frac{p^-}{p^--1}} dx\right)^{\frac{p^--1}{p^-}} \left(\int_{\mathbb{R}} w_0(x)\,|u-w|^{p^-}\,dx\right)^{\frac{1}{p^-}}, \quad \forall\, w \in E_u,$$

then by (H_2), we can deduce that the function

$$w \longmapsto \int_{\mathbb{R}} f(x,u)(u-w)dx$$

is continuous on $(E_u,\ \|\cdot\|_{E_u})$. Using that fact together with Proposition 2.1, we can immediately see that the inequality (3.25) can be extended to the whole space E_u, i.e.

$$\begin{aligned}\int_{\mathbb{R}} f(x,u)(u-v)dx \geq \int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-2} v'(u'-v')dx \\ + \int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u)-2} v(u-v)dx, \qquad \forall\, v \in E_u.\end{aligned} \tag{3.26}$$

For $s>0$ and $w \in E_u$, choosing $v = u - sw$ as test function in (3.26), it yields

$$\begin{aligned}s\int_{\mathbb{R}} f(x,u)wdx \geq s\int_{\mathbb{R}} w_1(x)\left|u'-sw'\right|^{p(u)-2}(u'-sw')w'dx \\ + s\int_{\mathbb{R}} w_0(x)\left|u-sw\right|^{p(u)-2}(u-sw)wdx.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}} f(x,u)wdx - \int_{\mathbb{R}} w_1(x)\left|u'-sw'\right|^{p(u)-2}(u'-sw')w'dx \\ - \int_{\mathbb{R}} w_0(x)\left|u-sw\right|^{p(u)-2}(u-sw)wdx \geq 0.\end{aligned}$$

Tending s to 0^+ in that last inequality, we obtain

$$\int_{\mathbb{R}} f(x,u)wdx \geq \int_{\mathbb{R}} w_1(x)\left|u'\right|^{p(u)-2} u'w'dx + \int_{\mathbb{R}} w_0(x)\left|u\right|^{p(u)-2} uwdx.$$

Clearly, the same inequality holds for $(-w)$. Therefore,

$$\int_{\mathbb{R}} w_1(x)\left|u'\right|^{p(u)-2} u'w'dx + \int_{\mathbb{R}} w_0(x)\left|u\right|^{p(u)-2} uwdx = \int_{\mathbb{R}} f(x,u)wdx, \qquad \forall\, w \in E_u. \tag{3.27}$$

In order to conclude the proof of Theorem 1.2, we need to prove that $f(x,u(x)) = g(x)\,|u|^{p(x)-2}\,u(x)$ a.e. $x \in \mathbb{R}$. For that aim, we start by taking $w = (u-1)^+$ as test function in (3.27):

$$\int_{\mathbb{R}} w_1(x)\left|u'\right|^{p(u)-2} u'((u-1)^+)'dx + \int_{\mathbb{R}} w_0(x)\left|u\right|^{p(u)-2} u(u-1)^+dx = \int_{\mathbb{R}} f(x,u)(u-1)^+dx.$$

Since

$$\int_{\mathbb{R}} w_1(x) \left|u'\right|^{p(u)-2} u'((u-1)^+)'dx = \int_{u\geq 1} w_1(x) \left|u'\right|^{p(u)} dx \geq 0,$$

by (H_3) we get

$$\begin{aligned}\int_{\mathbb{R}} w_0(x) \left|u\right|^{p(u)-2} u(u-1)^+dx &\leq \int_{\mathbb{R}} f(x,u)(u-1)^+dx \\ &= \int_{u\geq 1} g(x)(u-1)^+dx \leq \int_{u\geq 1} w_0(x)(u-1)^+dx.\end{aligned}$$

Thus,

$$\int_{u\geq 1} w_0(x) \left(|u|^{p(u)-2}u - 1\right)(u-1)^+dx \leq 0.$$

We immediately deduce that $u(x) \leq 1$ a.e. $x \in \mathbb{R}$. On the other hand, it is easy to see that $u \geq 0$. In fact, taking $w = u^- = \min(u, 0)$ as test function in (3.27), we have

$$\int_{\mathbb{R}} w_1(x) \left|(u^-)'\right|^{p(u)} dx + \int_{\mathbb{R}} w_0(x) \left|u^-\right|^{p(u)} dx = \int_{\mathbb{R}} f(x,u)u^- dx \leq 0,$$

which immediately implies that $u^- = 0$ and by consequence $u(x) \geq 0$ a.e. $x \in \mathbb{R}$. Now, taking $w = (\alpha - u)^+$ as test function in (3.27) and having in mind that

$$\int_{\mathbb{R}} w_1(x) \left|u'\right|^{p(u)-2} u((\alpha-u)^+)'dx = -\int_{\alpha\geq u} w_1(x) \left|u'\right|^{p(u)} dx \leq 0,$$

by (H_3) it yields

$$\begin{aligned}\int_{\mathbb{R}} w_0(x) u^{p(u)-1}(\alpha-u)^+dx &\geq \int_{\mathbb{R}} f(x,u)(\alpha-u)^+dx \\ &= \int_{\alpha\geq u} g(x)\alpha^{p(x)-1}(\alpha-u)^+dx \\ &\geq \int_{\alpha\geq u} w_0(x)\alpha^{p(\alpha)-1}(\alpha-u)^+dx.\end{aligned}$$

Hence,

$$\int_{\alpha\geq u} w_0(x) \left(\alpha^{p(\alpha)-1} - u^{p(u)-1}\right)(\alpha-u)^+dx \leq 0. \tag{3.28}$$

In view of (H_1), we know that $\alpha^{p(\alpha)-1} \geq u^{p(u)-1}$ on the set $\{x \in \mathbb{R},\ \alpha \geq u(x)\}$. From (3.28), we deduce that $(\alpha - u)^+ = 0$ and by consequence $u(x) \geq \alpha$ a.e. $x \in \mathbb{R}$. Finally, we conclude that $u \neq 0$ and $f(x, u(x)) = g(x)u^{p(x)-1}$ a.e. $x \in \mathbb{R}$. This ends the proof of Theorem 1.2. □

References

[1] B. Andreianov, M. Bendahmane, S. Ouaro, Structural stability for variable exponent elliptic problems II. The $p(u)$-Laplacian and coupled problems, *Nonlinear Anal.* **72**(2010) 4649–4660. https://doi.org/10.1016/j.na.2010.02.044; MR2639212; Zbl 1188.35068

[2] S. Antontsev, S. Shmarev, On a class of nonlocal evolutions equations with the $p[u(x,t)]$-Laplace operator, *Nonlinear Anal. Real World Appl.* **56**(2020) 103165. https://doi.org/10.1016/j.nonrwa.2020.103165; MR4107095; Zbl 1451.35080

[3] S. Antontsev, S. Shmarev, Nonlocal evolution equations with $p[u(x,t)]$-Laplacian and lower-order terms, *J. Elliptic Parabol. Equ.* **6**(2020), 211–237. https://doi.org/10.1007/s41808-020-00065-x; MR4114350; Zbl 1443.35078

[4] S. Antontsev, I. Kuznetsov, S. Shmarev, On a class of nonlocal evolution equations with the $p[\nabla u]$-Laplace operator, *J. Math. Anal. Appl.* **501**(2021), No. 2, Paper No. 125221, 26 pp. https://doi.org/10.1016/j.jmaa.2021.125221; MR4241625; Zbl 1468.35089

[5] S. Aouaoui, On Some Nonhomogeneous degenerate quasilinear equations arising in the continuum mechanics, *Comm. Appl. Nonlinear Anal.* **20**(2013), 87–108. MR3154460; Zbl 1298.35087

[6] S. Aouaoui, On some one-dimensional eigenvalue problem involving variable exponents, *Comm. Appl. Nonlinear Anal.* **21**(2014), 77–86. MR3202449; Zbl 1309.34019

[7] S. Aouaoui, A. E. Bahrouni, On some equation defined on the whole Euclidean space $\mathbb{R}^N$ and involving the $p(u)$-Laplacian, *Math. Meth. Appl. Sci.*, published online, 2021. https://doi.org/10.1002/mma.7291

[8] P. Blomgren, T. F. Chan, P. Mulet, L. Vese, W. L. Wan, Variational PDE models and methods for image processing, in: D. F. Griffiths, G. A. Watson (Eds.), *Numerical analysis 1999. Proceedings of the 18th Biennial Conference held at the University of Dundee, Dundee, June 29–July 2, 1999*, Chapman & Hall/CRC Research Notes in Mathematics, Vol. 420, Chapman & Hall/CRC, Boca Raton, FL, 2000, pp. 43–67. MR1751108; Zbl 0953.68621

[9] E. M. Bollt, R. Chartrand, S. Esedoḡlu, P. Schultz, K. R. Vixie, Graduated adaptive image denoising: local compromise between total variation and isotropic diffusion, *Adv. Comput. Math.* **31**(2009), 61–85. https://doi.org/10.1007/s10444-008-9082-7; MR2511574; Zbl 1169.94302

[10] Y. Chen, S. Levin, M. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM J. Appl. Math.* **66**(2006), 1383–1406. https://doi.org/10.1137/050624522; MR2246061; Zbl 1102.49010

[11] M. Chipot, H. B. de Oliveira, Some results on the $p(u)$-Laplacian problem, *Math. Ann.* **375**(2019), 283–306. https://doi.org/10.1007/s00208-019-01859-8; MR4000242; Zbl 1430.35106

[12] M. Chipot, H. B. de Oliveira, Correction to: Some results on the $p(u)$-Laplacian problem, *Math. Ann.* **375**(2019), 307–313. https://doi.org/10.1007/s00208-019-01859-8; MR4000243; Zbl 07104244

[13] D. E. Edmunds, J. Rákosník, Density of smooth functions in $W^{k,p(x)}(\Omega)$, *Proc. R. Soc. London Ser. A* **437**(1992), 229–236. https://doi.org/10.1098/rspa.1992.0059; MR1177754 ; Zbl 0779.46027

[14] S. Ouaro, N. Sawadogo, Structural stability for nonlinear Neumann boundary $p(u)$-Laplacian problem, *Discuss. Math. Differ. Incl. Control Optim.* **39**(2019), 81–117. https://doi.org/10.7151/dmdico.1209; MR4285511

[15] V. Rădulescu, Nonlinear elliptic equations with variable exponent: Old and new, *Nonlinear Anal.* **121**(2015), 336–369. https://doi.org/10.1016/j.na.2014.11.007; MR3348928; Zbl 1321.35030

[16] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, Vol. 1748, Springer-Verlag, Berlin, 2000. https://doi.org/10.1007/BFb0104029; MR1810360

[17] J. Tiirola, Image denoising using directional adaptive variable exponents model, *J. Math. Imaging Vision* **57**(2017), 56–74. https://doi.org/10.1007/s10851-016-0666-4; MR3592843; Zbl 1384.94009

[18] E. Zeidler, *Nonlinear functional analysis and its applications. I. Fixed-point theorems*, Springer, New York, 1986. https://doi.org/10.1007/978-1-4612-4838-5; MR816732

[19] E. Zeidler, *Nonlinear functional analysis and its applications. II/A. Linear monotone operators*, Springer, New York, 1990. https://doi.org/10.1007/978-1-4612-0985-0; MR1033497

[20] C. Zhang, X. Zhang, Some further results on the nonlocal p-Laplacian type problems, *Proc. Roy. Soc. Edinburgh Sect. A* **151**(2020), 953–970. https://doi.org/10.1017/prm.2020.45; MR4259332; Zbl 1466.35234

[21] C. Zhang, L. Wang, Y. H. Kim, Global bifurcation for a class of degenerate elliptic equations with variable exponents, *J. Math. Anal. Appl.* **371**(2010), 624–637. https://doi.org/10.1016/j.jmaa.2010.05.058; MR2670139; Zbl 1198.35089

[22] V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR Izv.* **29**(1987), 33–66. https://doi.org/10.1070/IM1987v029n01ABEH000958; MR864171; Zbl 0599.49031

[23] V. Zhikov, On the technique for passing to the limit in nonlinear elliptic equations, *Funct. Anal. Appl.* **43**(2009), No. 2, 96–112. https://doi.org/10.1007/s10688-009-0014-1; MR2542272; Zbl 1271.35033