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Positive solutions to three classes of non-local fourth-order problems with derivative-dependent nonlinearities

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Abstract. In the article, we investigate three classes of fourth-order boundary value problems with dependence on all derivatives in nonlinearities under the boundary conditions involving Stieltjes integrals. A Gronwall-type inequality is employed to get an a priori bound on the third-order derivative term, and the theory of fixed-point index is used on suitable open sets to obtain the existence of positive solutions. The nonlinearities have quadratic growth in the third-order derivative term. Previous results in the literature are not applicable in our case, as shown by our examples.

Keywords: positive solution, fixed point index, cone, Gronwall inequality.

2020 Mathematics Subject Classification: 34B18, 34B10, 34B15.

1 Introduction

In the article, we investigate the existence of positive solutions to the following three classes of fourth-order boundary value problems (BVPs) with dependence on all derivatives in nonlinearities under the boundary conditions involving Stieltjes integrals

$$\begin{cases} u^{(4)}(t) = f(t,u(t),u'(t),u''(t),u'''(t)), \quad t\in[0,1], \\ u'(0)+\alpha_1[u]=0,\ u''(0)+\alpha_2[u]=0,\ u(1)=\alpha_3[u],\ u'''(1)=0, \end{cases} \tag{1.1}$$

$$\begin{cases} -u^{(4)}(t) = \widetilde{f}(t,u(t),u'(t),u''(t),u'''(t)), \quad t\in[0,1], \\ u(0)=\beta_1[u],\ u'(0)=\beta_2[u],\ u''(0)=\beta_3[u],\ u'''(1)=0, \end{cases} \tag{1.2}$$

and

$$\begin{cases} u^{(4)}(t) = \overline{f}(t,u(t),u'(t),u''(t),u'''(t)), \quad t\in[0,1], \\ u(0)=u(1)=\eta_1[u],\ u''(0)+\eta_2[u]=0,\ u''(1)+\eta_2[u]=0, \end{cases} \tag{1.3}$$

where

$$\alpha_i[u]=\int_0^1 u(t)dA_i(t)\quad (i=1,2,3),\qquad \beta_i[u]=\int_0^1 u(t)dB_i(t)\quad (i=1,2,3),$$

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$$\eta_i[u] = \int_0^1 u(t)dH_i(t) \quad (i = 1, 2)$$

are Stieltjes integrals with A_i, B_i, H_i of bounded variation.

The BVPs (1.1) and (1.2) share the common features that the derivatives of Green's functions, from first to third order in t, do not change sign, however the first-and third-order derivatives of Green's function for the BVP (1.3) are sign-changing. The existence of positive solutions for the BVPs (1.1), (1.2) and (1.3) have been studied respectively in [9] and [5] with $\bar{f}(t, u(t), u''(t))$. The BVP (1.3) with $\eta_1[u] = \eta_2[u] = 0$ is also considered by [16] in which the fourth-order equation is transformed into a second-order problem by order reduction method. The authors in [10] discuss the second-order BVP with non-local boundary conditions

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), \quad t \in [0, 1], \\ au(0) - bu'(0) = \alpha[u], \ cu(1) + du'(1) = \beta[u], \end{cases} \tag{1.4}$$

where a, b, c and d are nonnegative constants with $\rho = ac + ad + bc > 0$. It is supposed in [10] that the nonlinear term has linear growth in both u and u' and some conditions related to the spectral radius of a related linear operator are used, moreover, the Nagumo condition is applied in one of their results. The BVP (1.4) with $a = d = 1, b = c = 0$ is studied by Zhang et al. [17], but the conditions of theorems in [10] can not contain the ones in [17], see [10, Remark 3.10, Remark 3.11].

Recently, Webb in [12] employs a Gronwall-type inequality proved in [13] to deal with a second order equation with nonlinearity having quadratic dependence in derivative terms, but no growth restriction in the function term. This new Gronwall inequality is used instead of a Nagumo condition to get an a priori bound on the derivative term. The theory of fixed-point index on suitable open sets is applied to obtain the existence of positive solutions to second-order non-local problems.

Motivated by the works mentioned above, in the present paper we adopt the idea and the techniques provided in [12] to consider the positive solutions of the fourth-order BVPs (1.1), (1.2) and (1.3). The nonlinearities contain all terms of the derivatives, and there is quadratic growth in u''' but no growth restriction in u, u' and u''. Li and Chen in [8] investigate the nontrivial solutions to fourth-order BVP with quadratic growth subject to local boundary conditions. In [9], the nonlinearity has linear growth in u and all derivatives with some conditions related to the spectral radius of a related linear operator, the results are not valid for the problems presented in this paper although the Nagumo condition also allows quadratic growth (see Example 2.7 and Remark 2.8). Making use of several different methods, the authors in [3, 5, 11] discuss the existence of positive solutions to some fourth-order BVPs, however not all of the derivatives is included in the nonlinearities since some derivatives of the Green's functions are sign-changing. Some relevant works may refer to [1] for fourth-order BVP with local boundary conditions via an application of contraction mapping theorem, [6] for certain perturbed Hammerstein integral equations with first-order derivative dependence, [7] for fourth-order BVP with local boundary conditions.

We recall the basic properties of fixed point index that we use.

Lemma 1.1 ([2, 4]). *Let Ω be a bounded open set relative to a cone P in Banach space X with $0 \in \Omega$. If $A : \overline{\Omega} \to P$ is a completely continuous operator, and $Au \neq \lambda u$ for $u \in \partial_P\Omega$, $\lambda \geq 1$, then the fixed point index $i(A, \Omega, P) = 1$, where $\overline{\Omega}$ and $\partial_P\Omega$ are respectively the closure and boundary of Ω relative to P.*

Lemma 1.2 ([2,4]). *Let Ω be a bounded open set relative to a cone P in Banach space X. If $A : \overline{\Omega} \to P$ is a completely continuous operator, and there exists $v_0 \in P \setminus \{0\}$ such that $u - Au \neq \sigma v_0$ for $u \in \partial_P \Omega$ and $\sigma \geq 0$, then the fixed point index $i(A, \Omega, P) = 0$.*

2 Positive solutions to the BVP (1.1)

Let $[\alpha, \beta] \subset [0, 1]$, we write $L^p_+[\alpha, \beta]$ $(1 \leq p \leq \infty)$ to denote functions that are non-negative almost everywhere (a.e.) and belong to $L^p[\alpha, \beta]$. The proof of the following lemma is completely similar to the method in [12].

Lemma 2.1. *Suppose that there are a constant $d_0 > 0$ and functions $d_1, d_2 \in L^1_+[\alpha, \beta]$ such that $u \in L^\infty_+[\alpha, \beta]$ satisfies*

$$u(t) \leq d_0 + \int_t^\beta d_1(s)u(s)ds + \int_t^\beta d_2(s)u^2(s)ds \quad \text{for a.e. } t \in [\alpha, \beta].$$

If there is a constant $R > 0$ such that $\int_\alpha^\beta d_2(s)u(s)ds \leq R$, then $u(t) \leq d_0 \exp(R) \exp(D_1(t))$ for a.e. $t \in [\alpha, \beta]$, where $D_1(t) := \int_t^\beta d_1(s)ds$.

Proof. Let $v(t) := d_0 + \int_t^\beta d_1(s)u(s)ds + \int_t^\beta d_2(s)u^2(s)ds$. Then v is absolutely continuous, $v(\beta) = d_0, v(t) \geq d_0 > 0$ for all $t \in [\alpha, \beta]$, and $u(t) \leq v(t)$ for a.e. $t \in [\alpha, \beta]$. Moreover, we have

$$v'(t) = -d_1(t)u(t) - d_2(t)u^2(t) \geq -d_1(t)v(t) - d_2(t)u(t)v(t) \quad \text{for a.e. } t \in [\alpha, \beta].$$

Then $v'(t)/v(t) \geq -d_1(t) - d_2(t)u(t)$ which can be integrated to give

$$\ln\left(\frac{v(\beta)}{v(t)}\right) \geq -D_1(t) - \int_t^\beta d_2(s)u(s)ds,$$

hence $u(t) \leq v(t) \leq d_0 \exp(R) \exp(D_1(t))$ for a.e. $t \in [\alpha, \beta]$. □

For BVP (1.1)

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1], \\ u'(0) + \alpha_1[u] = 0, \ u''(0) + \alpha_2[u] = 0, \ u(1) = \alpha_3[u], \ u'''(1) = 0, \end{cases}$$

we make the following assumptions:

(C_1) $f : [0,1] \times [0, \infty) \times (-\infty, 0]^3 \to [0, \infty)$ is continuous;

(C_2) A_i is of bounded variation, moreover

$$\mathcal{K}_i(s) := \int_0^1 G_0(t,s)dA_i(t) \geq 0, \qquad \forall s \in [0,1] \quad (i = 1,2,3),$$

where

$$G_0(t,s) = \begin{cases} \frac{1}{2}s(1-s) + \frac{1}{6}(s^3 - t^3), & 0 \leq t \leq s \leq 1, \\ \frac{1}{2}s(1-s) - \frac{1}{2}ts(t-s), & 0 \leq s \leq t \leq 1; \end{cases}$$

(C_3) The 3×3 matrix $[A]$ is positive whose (i,j)th entry is $\alpha_i[\gamma_j]$, i.e., it has nonnegative entries, where $\gamma_1(t) = 1-t$, $\gamma_2(t) = \frac{1}{2}(1-t^2)$ and $\gamma_3(t) = 1$ are the solutions of $u^{(4)} = 0$ respectively subject to boundary conditions:

$$u'(0)+1=0, \quad u''(0)=0, \quad u(1)=0, \quad u'''(1)=0;$$

$$u'(0)=0, \quad u''(0)+1=0, \quad u(1)=0, \quad u'''(1)=0;$$

$$u'(0)=0, \quad u''(0)=0, \quad u(1)=1, \quad u'''(1)=0.$$

Furthermore assume that its spectral radius $r([A]) < 1$.

Webb and Infante [14] in a general framework convert the BVP

$$\begin{cases} u^{(4)}(t) = f(t,u(t)), \quad t \in [0,1], \\ u'(0)+\alpha_1[u]=0,\ u''(0)+\alpha_2[u]=0,\ u(1)=\alpha_3[u],\ u'''(1)=0 \end{cases} \tag{2.1}$$

into the perturbed Hammerstein integral equation of the type

$$u(t) = \sum_{i=1}^{3} \gamma_i(t)\alpha_i[u] + \int_0^1 G_0(t,s)f(s,u(s))ds,$$

where $G_0(t,s)$ is the Green's function associated with

$$\begin{cases} u^{(4)}(t) = f(t,u(t)), \quad t \in [0,1], \\ u'(0) = u''(0) = u(1) = u'''(1) = 0. \end{cases}$$

Immediately after this they prove that if (C_1)–(C_3) are satisfied, (2.1) is equivalent to

$$u(t) = \int_0^1 G_1(t,s)f(s,u(s))ds,$$

where

$$G_1(t,s) = \langle (I-[A])^{-1}\mathcal{K}(s), \gamma(t)\rangle + G_0(t,s) = \sum_{i=1}^{3}\kappa_i(s)\gamma_i(t) + G_0(t,s),$$

$\langle (I-[A])^{-1}\mathcal{K}(s), \gamma(t)\rangle$ is the inner product in $\mathbb{R}^3$, $\kappa_i(s)$ is the ith component of $(I-[A])^{-1}\mathcal{K}(s)$.

Similar to the method of Webb–Infante, we define the operator S as

$$(Su)(t) = \int_0^1 G_1(t,s)f(s,u(s),u'(s),u''(s),u'''(s))ds.$$

Lemma 2.2. *If (C_2) and (C_3) hold, then $\kappa_i(s) \geq 0$ ($i=1,2,3$) and for $t,s \in [0,1]$,*

$$c_0(t)\Phi_0(s) \leq G_1(t,s) \leq \Phi_0(s), \tag{2.2}$$

where

$$\Phi_0(s) = \sum_{i=1}^{3}\kappa_i(s) + \frac{1}{2}s(1-s) + \frac{1}{6}s^3, \qquad c_0(t) = \frac{1}{2}(1-t^2),$$

and

$$c_1(t)\Phi_1(s) \leq -\frac{\partial G_1(t,s)}{\partial t} \leq \Phi_1(s), \qquad c_2(t)\Phi_2(s) \leq -\frac{\partial^2 G_1(t,s)}{\partial t^2} \leq \Phi_2(s), \tag{2.3}$$

where

$$\frac{\partial G_1(t,s)}{\partial t} = -\kappa_1(s) - t\kappa_2(s) - \frac{1}{2}\begin{cases} t^2, & 0 \leq t \leq s \leq 1, \\ s(2t-s), & 0 \leq s \leq t \leq 1, \end{cases}$$

$$\frac{\partial^2 G_1(t,s)}{\partial t^2} = -\kappa_2(s) - \begin{cases} t, & 0 \le t \le s \le 1, \\ s, & 0 \le s \le t \le 1, \end{cases}$$

$$\Phi_1(s) = \sum_{i=1}^{2} \kappa_i(s) + \frac{1}{2}s(2-s), \qquad c_1(t) = t^2, \ \ \Phi_2(s) = \kappa_2(s) + s, \qquad c_2(t) = t.$$

Proof. $\kappa_i(s) \ge 0 (i = 1,2,3)$ by hypotheses (C_2) and (C_3). For $0 \le s \le t \le 1$, $\frac{\partial}{\partial t}G_0(t,s) = \frac{1}{2}s(s-2t) \le 0$ which implies that

$$G_0(t,s) \le G_0(s,s) = \frac{1}{2}s(1-s);$$

For $0 \le t < s \le 1$, $\frac{\partial}{\partial t}G_0(t,s) = -\frac{1}{2}t^2 \le 0$ which implies that

$$G_0(t,s) \le G_0(0,s) = \frac{1}{2}s(1-s) + \frac{1}{6}s^3.$$

Then $G_0(t,s) \le \frac{1}{2}s(1-s) + \frac{1}{6}s^3$, $\forall (t,s) \in [0,1] \times [0,1]$.

Now we find the best function $C_0(t)$ such that $G_0(t,s) \ge C_0(t)\left(\frac{1}{2}s(1-s) + \frac{1}{6}s^3\right)$, $\forall (t,s) \in [0,1] \times [0,1]$.

For $0 \le s \le t \le 1$, this is

$$\frac{1}{2}s(1-s) - \frac{1}{2}ts(t-s) \ge C_0(t)\left(\frac{1}{2}s(1-s) + \frac{1}{6}s^3\right),$$

thus

$$C_0(t) \le \frac{3(1-t)(1+t-s)}{3-3s+s^2}.$$

Denote

$$g_1(t,s) = \frac{(1-t)(1+t-s)}{3-3s+s^2},$$

from

$$\frac{\partial}{\partial s}g_1(t,s) = \frac{(1-t)(s^2-2s(1+t)+3t)}{(3-3s+s^2)^2} \ge 0$$

it follows that $C_0(t) \le 3g_1(t,0) = 1-t^2$.

For $0 \le t < s \le 1$, this is

$$\frac{1}{2}s(1-s) + \frac{1}{6}(s^3-t^3) \ge C_0(t)\left(\frac{1}{2}s(1-s) + \frac{1}{6}s^3\right),$$

thus

$$C_0(t) \le \frac{3s-3s^2+s^3-t^3}{s(3-3s+s^2)}.$$

Denote

$$g_2(t,s) = \frac{3s-3s^2+s^3-t^3}{s(3-3s+s^2)},$$

from

$$\frac{\partial}{\partial s}g_2(t,s) = \frac{3(1-s)^2t^3}{s^2(3-3s+s^2)^2} \ge 0$$

it follows that $C_0(t) \le g_2(t,t) = \frac{3(1-t)}{3-3t+t^2}$.

Therefore

$$C_0(t)=\min\left\{1-t^2,\ \frac{3(1-t)}{3-3t+t^2}\right\}=1-t^2.$$

Since

$$\frac{1}{2}(1-t^2)\sum_{i=1}^{3}\kappa_i(s)\le\sum_{i=1}^{3}\kappa_i(s)\gamma_i(t)\le\sum_{i=1}^{3}\kappa_i(s),$$

$$\frac{1}{2}(1-t^2)\left(\frac{1}{2}s(1-s)+\frac{1}{6}s^3\right)\le(1-t^2)\left(\frac{1}{2}s(1-s)+\frac{1}{6}s^3\right)\le G_0(t,s)\le\frac{1}{2}s(1-s)+\frac{1}{6}s^3,$$

we know that (2.2) holds. As for (2.3), it comes directly from the inequalities

$$t^2\sum_{i=1}^{2}\kappa_i(s)\le t\sum_{i=1}^{2}\kappa_i(s)\le-\sum_{i=1}^{3}\kappa_i(s)\gamma_i'(t)\le\sum_{i=1}^{2}\kappa_i(s),$$

$$\frac{1}{2}t^2s(2-s)\le-\frac{\partial G_0(t,s)}{\partial t}\le\frac{1}{2}s(2-s),$$

$$t\kappa_2(s)\le\kappa_2(s)=-\sum_{i=1}^{3}\kappa_i(s)\gamma_i''(t),\quad ts\le-\frac{\partial^2G_0(t,s)}{\partial t^2}\le s$$

for $t,s\in[0,1]$. □

Let $C^3[0,1]$ be the Banach space which consists of all third-order continuously differentiable functions on $[0,1]$ with the norm $\|u\|_{C^3}=\max\{\|u\|_C,\|u'\|_C,\|u''\|_C,\|u'''\|_C\}$. In $C^3[0,1]$ we define the cone

$$K=\big\{u\in C^3[0,1]:u(t)\ge c_0(t)\|u\|_C,\ -u'(t)\ge c_1(t)\|u'\|_C,\\ -u''(t)\ge c_2(t)\|u''\|_C,\ \forall t\in[0,1];\ u'''(1)=0\big\}. \tag{2.4}$$

Lemma 2.3. *If* (C_1)–(C_3) *hold, then* $S:K\to K$ *is completely continuous and the positive solutions to BVP* (1.1) *are equivalent to the fixed points of* S *in* K.

Proof. Because $G_1(t,s)$, and the first- and second-order derivatives are continuous, the third order derivative is integrable in s, from Lemma 2.2 it is easy to prove that $S:K\to K$ is continuous. Let F be a bounded set in K, then there exists $M>0$ such that $\|u\|_{C^3}\le M$ for all $u\in K$. Denote

$$C=\max_{(t,x_0,x_1,x_2,x_3)\in[0,1]\times[0,M]\times[-M,0]^3}f(t,x_0,x_1,x_2,x_3).$$

By (C_1) and Lemma 2.2 we have that $\forall u\in F$ and $t\in[0,1]$,

$$|(Su)(t)|\le C\int_0^1\Phi_0(s)ds,\qquad|(Su)'(t)|\le C\int_0^1\left|\frac{\partial G_1(t,s)}{\partial t}\right|ds\le C\int_0^1\Phi_1(s)ds,$$

$$|(Su)''(t)|\le C\int_0^1\left|\frac{\partial^2G_1(t,s)}{\partial t^2}\right|ds\le C\int_0^1\Phi_2(s)ds,\qquad|(Su)'''(t)|\le C\int_0^1\left|\frac{\partial^3G_1(t,s)}{\partial t^3}\right|ds\le C,$$

then $S(F)$ is uniformly bounded in $C^3[0,1]$. Moreover $\forall u\in F$ and $t_1,t_2\in[0,1]$ with $t_1<t_2$,

$$\begin{aligned}|(Su)(t_1)-(Su)(t_2)|&\le\int_0^1|G_1(t_1,s)-G_1(t_2,s)|f(s,u(s),u'(s),u''(s),u'''(s))ds\\&\le C\int_0^1|G_1(t_1,s)-G_1(t_2,s)|ds,\end{aligned}$$

$$
\begin{aligned}
|(Su)'(t_1)-(Su)'(t_2)| &\le \int_0^1 \left|\frac{\partial G_1}{\partial t}(t_1,s)-\frac{\partial G_1}{\partial t}(t_2,s)\right| f(s,u(s),u'(s),u''(s),u'''(s))ds\\
&\le C\int_0^1 \left|\frac{\partial G_1}{\partial t}(t_1,s)-\frac{\partial G_1}{\partial t}(t_2,s)\right| ds,
\end{aligned}
$$

$$
\begin{aligned}
|(Su)''(t_1)-(Su)''(t_2)| &\le \int_0^1 \left|\frac{\partial^2 G_1}{\partial t^2}(t_1,s)-\frac{\partial^2 G_1}{\partial t^2}(t_2,s)\right| f(s,u(s),u'(s),u''(s),u'''(s))ds\\
&\le C\int_0^1 \left|\frac{\partial^2 G_1}{\partial t^2}(t_1,s)-\frac{\partial^2 G_1}{\partial t^2}(t_2,s)\right| ds,
\end{aligned}
$$

$$
\begin{aligned}
&|(Su)'''(t_1)-(Su)'''(t_2)|\\
&\quad= \left|\int_{t_1}^1 f(s,u(s),u'(s),u''(s),u'''(s))ds-\int_{t_2}^1 f(s,u(s),u'(s),u''(s),u'''(s))ds\right| \le C(t_2-t_1),
\end{aligned}
$$

thus $S(F)$ and $S^{(i)}(F) =: \{v^{(i)} : v^{(i)}(t) = (Su)^{(i)}(t), u \in F\}$ $(i = 1,2,3)$ are equicontinuous.

Therefore $S : K \to K$ is completely continuous by the Arzelà–Ascoli theorem. Similar to [14], the positive solutions to BVP (1.1) are equivalent to the fixed points of S in K. □

Lemma 2.4. *Suppose that* (C_1)–(C_3) *hold, there exist constants* $p_0 > 0, p_3 \ge 0$ *and functions* $p_1, p_2 \in L^1_+[0,1]$ *such that for all* $(t,x_0,x_1,x_2,x_3) \in [0,1]\times[0,\infty)\times(-\infty,0]^3$,

$$
f(t,x_0,x_1,x_2,x_3) \le p_0 + p_1(t)g(x_0,x_1,x_2) + p_2(t)|x_3| + p_3|x_3|^2, \tag{2.5}
$$

where $g : [0,\infty)\times(-\infty,0]^2 \to [0,\infty)$ *is continuous, non-decreasing in the first variable, and non-increasing in the second and third variables. Let* $\lambda \ge 1, \sigma \ge 0, r > 0$*, define* $D_i := \int_0^1 p_i(s)ds$ $(i = 1,2)$ *and*

$$
Q(r) := (p_0 + g(r,-r,-r)D_1)\exp(D_2)\exp(p_3 r). \tag{2.6}
$$

If $u \in K$ *with* $\|u\|_{C^2} \le r$ *such that* $\lambda u(t) = (Su)(t) + \sigma$*, then* $\|u'''\|_C \le Q(r)$.

Proof. Since $u \in K$ and $\lambda u(t) = (Su)(t) + \sigma$, we have that $\lambda u^{(4)}(t) = f(t,u(t),u'(t),u''(t),u'''(t))$ and $\lambda u'''(t) = -\int_t^1 f(s,u(s),u'(s),u''(s),u'''(s))ds \le 0$. From $\|u\|_{C^2} \le r$ it follows that

$$
\begin{aligned}
|u'''(t)| \le \lambda|u'''(t)| &= \int_t^1 f(s,u(s),u'(s),u''(s),u'''(s))ds\\
&\le \int_t^1 \left(p_0 + p_1(s)g(u(s),u'(s),u''(s)) + p_2(s)|u'''(s)| + p_3|u'''(s)|^2\right) ds\\
&\le (p_0 + g(r,-r,-r)D_1) + \int_t^1 \left(p_2(s)|u'''(s)| + p_3|u'''(s)|^2\right) ds
\end{aligned}
$$

and $\int_0^1 p_3|u'''(s)|ds = -\int_0^1 p_3 u'''(s)ds = p_3(u''(0)-u''(1)) \le p_3 r$. By Lemma 2.1, we deduce that

$$
|u'''(t)| \le (p_0 + g(r,-r,-r)D_1)\exp(D_2)\exp(p_3 r) = Q(r),
$$

the proof is complete. □

Let $[a,b]$ be a subset of $(0,1)$ and denote

$$
\gamma := \min\left\{\min_{t\in[a,b]} c_0(t), \min_{t\in[a,b]} c_1(t), \min_{t\in[a,b]} c_2(t)\right\} = \min\left\{\frac{1}{2}(1-b^2), a^2\right\},
$$

$$\frac{1}{m} := \max\left\{\int_0^1 \Phi_0(s)ds, \int_0^1 \Phi_1(s)ds, \int_0^1 \Phi_2(s)ds\right\},$$
$$\frac{1}{M} := \min\left\{\int_a^b \Phi_0(s)ds, \int_a^b \Phi_1(s)ds, \int_a^b \Phi_2(s)ds\right\},$$

where $c_i(t)$ and $\Phi_i(s)(i=0,1,2)$ are provided in Lemma 2.2. Obviously, $\gamma \in (0,1/2)$ and $m < M$.

Theorem 2.5. *Suppose that* (C_1)-(C_3) *hold and* f *satisfies the growth assumption* (2.5). *The BVP* (1.1) *has at least one positive solution* $u \in K$ *if either of the following conditions* (F_1), (F_2) *holds, where* Q *is given by* (2.6).

(F_1) *There exist* $0 < r_1 < r_2$ *with* $r_1 < r_2\gamma$, *such that*

(F_1a) *for* $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0, r_1] \times [-r_1, 0]^2 \times [-Q(r_1), 0]$,

$$f(t, x_0, x_1, x_2, x_3) < mr_1; \tag{2.7}$$

(F_1b) *for* $(t, x_0, x_1, x_2, x_3) \in W_1 := W_{1,0} \cup W_{1,1} \cup W_{1,2}$,

$$f(t, x_0, x_1, x_2, x_3) > Mr_2, \tag{2.8}$$

where

$$W_{1,0} = [a,b] \times [r_2\gamma, r_2] \times [-r_2, 0]^2 \times [-Q(r_2), 0],$$
$$W_{1,1} = [a,b] \times [0, r_2] \times [-r_2, -r_2\gamma] \times [-r_2, 0] \times [-Q(r_2), 0],$$
$$W_{1,2} = [a,b] \times [0, r_2] \times [-r_2, 0] \times [-r_2, -r_2\gamma] \times [-Q(r_2), 0].$$

(F_2) *There exist* $0 < r_1 < r_2$ *with* $Mr_1 < mr_2$, *such that*

(F_2a) *for* $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0, r_2] \times [-r_2, 0]^2 \times [-Q(r_2), 0]$,

$$f(t, x_0, x_1, x_2, x_3) < mr_2; \tag{2.9}$$

(F_2b) *for* $(t, x_0, x_1, x_2, x_3) \in W_2 := W_{2,0} \cup W_{2,1} \cup W_{2,2}$,

$$f(t, x_0, x_1, x_2, x_3) > Mr_1, \tag{2.10}$$

where

$$W_{2,0} = [a,b] \times [r_1\gamma, r_1] \times [-r_1, 0]^2 \times [-Q(r_1), 0],$$
$$W_{2,1} = [a,b] \times [0, r_1] \times [-r_1, -r_1\gamma] \times [-r_1, 0] \times [-Q(r_1), 0],$$
$$W_{2,2} = [a,b] \times [0, r_1] \times [-r_1, 0] \times [-r_1, -r_1\gamma] \times [-Q(r_1), 0].$$

Proof. Suppose that (F_1) holds. Define an open (relative to K) bounded set

$$U_{r_1} := \left\{u \in K : \|u\|_{C^2} < r_1, \|u'''\|_C < Q(r_1) + 1\right\}.$$

Then the boundary $\partial_K U_{r_1}$ of U_{r_1} (relative to K) satisfies $\partial_K U_{r_1} \subset U_{r_1,0} \cup U_{r_1,1} \cup U_{r_1,2}$, where

$$U_{r_1,0} := \{u \in K : \|u\|_C = r_1, \|u'\|_C \le r_1, \|u''\|_C \le r_1, \|u'''\|_C \le Q(r_1) + 1\},$$

$$U_{r_1,1} := \{u \in K : \|u\|_C \le r_1, \|u'\|_C = r_1, \|u''\|_C \le r_1, \|u'''\|_C \le Q(r_1) + 1\},$$
$$U_{r_1,2} := \{u \in K : \|u\|_C \le r_1, \|u'\|_C \le r_1, \|u''\|_C = r_1, \|u'''\|_C \le Q(r_1) + 1\}.$$

We will show that $Su \neq \lambda u$ for all $u \in \partial_K U_{r_1}$ and all $\lambda \ge 1$. If not, there exist $u \in \partial_K U_{r_1}$ and $\lambda \ge 1$ such that $\lambda u(t) = (Su)(t)$. It is clear that $\|u'''\|_C \le Q(r_1)$ by Lemma 2.4.

From Lemma 2.2 and (2.7) it follows that when $u \in U_{r_1,0}$,

$$\lambda u(t) = \int_0^1 G_1(t,s) f(s, u(s), u'(s), u''(s), u'''(s))ds < \int_0^1 \Phi_0(s) m r_1 ds \le r_1;$$

when $u \in U_{r_1,1}$,

$$-\lambda u'(t) = -\int_0^1 \frac{\partial G_1(t,s)}{\partial t} f(s, u(s), u'(s), u''(s), u'''(s))ds < \int_0^1 \Phi_1(s) m r_1 ds \le r_1;$$

when $u \in U_{r_1,2}$,

$$-\lambda u''(t) = -\int_0^1 \frac{\partial^2 G_1(t,s)}{\partial t^2} f(s, u(s), u'(s), u''(s), u'''(s))ds < \int_0^1 \Phi_2(s) m r_1 ds \le r_1.$$

Taking the maximum over $[0,1]$ we give a contradiction $\lambda r_1 < r_1$.

By Lemma 1.1 the fixed point index $i(S, U_{r_1}, K) = 1$.

Define an open (relative to K) set

$$V_{r_2} := \left\{ u \in K : \min_{t\in[a,b]} u(t) < r_2\gamma, \min_{t\in[a,b]} (-u'(t)) < r_2\gamma, \min_{t\in[a,b]} (-u''(t)) < r_2\gamma, \|u'''\|_C < Q(r_2) + 1 \right\}.$$

It is clear that $\overline{U}_{r_1} \subset V_{r_2}$ by $r_1 < r_2\gamma$ and $Q(r_1) < Q(r_2)$. Since $\|u\|_{C^2} \le r_2$ for $u \in V_{r_2}$ by (2.4), V_{r_2} is bounded. The boundary $\partial_K V_{r_2}$ of V_{r_2} (relative to K) satisfies $\partial_K V_{r_2} \subset V_{r_2,0} \cup V_{r_2,1} \cup V_{r_2,2}$, where

$$V_{r_2,0} := \left\{ u \in K : \min_{t\in[a,b]} u(t) = r_2\gamma, \min_{t\in[a,b]} (-u'(t)) \le r_2\gamma, \min_{t\in[a,b]} (-u''(t)) \le r_2\gamma, \|u'''\|_C \le Q(r_2) + 1 \right\},$$

$$V_{r_2,1} := \left\{ u \in K : \min_{t\in[a,b]} u(t) \le r_2\gamma, \min_{t\in[a,b]} (-u'(t)) = r_2\gamma, \min_{t\in[a,b]} (-u''(t)) \le r_2\gamma, \|u'''\|_C \le Q(r_2) + 1 \right\},$$

$$V_{r_2,2} := \left\{ u \in K : \min_{t\in[a,b]} u(t) \le r_2\gamma, \min_{t\in[a,b]} (-u'(t)) \le r_2\gamma, \min_{t\in[a,b]} (-u''(t)) = r_2\gamma, \|u'''\|_C \le Q(r_2) + 1 \right\}.$$

Let $v_0(t) \equiv 1$ and note that $v_0 \in K$. We claim that $u \neq Su + \sigma v_0$ for all $u \in \partial_K V_{r_2}$ and all $\sigma \ge 0$. If the claim is false, there exist $u \in \partial_K V_{r_2}$ and $\sigma \ge 0$ such that $u = Su + \sigma v_0$. Thus $\|u'''\|_C \le Q(r_2)$ for $u \in V_{r_2}$ by Lemma 2.4. From Lemma 2.2 and (2.8) we have the following contradictions. When $u \in V_{r_2,0}$,

$$u(t) = \int_0^1 G_1(t,s) f(s, u(s), u'(s), u''(s), u'''(s))ds + \sigma > \int_a^b c_0(t)\Phi_0(s) M r_2 ds + \sigma \ge r_2\gamma + \sigma,$$

taking the minimum for $t \in [a,b]$ gives the contradiction $r_2\gamma > r_2\gamma + \sigma$. When $u \in V_{r_2,1}$,

$$-u'(t) = -\int_0^1 \frac{\partial G_1(t,s)}{\partial t} f(s, u(s), u'(s), u''(s), u'''(s))ds > \int_a^b c_1(t)\Phi_1(s) M r_2 ds \ge r_2\gamma,$$

taking the minimum for $t \in [a,b]$ gives the contradiction $r_2\gamma > r_2\gamma$. When $u \in V_{r_2,2}$,

$$-u''(t) = -\int_0^1 \frac{\partial^2 G_1(t,s)}{\partial t^2} f(s,u(s),u'(s),u''(s),u'''(s))ds > \int_a^b c_2(t)\Phi_2(s)Mr_2 ds \geq r_2\gamma,$$

taking the minimum for $t \in [a,b]$ also gives the contradiction $r_2\gamma > r_2\gamma$.

By Lemma 1.2 the fixed point index $i(S, V_{r_2}, K) = 0$.

From the additivity property of fixed point index we have $i(S, V_{r_2}\backslash\overline{U}_{r_1}, K) = -1$. So there is a fixed point of S in the set $V_{r_2}\backslash\overline{U}_{r_1}$ which is clearly nonzero and the positive solutions to BVP (1.1) by Lemma 2.3.

Suppose that (F_2) holds, notice that f is well defined since $Mr_1 < mr_2$. Define open (relative to K) bounded sets $U_{r_2} := \{u \in K : \|u\|_{C^2} < r_2, \ \|u'''\|_C < Q(r_2)+1\}$ and

$$V_{r_1} := \left\{u \in K : \min_{t\in[a,b]} u(t) < r_1\gamma, \min_{t\in[a,b]}(-u'(t)) < r_1\gamma, \min_{t\in[a,b]}(-u''(t)) < r_1\gamma, \|u'''\|_C < Q(r_1)+1\right\}.$$

It is clear that $\overline{V}_{r_1} \subset U_{r_2}$. The rest of proof is similar to the above. □

Example 2.6. Consider the following fourth-order boundary problems under mixed multi-point and integral boundary conditions with sign-changing coefficients and kernel functions.

$$\begin{cases} u^{(4)}(t) = f(t,u(t),u'(t),u''(t),u'''(t)), \quad t \in [0,1], \\ u'(0) + \frac{1}{4}u(\frac{1}{4}) - \frac{1}{12}u(\frac{3}{4}) = 0, \quad u''(0) + \int_0^1 u(t)\cos(\pi t)dt = 0, \\ u(1) = \frac{1}{2}u(\frac{1}{2}) - \frac{1}{4}u(\frac{3}{4}), \quad u'''(1) = 0, \end{cases} \tag{2.11}$$

thus $\alpha_1[u] = \frac{1}{4}u(\frac{1}{4}) - \frac{1}{12}u(\frac{3}{4})$, $\alpha_2[u] = \int_0^1 u(t)\cos(\pi t)dt$, $\alpha_3[u] = \frac{1}{2}u(\frac{1}{2}) - \frac{1}{4}u(\frac{3}{4})$. Then

$$\begin{aligned} 0 \leq \mathcal{K}_1(s) &= \tfrac{1}{4}G_0\left(\tfrac{1}{4},s\right) - \tfrac{1}{12}G_0\left(\tfrac{3}{4},s\right) \\ &= \begin{cases} -\frac{1}{12}s^2 + \frac{19}{192}s, & 0 \leq s \leq \frac{1}{4}, \\ \frac{1}{24}s^3 - \frac{11}{96}s^2 + \frac{41}{384}s - \frac{1}{1536}, & \frac{1}{4} < s \leq \frac{3}{4}, \\ \frac{1}{36}s^3 - \frac{1}{12}s^2 + \frac{1}{12}s + \frac{1}{192}, & \frac{3}{4} < s \leq 1, \end{cases} \end{aligned}$$

$$\mathcal{K}_2(s) = \int_0^1 G_0(t,s)\cos(\pi t)dt = \frac{2s - s^2}{2\pi^2} + \frac{\cos \pi s}{\pi^4} - \frac{1}{\pi^4} \geq 0 \qquad (0 \leq s \leq 1),$$

$$\begin{aligned} 0 \leq \mathcal{K}_3(s) &= \tfrac{1}{2}G_0\left(\tfrac{1}{2},s\right) - \tfrac{1}{4}G_0\left(\tfrac{3}{4},s\right) \\ &= \begin{cases} -\frac{3}{32}s^2 + \frac{17}{128}s, & 0 \leq s \leq \frac{1}{2}, \\ \frac{1}{12}s^3 - \frac{7}{32}s^2 + \frac{25}{128}s - \frac{1}{96}, & \frac{1}{2} < s \leq \frac{3}{4}, \\ \frac{1}{24}s^3 - \frac{1}{8}s^2 + \frac{1}{8}s + \frac{11}{1536}, & \frac{3}{4} < s \leq 1, \end{cases} \end{aligned}$$

the 3×3 matrix

$$[A] = \begin{pmatrix} \alpha_1[\gamma_1] & \alpha_1[\gamma_2] & \alpha_1[\gamma_3] \\ \alpha_2[\gamma_1] & \alpha_2[\gamma_2] & \alpha_2[\gamma_3] \\ \alpha_3[\gamma_1] & \alpha_3[\gamma_2] & \alpha_3[\gamma_3] \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{19}{192} & \frac{1}{6} \\ \frac{2}{\pi^2} & \frac{1}{\pi^2} & 0 \\ \frac{3}{16} & \frac{17}{128} & \frac{1}{4} \end{pmatrix}$$

and its spectral radius $r([A]) \approx 0.4479 < 1$ (Some values here and later are calculated using the mathematical software *Mathematica*). Therefore, (C_2) and (C_3) are satisfied. We choose $[a,b] = [1/4, 3/4]$ and note that $\gamma = 1/16$,

$$\kappa_1(s) = \begin{cases} \frac{-74+2\pi^4(37-30s)s+23\pi^2s^2+74\cos(\pi s)}{4\pi^2(-151+114\pi^2)}, & 0 \le s \le \frac{1}{4}, \\ \frac{-592+\pi^2(3-36s+328s^2-192s^3)+\pi^4(-3+628s-624s^2+192s^3)+592\cos(\pi s)}{32\pi^2(-151+114\pi^2)}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{-1776+\pi^2(41-300s+1368s^2-832s^3)+\pi^4(-41+2076s-2256s^2+832s^3)+1776\cos(\pi s)}{96\pi^2(-151+114\pi^2)}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{-888+\pi^2(-47+120s+324s^2-256s^3)+\pi^4(47+768s-768s^2+256s^3)+888\cos(\pi s)}{48\pi^2(-151+114\pi^2)}, & \frac{3}{4} < s \le 1, \end{cases}$$

$$\kappa_2(s) = \begin{cases} \frac{-114+\pi^2(151-87s)s+114\cos(\pi s)}{\pi^2(-151+114\pi^2)}, & 0 \le s \le \frac{1}{4}, \\ \frac{-1824+\pi^2(-3+2452s-1536s^2+192s^3)+1824\cos(\pi s)}{16\pi^2(-151+114\pi^2)}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{-5472+\pi^2(-41+7548s-4992s^2+832s^3)+5472\cos(\pi s)}{48\pi^2(-151+114\pi^2)}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{-2736+\pi^2(47+3504s-2136s^2+256s^3)+2736\cos(\pi s)}{24\pi^2(-151+114\pi^2)}, & \frac{3}{4} < s \le 1, \end{cases}$$

$$\kappa_3(s) = \begin{cases} \frac{-794+157\pi^2s^2-2\pi^4s(-397+288s)+794\cos(\pi s)}{32\pi^2(-151+114\pi^2)}, & 0 \le s \le \frac{1}{4}, \\ \frac{-12704+4\pi^4(-3+3212s-2448s^2+192s^3)+\pi^2(-5+60s+2272s^2+320s^3)+12704\cos(\pi s)}{512\pi^2(-151+114\pi^2)}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{-38112+\pi^2(3153-18828s+44832s^2-24384s^3)+4\pi^4(-649+13476s-15024s^2+5696s^3)+38112\cos(\pi s)}{1536\pi^2(-151+114\pi^2)}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{-19056+\pi^2(-1029+1008s+8520s^2-6016s^3)+256\pi^4(4+69s-69s^2+23s^3)+19056\cos(\pi s)}{768\pi^2(-151+114\pi^2)}, & \frac{3}{4} < s \le 1, \end{cases}$$

and hence

$$\int_0^1 \Phi_0(s)ds = \frac{-483264 + 96736\pi^2 + 79071\pi^4}{57984\pi^2 - 43776\pi^4},$$

$$\int_0^1 \Phi_1(s)ds = \frac{50880 + 539\pi^2 - 16421\pi^4}{3072\pi^2(-151 + 114\pi^2)},$$

$$\int_0^1 \Phi_2(s)ds = \frac{21888 + 5371\pi^2 - 10944\pi^4}{28992\pi^2 - 21888\pi^4)},$$

$$\frac{1}{m} = \max\left\{\int_0^1 \Phi_0(s)ds, \int_0^1 \Phi_1(s)ds, \int_0^1 \Phi_2(s)ds\right\} = \frac{21888 + 5371\pi^2 - 10944\pi^4}{28992\pi^2 - 21888\pi^4)},$$

$$\int_{1/4}^{3/4} \Phi_0(s)ds = \frac{-483264 + 103739\pi^2 + 89136\pi^4}{6144\pi^2(-151 + 114\pi^2)},$$

$$\int_{1/4}^{3/4} \Phi_1(s)ds = \frac{25440 + 262\pi^2 - 9013\pi^4}{57984\pi^2 - 43776\pi^4},$$

$$\int_{1/4}^{3/4} \Phi_2(s)ds = \frac{10944 + 2225\pi^2 - 5472\pi^4}{28992\pi^2 - 21888\pi^4)},$$

$$\frac{1}{M} = \min\left\{\int_{1/4}^{3/4} \Phi_0(s)ds, \int_{1/4}^{3/4} \Phi_1(s)ds, \int_{1/4}^{3/4} \Phi_2(s)ds\right\} = \frac{-483264 + 103739\pi^2 + 89136\pi^4}{6144\pi^2(-151 + 114\pi^2)},$$

$m \approx 1.8624, M \approx 6.4045$.

Let $f(t,x_0,x_1,x_2,x_3) = d\big(x_0^{k_0} + (-x_1)^{k_1} + (-x_2)^{k_2} + x_3^2\big)$ for $(t,x_0,x_1,x_2,x_3) \in [0,1] \times [0,\infty) \times (-\infty,0]^3$, here $k_i > 1$ $(i = 0,1,2)$, and $d > 0$ is a constant which is determined by the next step. Clearly (C_1) holds. For a given $r_1 > 0$, choosing $d_0 > 0$ and d sufficiently small such that

$$d\left(r_1^{k_0} + r_1^{k_1} + r_1^{k_2} + \left(\left(d_0 + \left(r_1^{k_0} + r_1^{k_1} + r_1^{k_2}\right)d\right)\exp(dr_1)\right)^2\right) < mr_1,$$

we have that (2.5) and (2.7) are satisfied with $g(x_0,x_1,x_2) = x_0^{k_0} + (-x_1)^{k_1} + (-x_2)^{k_2}$. Choosing r_2 large enough such that $r_2 > r_1/\gamma$ and $r_2^{k_i - 1} > Md^{-1}\gamma^{-k_i}$ $(i = 0,1,2)$, we have that for $(t,x_0,x_1,x_2,x_3) \in W_{1,i}$ (see Theorem 2.5),

$$f(t,x_0,x_1,x_2,x_3) \geq d\,(r_2\gamma)^{k_i} > Mr_2 \qquad (i = 0,1,2),$$

i.e., (2.8) is satisfied. By Theorem 2.5 the BVP (2.11) has at least one positive solution. Of course 0 is also a solution of this problem. Especially, if $r_1 = 0.01, d_0 = 0.01$ and $k_0 = k_1 = k_2 = 2$, we may take $d = 20$.

Example 2.7. Consider BVP (2.11) with $f(t,x_0,x_1,x_2,x_3) = d\left(x_0^{k_0} + (-x_1)^{k_1} + (-x_2)^{k_2} + x_3^2\right)$ for $(t,x_0,x_1,x_2,x_3) \in [0,1] \times [0,\infty) \times (-\infty,0]^3$, here $k_i \in (0,1)$ $(i = 0,1,2)$, and $d > 0$ is a constant which is determined by the next step. Clearly (C_1) holds. For a given $r_2 > 0$, choosing $d_0 > 0$ and d sufficiently small such that

$$d\left(r_2^{k_0} + r_2^{k_1} + r_2^{k_2} + \left(\left(d_0 + \left(r_2^{k_0} + r_2^{k_1} + r_2^{k_2}\right)d\right)\exp(dr_2)\right)^2\right) < mr_2,$$

we have that (2.5) and (2.9) are satisfied with $g(x_0,x_1,x_2) = x_0^{k_0} + (-x_1)^{k_1} + (-x_2)^{k_2}$. Choosing r_1 small enough such that $r_1 < mr_2M^{-1}$ and $r_1^{1-k_i} < d\gamma^{k_i}M^{-1}$ $(i = 0,1,2)$, we have that for $(t,x_0,x_1,x_2,x_3) \in W_{2,i}$ (see Theorem 2.5),

$$f(t,x_0,x_1,x_2,x_3) \geq d\,(r_1\gamma)^{k_i} > Mr_1 \qquad (i = 0,1,2),$$

i.e., (2.10) is satisfied. By Theorem 2.5 the BVP (2.11) has at least one positive solution. Of course 0 is also a solution of this problem. Especially, if $r_2 = 1, d_0 = 0.01$ and $k_0 = k_1 = k_2 = 1/2$, we may take $d = 7/20$.

Remark 2.8. For f as in Example 2.7, if $x_0 = x_1 = x_2 = 0, x_3 \to -\infty$,

$$f(t,x_0,x_1,x_2,x_3) \leq a_0x_0 - a_1x_1 - a_2x_2 - a_3x_3 + C_0$$

does not hold; if $x_0 \to 0^+, x_1 = x_2 = x_3 = 0$,

$$f(t,x_0,x_1,x_2,x_3) \leq b_0x_0 - b_1x_1 - b_2x_2 - b_3x_3$$

does not hold, where $a_i, b_i (i = 0,1,2,3)$ and C_0 are positive constants. Therefore, the conditions in [9, Theorem 2.1, Theorem 2.2] are not satisfied and the results in [9] can not be applied.

3 Positive solutions to the BVP (1.2)

For BVP (1.2)

$$\begin{cases} -u^{(4)}(t) = \widetilde{f}(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1], \\ u(0) = \beta_1[u], \ u'(0) = \beta_2[u], \ u''(0) = \beta_3[u], \ u'''(1) = 0, \end{cases}$$

we make the following assumptions:

$(\widetilde{C}_1)$ $\widetilde{f} : [0,1] \times [0,\infty)^4 \to [0,\infty)$ is continuous;

$(\widetilde{C}_2)$ B_i is of bounded variation, moreover

$$\widetilde{\mathcal{K}}_i(s) := \int_0^1 \widetilde{G}_0(t,s) dB_i(t) \geq 0, \qquad \forall s \in [0,1] \quad (i = 1,2,3),$$

where

$$\widetilde{G}_0(t,s) = \begin{cases} \frac{1}{6} t^3, & 0 \leq t \leq s \leq 1, \\ \frac{1}{6} s(3t^2 - 3ts + s^2), & 0 \leq s \leq t \leq 1; \end{cases}$$

$(\widetilde{C}_3)$ The 3×3 matrix $[B]$ is positive whose (i,j)th entry is $\beta_i[\delta_j]$, where $\delta_1(t) = 1$, $\delta_2(t) = t$ and $\delta_3(t) = \frac{1}{2}t^2$ are the solutions of $u^{(4)} = 0$ respectively subject to boundary conditions:

$$u'(0) = 1, \quad u''(0) = 0, \quad u(1) = 0, \quad u'''(1) = 0;$$

$$u'(0) = 0, \quad u''(0) = 1, \quad u(1) = 0, \quad u'''(1) = 0;$$

$$u'(0) = 0, \quad u''(0) = 0, \quad u(1) = 1, \quad u'''(1) = 0.$$

Furthermore assume that its spectral radius $r([B]) < 1$.

Define the operator $\widetilde{S}$ as

$$(\widetilde{S}u)(t) = \int_0^1 G_2(t,s) \widetilde{f}(s, u(s), u'(s), u''(s), u'''(s)) ds,$$

where

$$G_2(t,s) = \langle (I - [B])^{-1} \widetilde{\mathcal{K}}(s), \delta(t) \rangle + \widetilde{G}_0(t,s) = \sum_{i=1}^{3} \widetilde{\kappa}_i(s) \delta_i(t) + \widetilde{G}_0(t,s),$$

$\langle (I - [B])^{-1} \widetilde{\mathcal{K}}(s), \delta(t) \rangle$ is the inner product in $\mathbb{R}^3$, $\widetilde{\kappa}_i(s)$ is the ith component of $(I - [B])^{-1} \widetilde{\mathcal{K}}(s)$.

Lemma 3.1. *If $(\widetilde{C}_2)$ and $(\widetilde{C}_3)$ hold, then $\widetilde{\kappa}_i(s) \geq 0$ $(i = 1,2,3)$ and, for $t, s \in [0,1]$,*

$$\widetilde{c}_0(t) \widetilde{\Phi}_0(s) \leq G_2(t,s) \leq \widetilde{\Phi}_0(s), \tag{3.1}$$

where

$$\widetilde{\Phi}_0(s) = \sum_{i=1}^{3} \widetilde{\kappa}_i(s) + \frac{1}{6} s^3 + \frac{1}{2} s(1-s), \qquad \widetilde{c}_0(t) = \frac{1}{2} t^3,$$

and

$$\widetilde{c}_1(t) \widetilde{\Phi}_1(s) \leq \frac{\partial G_2(t,s)}{\partial t} \leq \widetilde{\Phi}_1(s), \qquad \widetilde{c}_2(t) \widetilde{\Phi}_2(s) \leq \frac{\partial^2 G_2(t,s)}{\partial t^2} \leq \widetilde{\Phi}_2(s), \tag{3.2}$$

where

$$\frac{\partial G_2(t,s)}{\partial t} = \widetilde{\kappa}_2(s) + t\widetilde{\kappa}_3(s) + \frac{1}{2}\begin{cases} t^2, & 0 \le t \le s \le 1, \\ s(2t-s), & 0 \le s \le t \le 1, \end{cases}$$

$$\frac{\partial^2 G_2(t,s)}{\partial t^2} = \widetilde{\kappa}_3(s) + \begin{cases} t, & 0 \le t \le s \le 1, \\ s, & 0 \le s \le t \le 1, \end{cases}$$

$$\widetilde{\Phi}_1(s) = \sum_{i=2}^{3} \widetilde{\kappa}_i(s) + \frac{1}{2}s(2-s), \qquad \widetilde{c}_1(t) = t^2, \qquad \widetilde{\Phi}_2(s) = \widetilde{\kappa}_3(s) + s, \qquad \widetilde{c}_2(t) = t.$$

Proof. $\widetilde{\kappa}_i(s) \ge 0$ $(i = 1,2,3)$ by hypotheses $(\widetilde{C}_2)$ and $(\widetilde{C}_3)$. For $0 \le s \le t \le 1$, $\frac{\partial}{\partial t}\widetilde{G}_0(t,s) = \frac{1}{2}s(2t-s) \ge 0$ which implies that

$$\widetilde{G}_0(t,s) \le \widetilde{G}_0(1,s) = \frac{1}{6}s^3 + \frac{1}{2}s(1-s);$$

For $0 \le t < s \le 1$, $\frac{\partial}{\partial t}\widetilde{G}_0(t,s) = \frac{1}{2}t^2 \ge 0$ which implies that

$$\widetilde{G}_0(t,s) \le \widetilde{G}_0(s,s) = \frac{1}{6}s^3.$$

Then $\widetilde{G}_0(t,s) \le \frac{1}{6}s^3 + \frac{1}{2}s(1-s)$, $\forall (t,s) \in [0,1] \times [0,1]$.

Now we find the best function $\widetilde{C}_0(t)$ such that $\widetilde{G}_0(t,s) \ge \widetilde{C}_0(t)\left(\frac{1}{6}s^3 + \frac{1}{2}s(1-s)\right)$, $\forall (t,s) \in [0,1] \times [0,1]$.

For $0 \le s \le t \le 1$, this is

$$\frac{1}{6}s(3t^2 - 3ts + s^2) \ge \widetilde{C}_0(t)\left(\frac{1}{6}s^3 + \frac{1}{2}s(1-s)\right),$$

thus

$$\widetilde{C}_0(t) \le \frac{3t^2 - 3ts + s^2}{3 - 3s + s^2}.$$

Denote

$$\widetilde{g}_1(t,s) = \frac{3t^2 - 3ts + s^2}{3 - 3s + s^2},$$

from

$$\frac{\partial}{\partial s}\widetilde{g}_1(t,s) = \frac{3(t-1)(s^2 - 2s(1+t) + 3t)}{(3 - 3s + s^2)^2} \le 0$$

it follows that $\widetilde{C}_0(t) \le \widetilde{g}_1(t,t) = \frac{t^2}{3-3t+t^2}$.

For $0 \le t < s \le 1$, this is

$$\frac{1}{6}t^3 \ge \widetilde{C}_0(t)\left(\frac{1}{6}s^3 + \frac{1}{2}s(1-s)\right),$$

thus

$$\widetilde{C}_0(t) \le \frac{t^3}{s(3 - 3s + s^2)}.$$

Denote

$$\widetilde{g}_2(t,s) = \frac{1}{s(3 - 3s + s^2)},$$

from

$$\frac{\partial}{\partial s}\widetilde{g}_2(t,s) = -\frac{3(1-s)^2}{s^2(3 - 3s + s^2)^2} \le 0$$

it follows that $\widetilde{C}_0(t) \le t^3 \widetilde{g}_2(t,1) = t^3$.

Therefore

$$\widetilde{C}_0(t) = \min\left\{\frac{t^2}{3-3t+t^2},\ t^3\right\} = t^3.$$

Since

$$\frac{1}{2}t^3\sum_{i=1}^{3}\widetilde{\kappa}_i(s) \le \frac{1}{2}t^2\sum_{i=1}^{3}\widetilde{\kappa}_i(s) \le \sum_{i=1}^{3}\widetilde{\kappa}_i(s)\delta_i(t) \le \sum_{i=1}^{3}\widetilde{\kappa}_i(s),$$

$$\frac{1}{2}t^3\left(\frac{1}{6}s^3+\frac{1}{2}s(1-s)\right) \le t^3\left(\frac{1}{6}s^3+\frac{1}{2}s(1-s)\right) \le \widetilde{G}_0(t,s) \le \frac{1}{6}s^3+\frac{1}{2}s(1-s),$$

we know that (3.1) holds. As for (3.2), it comes directly from the inequalities

$$t^2\sum_{i=2}^{3}\widetilde{\kappa}_i(s) \le t\sum_{i=2}^{3}\widetilde{\kappa}_i(s) \le \sum_{i=1}^{3}\widetilde{\kappa}_i(s)\delta_i'(t) \le \sum_{i=2}^{3}\widetilde{\kappa}_i(s),$$

$$\frac{1}{2}t^2 s(2-s) \le \frac{\partial \widetilde{G}_0(t,s)}{\partial t} \le \frac{1}{2}s(2-s),$$

$$t\widetilde{\kappa}_3(s) \le \widetilde{\kappa}_3(s) = \sum_{i=1}^{3}\widetilde{\kappa}_i(s)\delta_i''(t), \qquad ts \le \frac{\partial^2 \widetilde{G}_0(t,s)}{\partial t^2} \le s$$

for $t,s \in [0,1]$. □

In $C^3[0,1]$ we define the cone

$$\widetilde{K} = \big\{u \in C^3[0,1] : u(t) \ge \widetilde{c}_0(t)\|u\|_C,\ u'(t) \ge \widetilde{c}_1(t)\|u'\|_C,\\ u''(t) \ge \widetilde{c}_2(t)\|u''\|_C,\ \forall t \in [0,1];\ u'''(1) = 0\big\}. \tag{3.3}$$

Lemma 3.2. *If $(\widetilde{C}_1)$–$(\widetilde{C}_3)$ hold, then $\widetilde{S} : \widetilde{K} \to \widetilde{K}$ is completely continuous and the positive solutions to BVP (1.2) are equivalent to the fixed points of $\widetilde{S}$ in $\widetilde{K}$.*

Proof. Because $G_2(t,s)$, and the first- and second-order derivatives are continuous, the third-order derivative is integrable in s, from Lemma 3.1 it is easy to prove that $\widetilde{S} : \widetilde{K} \to \widetilde{K}$ is continuous. Let F be a bounded set in $\widetilde{K}$, then there exists $M > 0$ such that $\|u\|_{C^3} \le M$ for all $u \in \widetilde{K}$. Denote

$$C = \max_{(t,x_0,x_1,x_2,x_3)\in[0,1]\times[0,M]^4} \widetilde{f}(t,x_0,x_1,x_2,x_3).$$

By $(\widetilde{C}_1)$ and Lemma 3.1 we have that $\forall u \in F$ and $t \in [0,1]$,

$$|(\widetilde{S}u)(t)| \le C\int_0^1 \widetilde{\Phi}_0(s)ds, \qquad |(\widetilde{S}u)'(t)| \le C\int_0^1 \left|\frac{\partial G_2(t,s)}{\partial t}\right|ds \le C\int_0^1 \widetilde{\Phi}_1(s)ds,$$

$$|(\widetilde{S}u)''(t)| \le C\int_0^1 \left|\frac{\partial^2 G_2(t,s)}{\partial t^2}\right|ds \le C\int_0^1 \widetilde{\Phi}_2(s)ds, \qquad |(\widetilde{S}u)'''(t)| \le C\int_0^1 \left|\frac{\partial^3 G_2(t,s)}{\partial t^3}\right|ds \le C,$$

then $\widetilde{S}(F)$ is uniformly bounded in $C^3[0,1]$. Moreover $\forall u \in F$ and $t_1,t_2 \in [0,1]$ with $t_1 < t_2$,

$$\begin{aligned}|(\widetilde{S}u)(t_1) - (\widetilde{S}u)(t_2)| &\le \int_0^1 |G_2(t_1,s) - G_2(t_2,s)|\widetilde{f}(s,u(s),u'(s),u''(s),u'''(s))ds\\ &\le C\int_0^1 |G_2(t_1,s) - G_2(t_2,s)|ds,\end{aligned}$$

$$|(\widetilde{S}u)'(t_1)-(\widetilde{S}u)'(t_2)| \le \int_0^1 \left|\frac{\partial G_2}{\partial t}(t_1,s)-\frac{\partial G_2}{\partial t}(t_2,s)\right| \widetilde{f}(s,u(s),u'(s),u''(s),u'''(s))ds$$
$$\le C\int_0^1 \left|\frac{\partial G_2}{\partial t}(t_1,s)-\frac{\partial G_2}{\partial t}(t_2,s)\right| ds,$$

$$|(\widetilde{S}u)''(t_1)-(\widetilde{S}u)''(t_2)| \le \int_0^1 \left|\frac{\partial^2 G_2}{\partial t^2}(t_1,s)-\frac{\partial^2 G_2}{\partial t^2}(t_2,s)\right| \widetilde{f}(s,u(s),u'(s),u''(s),u'''(s))ds$$
$$\le C\int_0^1 \left|\frac{\partial^2 G_2}{\partial t^2}(t_1,s)-\frac{\partial^2 G_2}{\partial t^2}(t_2,s)\right| ds,$$

$$|(\widetilde{S}u)'''(t_1)-(Su)'''(t_2)|$$
$$=\left|\int_{t_1}^1 \widetilde{f}(s,u(s),u'(s),u''(s),u'''(s))ds-\int_{t_2}^1 \widetilde{f}(s,u(s),u'(s),u''(s),u'''(s))ds\right| \le C(t_2-t_1),$$

thus $\widetilde{S}(F)$ and $\widetilde{S}^{(i)}(F) =: \{v^{(i)} : v^{(i)}(t) = (\widetilde{S}u)^{(i)}(t), u \in F\}$ $(i = 1,2,3)$ are equicontinuous.

Therefore $\widetilde{S} : \widetilde{K} \to \widetilde{K}$ is completely continuous by the Arzelà–Ascoli theorem. Similar to [14], the positive solutions to BVP (1.2) are equivalent to the fixed points of $\widetilde{S}$ in $\widetilde{K}$. □

Lemma 3.3. *Suppose that* $(\widetilde{C}_1)$–$(\widetilde{C}_3)$ *hold, there exist constants* $\widetilde{p}_0 > 0, \widetilde{p}_3 \ge 0$ *and functions* $\widetilde{p}_1, \widetilde{p}_2 \in L^1_+[0,1]$ *such that for all* $(t,x_0,x_1,x_2,x_3) \in [0,1]\times[0,\infty)^4$,

$$\widetilde{f}(t,x_0,x_1,x_2,x_3) \le \widetilde{p}_0+\widetilde{p}_1(t)\widetilde{g}(x_0,x_1,x_2)+\widetilde{p}_2(t)x_3+\widetilde{p}_3x_3^2, \tag{3.4}$$

where $\widetilde{g} : [0,\infty)^3 \to [0,\infty)$ *is continuous, non-decreasing in every variable. Let* $\lambda \ge 1, \sigma \ge 0, r > 0$, *define* $\widetilde{D}_i := \int_0^1 \widetilde{p}_i(s)ds$ $(i = 1,2)$ *and*

$$\widetilde{Q}(r) := (\widetilde{p}_0+\widetilde{g}(r,r,r)\widetilde{D}_1)\exp(\widetilde{D}_2)\exp(\widetilde{p}_3 r). \tag{3.5}$$

If $u \in \widetilde{K}$ *with* $\|u\|_{C^2} \le r$ *such that* $\lambda u(t) = (\widetilde{S}u)(t)+\sigma$, *then* $\|u'''\|_C \le \widetilde{Q}(r)$.

Let $[a,b]$ be a subset of $(0,1)$ and denote

$$\widetilde{\gamma} := \min\left\{\min_{t\in[a,b]}\widetilde{c}_0(t), \min_{t\in[a,b]}\widetilde{c}_1(t), \min_{t\in[a,b]}\widetilde{c}_2(t)\right\} = \frac{1}{2}a^3,$$
$$\frac{1}{\widetilde{m}} := \max\left\{\int_0^1 \widetilde{\Phi}_0(s)ds, \int_0^1 \widetilde{\Phi}_1(s)ds, \int_0^1 \widetilde{\Phi}_2(s)ds\right\},$$
$$\frac{1}{\widetilde{M}} := \min\left\{\int_a^b \widetilde{\Phi}_0(s)ds, \int_a^b \widetilde{\Phi}_1(s)ds, \int_a^b \widetilde{\Phi}_2(s)ds\right\},$$

where $\widetilde{c}_i(t)$ and $\widetilde{\Phi}_i(s) (i = 0,1,2)$ are provided in Lemma 3.1. Obviously, $\widetilde{\gamma} \in (0,1/2)$ and $\widetilde{m} < \widetilde{M}$.

Similar to the proof of Theorem 2.5, we have the next theorem.

Theorem 3.4. *Suppose that* $(\widetilde{C}_1)$-$(\widetilde{C}_3)$ *hold and* $\widetilde{f}$ *satisfies the growth assumption* (3.4). *The BVP* (1.2) *has at least one positive solution* $u \in \widetilde{K}$ *either of the following conditions* $(\widetilde{F}_1)$, $(\widetilde{F}_2)$ *holds, where* $\widetilde{Q}$ *is given by* (3.5).

$(\widetilde{F}_1)$ *There exist* $0 < r_1 < r_2$ *with* $r_1 < r_2\widetilde{\gamma}$, *such that*

$(\widetilde{F}_1 a)$ *for* $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0, r_1]^3 \times [0, \widetilde{Q}(r_1)]$,

$$\widetilde{f}(t, x_0, x_1, x_2, x_3) < \widetilde{m} r_1; \tag{3.6}$$

$(\widetilde{F}_1 b)$ *for* $(t, x_0, x_1, x_2, x_3) \in \widetilde{W}_1 := \widetilde{W}_{1,0} \cup \widetilde{W}_{1,1} \cup \widetilde{W}_{1,2}$,

$$\widetilde{f}(t, x_0, x_1, x_2, x_3) > \widetilde{M} r_2, \tag{3.7}$$

where

$$\begin{aligned}
\widetilde{W}_{1,0} &= [a,b] \times [r_2\widetilde{\gamma}, r_2] \times [0, r_2]^2 \times [0, \widetilde{Q}(r_2)],\\
\widetilde{W}_{1,1} &= [a,b] \times [0, r_2] \times [r_2\widetilde{\gamma}, r_2] \times [0, r_2] \times [0, \widetilde{Q}(r_2)],\\
\widetilde{W}_{1,2} &= [a,b] \times [0, r_2]^2 \times [r_2\widetilde{\gamma}, r_2] \times [0, \widetilde{Q}(r_2)].
\end{aligned}$$

$(\widetilde{F}_2)$ *There exist* $0 < r_1 < r_2$ *with* $\widetilde{M} r_1 < \widetilde{m} r_2$, *such that*

$(\widetilde{F}_2 a)$ *for* $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0, r_2]^3 \times [0, \widetilde{Q}(r_2)]$,

$$\widetilde{f}(t, x_0, x_1, x_2, x_3) < \widetilde{m} r_2; \tag{3.8}$$

$(\widetilde{F}_2 b)$ *for* $(t, x_0, x_1, x_2, x_3) \in \widetilde{W}_2 := \widetilde{W}_{2,0} \cup \widetilde{W}_{2,1} \cup \widetilde{W}_{2,2}$,

$$\widetilde{f}(t, x_0, x_1, x_2, x_3) > \widetilde{M} r_1, \tag{3.9}$$

where

$$\begin{aligned}
\widetilde{W}_{2,0} &= [a,b] \times [r_1\widetilde{\gamma}, r_1] \times [0, r_1]^2 \times [0, \widetilde{Q}(r_1)],\\
\widetilde{W}_{2,1} &= [a,b] \times [0, r_1] \times [r_1\widetilde{\gamma}, r_1] \times [0, r_1] \times [0, \widetilde{Q}(r_1)],\\
\widetilde{W}_{2,2} &= [a,b] \times [0, r_1]^2 \times [r_1\widetilde{\gamma}, r_1] \times [0, \widetilde{Q}(r_1)].
\end{aligned}$$

Example 3.5. Consider

$$\begin{cases}
-u^{(4)}(t) = \widetilde{f}(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1],\\
u(0) = \frac{1}{2}u(\frac{1}{4}) - \frac{1}{160}u(\frac{3}{4}), \quad u'(0) = \int_0^1 (t - \frac{1}{8})u(t)dt,\\
u''(0) = \frac{1}{2}u(\frac{1}{2}) - \frac{1}{14}u(\frac{3}{4}), \quad u'''(1) = 0,
\end{cases} \tag{3.10}$$

thus $\beta_1[u] = \frac{1}{2}u(\frac{1}{4}) - \frac{1}{160}u(\frac{3}{4})$, $\beta_2[u] = \int_0^1 (t - \frac{1}{8})u(t)dt$, $\beta_3[u] = \frac{1}{2}u(\frac{1}{2}) - \frac{1}{14}u(\frac{3}{4})$. Then

$$\begin{aligned}
0 \le \widetilde{\mathcal{K}}_1(s) &= \tfrac{1}{2}\widetilde{G}_0\left(\tfrac{1}{4}, s\right) - \tfrac{1}{160}\widetilde{G}_0\left(\tfrac{3}{4}, s\right)\\
&= \begin{cases}
\frac{79}{960}s^3 - \frac{77}{1280}s^2 + \frac{71}{5120}s, & 0 \le s \le \frac{1}{4},\\
\frac{1}{768} - \frac{9}{5120}s + \frac{3}{1280}s^2 - \frac{1}{960}s^3, & \frac{1}{4} < s \le \frac{3}{4},\\
\frac{53}{61440}, & \frac{3}{4} < s \le 1,
\end{cases}
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathcal{K}}_2(s) &= \frac{1}{6}\int_0^s \left(t - \frac{1}{8}\right) t^3 dt + \frac{1}{6}\int_s^1 \left(t - \frac{1}{8}\right) s(3t^2 - 3ts + s^2)dt\\
&= \frac{1}{960}s\left(100 - 130s + 60s^2 + 5s^3 - 8s^4\right) \ge 0 \qquad (0 \le s \le 1),
\end{aligned}$$

$$0 \le \widetilde{\mathcal{K}}_3(s) = \frac{1}{2}\widetilde{G}_0\left(\frac{1}{2}, s\right) - \frac{1}{14}\widetilde{G}_0\left(\frac{3}{4}, s\right)$$
$$= \begin{cases} \frac{1}{14}s^3 - \frac{11}{112}s^2 + \frac{19}{448}s, & 0 \le s \le \frac{1}{2}, \\ \frac{1}{96} - \frac{9}{448}s + \frac{3}{112}s^2 - \frac{1}{84}s^3, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{29}{5376}, & \frac{3}{4} < s \le 1, \end{cases}$$

the 3×3 matrix

$$[B] = \begin{pmatrix} \beta_1[\delta_1] & \beta_1[\delta_2] & \beta_1[\delta_3] \\ \beta_2[\delta_1] & \beta_2[\delta_2] & \beta_2[\delta_3] \\ \beta_3[\delta_1] & \beta_3[\delta_2] & \beta_3[\delta_3] \end{pmatrix} = \begin{pmatrix} \frac{79}{160} & \frac{77}{640} & \frac{71}{5120} \\ \frac{3}{8} & \frac{13}{48} & \frac{5}{48} \\ \frac{3}{7} & \frac{11}{56} & \frac{19}{448} \end{pmatrix}$$

and its spectral radius $r([B]) \approx 0.6600 < 1$. Therefore, $(\widetilde{C}_2)$ and $(\widetilde{C}_3)$ hold. We choose $[a,b] = [1/4, 3/4]$ and note that $\widetilde{\gamma} = 1/128$,

$$\widetilde{\kappa}_1(s) = \begin{cases} \frac{s(176400-459360s+504520s^2+4785s^3-7656s^4)}{2252880}, & 0 \le s \le \frac{1}{4}, \\ \frac{6875+93900s-129360s^2+64520s^3+4785s^4-7656s^5}{2252880}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{17425+165750s-214620s^2+99640s^3+9570s^4-15312s^5}{4505760}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{22025+3828s(100-130s+60s^2+5s^3-8s^4)}{9011520}, & \frac{3}{4} < s \le 1, \end{cases}$$

$$\widetilde{\kappa}_2(s) = \begin{cases} \frac{s(3986640-6524820s+4630400s^2+171615s^3-274584s^4)}{19900440}, & 0 \le s \le \frac{1}{4}, \\ \frac{36175+3552540s-4788420s^2+2315200s^3+171615s^4-274584s^5}{19900440}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{155405+6606750s-8580180s^2+3965960s^3+343230s^4-549168s^5}{39800880}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{181885+137292s(100-130s+60s^2+5s^3-8s^4)}{79601760}, & \frac{3}{4} < s \le 1, \end{cases}$$

$$\widetilde{\kappa}_3(s) = \begin{cases} \frac{s(299565-649440s+553600s^2+6765s^3-10824s^4)}{2487555}, & 0 \le s \le \frac{1}{4}, \\ \frac{4325+247665s-441840s^2+276800s^3+6765s^4-10824s^5}{2487555}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{66715+146940s-186900s^2+89080s^3+13530s^4-21648s^5}{4975110}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{17900+1353s(100-130s+60s^2+5s^3-8s^4)}{2487555}, & \frac{3}{4} < s \le 1, \end{cases}$$

and hence

$$\int_0^1 \widetilde{\Phi}_0(s)ds = \frac{1481629721}{7641768960}, \quad \int_0^1 \widetilde{\Phi}_1(s)ds = \frac{3339971}{8547840}, \quad \int_0^1 \widetilde{\Phi}_2(s)ds = \frac{41265293}{79601760},$$

$$\frac{1}{\widetilde{m}} = \max\left\{\int_0^1 \widetilde{\Phi}_0(s)ds, \int_0^1 \widetilde{\Phi}_1(s)ds, \int_0^1 \widetilde{\Phi}_2(s)ds\right\} = \frac{41265293}{79601760},$$

$$\int_{1/4}^{3/4} \widetilde{\Phi}_0(s)ds = \frac{6666545149}{61134151680}, \quad \int_{1/4}^{3/4} \widetilde{\Phi}_1(s)ds = \frac{14676709}{68382720}, \quad \int_{1/4}^{3/4} \widetilde{\Phi}_2(s)ds = \frac{331536539}{1273628160},$$

$$\frac{1}{\widetilde{M}} = \min\left\{\int_{1/4}^{3/4} \widetilde{\Phi}_0(s)ds, \int_{1/4}^{3/4} \widetilde{\Phi}_1(s)ds, \int_{1/4}^{3/4} \widetilde{\Phi}_2(s)ds\right\} = \frac{331536539}{1273628160},$$

$\widetilde{m} \approx 1.9290$, $\widetilde{M} \approx 9.1703$.

Let $\widetilde{f}(t, x_0, x_1, x_2, x_3) = \widetilde{d}\left(x_0^{k_0} + x_1^{k_1} + x_2^{k_2} + x_3^2\right)$, $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0,\infty)^4$, here $k_i > 1$ $(i = 0,1,2)$, and $\widetilde{d} > 0$ is a constant which is determined by the next step. Clearly $(\widetilde{C}_1)$ holds. For a given $r_1 > 0$, choosing $\widetilde{d}_0 > 0$ and $\widetilde{d}$ sufficiently small such that

$$\widetilde{d}\left(r_1^{k_0} + r_1^{k_1} + r_1^{k_2} + \left(\left(\widetilde{d}_0 + \left(r_1^{k_0} + r_1^{k_1} + r_1^{k_2}\right)\widetilde{d}\right)\exp\left(\widetilde{d}r_1\right)\right)^2\right) < \widetilde{m}r_1,$$

we have that (3.4) and (3.6) are satisfied with $\widetilde{g}(x_0, x_1, x_2) = x_0^{k_0} + x_1^{k_1} + x_2^{k_2}$. Choosing r_2 large enough such that $r_2 > r_1/\widetilde{\gamma}$ and $r_2^{k_i - 1} > \widetilde{M}\widetilde{d}^{-1}\widetilde{\gamma}^{-k_i}$ $(i = 0,1,2)$, we have that for $(t, x_0, x_1, x_2, x_3) \in \widetilde{W}_{1,i}$ (see Theorem 3.4),

$$f(t, x_0, x_1, x_2, x_3) \geq \widetilde{d}\,(r_2\widetilde{\gamma})^{k_i} > \widetilde{M}r_2 \qquad (i = 0,1,2),$$

i.e., (3.7) is satisfied. By Theorem 3.4 the BVP (3.10) has at least one positive solution. Of course 0 is also a solution of this problem. Especially, if $r_1 = 0.01, \widetilde{d}_0 = 0.01$ and $k_0 = k_1 = k_2 = 2$, we may take $\widetilde{d} = 23$.

Example 3.6. Consider (3.10) with $\widetilde{f}(t, x_0, x_1, x_2, x_3) = \widetilde{d}(x_0^{k_0} + x_1^{k_1} + x_2^{k_2} + x_3^2)$, $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0,\infty)^4$, here $k_i \in (0,1)$ $(i = 0,1,2)$, and $\widetilde{d} > 0$ is a constant which is determined by the next step. Clearly $(\widetilde{C}_1)$ holds. For a given $r_2 > 0$, choosing $\widetilde{d}_0 > 0$ and $\widetilde{d}$ sufficiently small such that

$$\widetilde{d}\left(r_2^{k_0} + r_2^{k_1} + r_2^{k_2} + \left(\left(\widetilde{d}_0 + \left(r_2^{k_0} + r_2^{k_1} + r_2^{k_2}\right)\widetilde{d}\right)\exp\left(\widetilde{d}r_2\right)\right)^2\right) < \widetilde{m}r_2,$$

we have that (3.4) and (3.8) are satisfied with $\widetilde{g}(x_0, x_1, x_2) = x_0^{k_0} + x_1^{k_1} + x_2^{k_2}$. Choosing r_1 small enough such that $r_1 < \widetilde{m}r_2\widetilde{M}^{-1}$ and $r_1^{1-k_i} < \widetilde{d}\widetilde{\gamma}^{k_i}\widetilde{M}^{-1}$ $(i = 0,1,2)$, we have that for $(t, x_0, x_1, x_2, x_3) \in \widetilde{W}_{2,i}$ (see Theorem 3.4),

$$f(t, x_0, x_1, x_2, x_3) \geq \widetilde{d}\,(r_1\widetilde{\gamma})^{k_i} > \widetilde{M}r_1 \qquad (i = 0,1,2),$$

i.e., (3.9) is satisfied. By Theorem 3.4 the BVP (3.10) has at least one positive solution. Of course 0 is also a solution of this problem. Especially, if $r_2 = 1, \widetilde{d}_0 = 0.01$ and $k_0 = k_1 = k_2 = 1/2$, we may take $\widetilde{d} = 7/20$.

Remark 3.7. For $\widetilde{f}$ as in Example 3.6, if $x_0 = x_1 = x_2 = 0, x_3 \to +\infty$,

$$\widetilde{f}(t, x_0, x_1, x_2, x_3) \leq \widetilde{a}_0x_0 + \widetilde{a}_1x_1 + \widetilde{a}_2x_2 + \widetilde{a}_3x_3 + \widetilde{C}_0$$

does not hold; if $x_0 \to 0^+, x_1 = x_2 = x_3 = 0$,

$$\widetilde{f}(t, x_0, x_1, x_2, x_3) \leq \widetilde{b}_0x_0 + \widetilde{b}_1x_1 + \widetilde{b}_2x_2 + \widetilde{b}_3x_3$$

does not hold, where $\widetilde{a}_i, \widetilde{b}_i (i = 0,1,2,3)$ and $\widetilde{C}_0$ are positive constants. Therefore, the conditions in [9, Theorem 3.1, Theorem 3.2] are not satisfied and the results in [9] can not be applied.

4 Positive Solutions to the BVP (1.3)

For BVP (1.3)

$$\begin{cases} u^{(4)}(t) = \overline{f}(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1], \\ u(0) = u(1) = \eta_1[u], \ u''(0) + \eta_2[u] = 0, \ u''(1) + \eta_2[u] = 0, \end{cases}$$

we make the following assumptions:

$(\overline{C}_1)$ $\overline{f} : [0,1] \times [0,\infty) \times (-\infty,\infty) \times (-\infty,0] \times (-\infty,\infty) \to [0,\infty)$ is continuous;

$(\overline{C}_2)$ H_i is of bounded variation, moreover

$$\overline{\mathcal{K}}_i(s) := \int_0^1 \overline{G}_0(t,s) dH_i(t) \geq 0, \ \forall s \in [0,1] (i = 1,2),$$

where

$$\overline{G}_0(t,s) = \begin{cases} \frac{1}{6} t(1-s)(2s - t^2 - s^2), & 0 \leq t \leq s \leq 1, \\ \frac{1}{6} s(1-t)(2t - s^2 - t^2), & 0 \leq s \leq t \leq 1; \end{cases}$$

$(\overline{C}_3)$ The 2×2 matrix $[H]$ is positive whose (i,j)th entry is $\eta_i[\xi_j]$, where $\xi_1(t) = 1$ and $\xi_2(t) = \frac{1}{2} t(1-t)$ are the solutions of $u^{(4)} = 0$ respectively subject to boundary conditions:

$$u(0) = u(1) = 1, \quad u''(0) = u''(1) = 0;$$
$$u(0) = u(1) = 0, \quad u''(0) = u''(1) = -1.$$

Furthermore assume that its spectral radius $r([H]) < 1$.

Define the operator $\overline{S}$ as

$$(\overline{S}u)(t) = \int_0^1 G_3(t,s) \overline{f}(s, u(s), u'(s), u''(s), u'''(s)) ds,$$

where

$$G_3(t,s) = \langle (I - [H])^{-1} \overline{\mathcal{K}}(s), \xi(t) \rangle + \overline{G}_0(t,s) = \sum_{i=1}^{2} \overline{\kappa}_i(s) \xi_i(t) + \overline{G}_0(t,s),$$

$\langle (I-[H])^{-1}\overline{\mathcal{K}}(s), \xi(t)\rangle$ is the inner product in $\mathbb{R}^2$, $\overline{\kappa}_i(s)$ is the ith component of $(I-[H])^{-1}\overline{\mathcal{K}}(s)$.

Lemma 4.1. *If $(\overline{C}_2)$ and $(\overline{C}_3)$ hold, then $\overline{\kappa}_i(s) \geq 0$ $(i = 1,2)$,*

$$G_3(0,s) = G_3(1,s) = \overline{\kappa}_1(s), \qquad \frac{\partial^2 G_3(0,s)}{\partial t^2} = \frac{\partial^2 G_3(1,s)}{\partial t^2} = -\overline{\kappa}_2(s)$$

and for $t, s \in [0,1]$,

$$\overline{c}_0(t) \overline{\Phi}_0(s) \leq G_3(t,s) \leq \overline{\Phi}_0(s), \tag{4.1}$$

where

$$\overline{\Phi}_0(s) = \overline{\kappa}_1(s) + \frac{1}{8} \overline{\kappa}_2(s) + \widehat{\Phi}_0(s),$$

$$\overline{c}_0(t) = \begin{cases} \frac{3\sqrt{3}}{2} t(1-t^2), & 0 \leq t \leq \frac{1}{2}, \\ \frac{3\sqrt{3}}{2} t(1-t)(2-t), & \frac{1}{2} < t \leq 1, \end{cases}$$

$$\widehat{\Phi}_0(s) = \begin{cases} \frac{\sqrt{3}}{27} s(1-s^2)^{3/2}, & 0 \le s \le \frac{1}{2}, \\ \frac{\sqrt{3}}{27}(1-s)s^{3/2}(2-s)^{3/2}, & \frac{1}{2} < s \le 1; \end{cases}$$

and

$$\overline{c}_1(t)\overline{\Phi}_1(s) \le -\frac{\partial^2 G_3(t,s)}{\partial t^2} \le \overline{\Phi}_1(s) \tag{4.2}$$

where

$$\frac{\partial^2 G_3(t,s)}{\partial t^2} = -\overline{\kappa}_2(s) - \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1, \end{cases}$$

$$\overline{\Phi}_1(s) = \overline{\kappa}_2(s) + s(1-s), \qquad \overline{c}_1(t) = \min\{t, 1-t\}.$$

Proof. $\overline{\kappa}_i(s) \ge 0$ by hypotheses $(\overline{C}_2)$ and $(\overline{C}_3)$, and the following inequality is proved in [15]

$$\overline{c}_0(t)\widehat{\Phi}_0(s) \le \overline{G}_0(t,s) \le \widehat{\Phi}_0(s).$$

From

$$G_3(t,s) = \sum_{i=1}^{2} \overline{\kappa}_i(s)\xi_i(t) + \overline{G}_0(t,s) \le \overline{\kappa}_1(s) + \frac{1}{8}\overline{\kappa}_2(s) + \widehat{\Phi}_0(s) = \overline{\Phi}_0(s)$$

and

$$\begin{aligned} G_3(t,s) &= \overline{\kappa}_1(s) + \frac{1}{8} \times 4t(1-t)\overline{\kappa}_2(s) + \overline{G}_0(t,s) \\ &\ge 4t(1-t)\left(\overline{\kappa}_1(s) + \frac{1}{8}\overline{\kappa}_2(s)\right) + \overline{c}_0(t)\widehat{\Phi}_0(s) \\ &\ge \frac{9\sqrt{3}}{4}t(1-t)\left(\overline{\kappa}_1(s) + \frac{1}{8}\overline{\kappa}_2(s)\right) + \overline{c}_0(t)\widehat{\Phi}_0(s) \ge \overline{c}_0(t)\overline{\Phi}_0(s), \end{aligned}$$

it follows that (4.1) hold. As for (4.2), it can be checked easily. □

In $C^3[0,1]$ we define the cone

$$\begin{aligned} \overline{K} = \big\{u \in C^3[0,1] : u(t) \ge \overline{c}_0(t)\|u\|_C,\ -u''(t) \ge \overline{c}_1(t)\|u''\|_C,\ \forall t \in [0,1]; \\ u(0) = u(1),\ u''(0) = u''(1)\big\}. \end{aligned} \tag{4.3}$$

Lemma 4.2. *If $(\overline{C}_1)$–$(\overline{C}_3)$ hold, then $\overline{S} : \overline{K} \to \overline{K}$ is completely continuous and the positive solutions to BVP (1.3) are equivalent to the fixed points of $\overline{S}$ in $\overline{K}$.*

Lemma 4.3. *Suppose that $(\overline{C}_1)$-$(\overline{C}_3)$ hold, there exist constants $\overline{p}_0 > 0, \overline{p}_3 \ge 0$ and functions $\overline{p}_1, \overline{p}_2 \in L^1_+[0,1]$ such that for all $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0,\infty) \times (-\infty,\infty) \times (-\infty,0] \times (-\infty,\infty)$,*

$$\overline{f}(t, x_0, x_1, x_2, x_3) \le \overline{p}_0 + \overline{p}_1(t)\overline{g}(x_0, x_1, x_2) + \overline{p}_2(t)|x_3| + \overline{p}_3|x_3|^2, \tag{4.4}$$

where $\overline{g} : [0,\infty) \times (-\infty,\infty) \times (-\infty,0] \to [0,\infty)$ is continuous, non-decreasing in the first variable, even and non-decreasing in $[0,\infty)$ in the second variable, non-increasing in the third variable. Let $\lambda \ge 1, \sigma \ge 0, r > 0$, define $\overline{D}_i := \int_0^1 \overline{p}_i(s)ds$ $(i = 1,2)$ and

$$\overline{Q}(r) := (\overline{p}_0 + \overline{g}(r, r, -r)\overline{D}_1)\exp(\overline{D}_2)\exp(\overline{p}_3 r). \tag{4.5}$$

If $u \in \overline{K}$ with $\|u\|_{C^2} \le r$ such that $\lambda u(t) = (\overline{S}u)(t) + \sigma$, then $\|u'''\|_C \le \overline{Q}(r)$.

Proof. Since $u \in \overline{K}$, there exists $t_0 \in (0,1)$ such that $u'''(t_0) = 0$. From $\lambda u(t) = (\overline{S}u)(t) + \sigma$, we have that $\lambda u^{(4)}(t) = \overline{f}(t, u(t), u'(t), u''(t), u'''(t)) \geq 0$. Therefore, $u'''(t) \leq 0$ $(t \in [0, t_0])$, $u'''(t) \geq 0$ $(t \in [t_0, 1])$ and

$$\lambda u'''(t) = \int_{t_0}^{t} \overline{f}(s, u(s), u'(s), u''(s), u'''(s))ds \qquad (t \in [0,1]). \tag{4.6}$$

If $t \leq t_0$, from $\|u\|_{C^2} \leq r$ and (4.6) it follows that

$$\begin{aligned}
|u'''(t)| \leq \lambda |u'''(t)| &= \left| \int_{t_0}^{t} \overline{f}(s, u(s), u'(s), u''(s), u'''(s))ds \right| \\
&\leq \int_{t}^{t_0} \left(\overline{p}_0 + \overline{p}_1(s)\overline{g}(u(s), u'(s), u''(s)) + \overline{p}_2(s)|u'''(s)| + \overline{p}_3|u'''(s)|^2 \right) ds \\
&\leq (\overline{p}_0 + \overline{g}(r, r, -r)\overline{D}_1) + \int_{t}^{t_0} \left(\overline{p}_2(s)|u'''(s)| + \overline{p}_3|u'''(s)|^2 \right) ds.
\end{aligned}$$

Since

$$\int_0^{t_0} \overline{p}_3|u'''(s)|ds = -\int_0^{t_0} \overline{p}_3 u'''(s)ds = \overline{p}_3(u''(0) - u''(t_0)) \leq \overline{p}_3 r,$$

by Lemma 2.1 we deduce that

$$|u'''(t)| \leq (\overline{p}_0 + \overline{g}(r, r, -r)\overline{D}_1) \exp(\overline{D}_2) \exp(\overline{p}_3 r) = \overline{Q}(r), \qquad t \in [0, t_0].$$

If $t \geq t_0$, we change the variable from s to $\sigma = t_0 + 1 - s$. Denote $w(\sigma) = u(t_0 + 1 - \sigma)$ and then $w'(\sigma) = -u'(s)$, $w''(\sigma) = u''(s)$, $w'''(\sigma) = -u'''(s)$. Setting $\tau = t_0 + 1 - t$, from $\|u\|_{C^2} \leq r$ and (4.6) we have that

$$\begin{aligned}
&|w'''(\tau)| = |-u'''(t)| \leq \lambda|u'''(t)| = \left| \int_{t_0}^{t} \overline{f}(s, u(s), u'(s), u''(s), u'''(s))ds \right| \\
&= \left| -\int_1^{\tau} \overline{f}(t_0 + 1 - \sigma, w(\sigma), -w'(\sigma), w''(\sigma), -w'''(\sigma))d\sigma \right| \\
&\leq \int_{\tau}^{1} (\overline{p}_0 + \overline{p}_1(t_0 + 1 - \sigma)\overline{g}(w(\sigma), -w'(\sigma), w''(\sigma)) + \overline{p}_2(t_0 + 1 - \sigma)|w'''(\sigma)| + \overline{p}_3|w'''(\sigma)|^2)d\sigma \\
&\leq (\overline{p}_0 + \overline{g}(r, r, -r)\overline{D}_1) + \int_{\tau}^{1} (\overline{p}_2(t_0 + 1 - \sigma)|w'''(\sigma)| + \overline{p}_3|w'''(\sigma)|^2)d\sigma.
\end{aligned}$$

Since

$$\int_{t_0}^{1} \overline{p}_3|w'''(\sigma)|d\sigma = -\int_{t_0}^{1} \overline{p}_3 u'''(s)ds = \overline{p}_3(u''(t_0) - u''(1)) \leq \overline{p}_3 r,$$

by Lemma 2.1 we deduce that

$$|w'''(\tau)| \leq (\overline{p}_0 + \overline{g}(r, r, -r)\overline{D}_1) \exp(\overline{D}_2) \exp(\overline{p}_3 r) = \overline{Q}(r), \qquad \tau \in [t_0, 1],$$

i.e. $|u'''(t)| \leq \overline{Q}(r)$, $t \in [t_0, 1]$.

So the proof is complete. □

Let $[a, b]$ be a subset of $(0, 1)$ and denote

$$\overline{\gamma} := \min\left\{ \min_{t\in[a,b]} \overline{c}_0(t), \min_{t\in[a,b]} \overline{c}_1(t) \right\} = \min\left\{ \frac{3\sqrt{3}}{2}a(1 - a^2), \frac{3\sqrt{3}}{2}b(1 - b)(2 - b), a, 1 - b \right\},$$

$$\frac{1}{\overline{m}} := \max\left\{ \int_0^1 \overline{\Phi}_0(s)ds, \int_0^1 \overline{\Phi}_1(s)ds \right\}, \qquad \frac{1}{\overline{M}} := \min\left\{ \int_a^b \overline{\Phi}_0(s)ds, \int_a^b \overline{\Phi}_1(s)ds \right\},$$

where $\overline{c}_i(t)$ and $\overline{\Phi}_i(s) (i = 0, 1)$ are provided in Lemma 4.1. Obviously, $\overline{\gamma} \in (0, 1/2)$ and $\overline{m} < \overline{M}$.

Theorem 4.4. *Suppose that* $(\overline{C}_1)$–$(\overline{C}_3)$ *hold and* $\overline{f}$ *satisfies the growth assumption* (4.4)*. The BVP* (1.3) *has at least one positive solution* $u \in \overline{K}$ *if either of the following conditions* $(\overline{F}_1)$*,* $(\overline{F}_2)$ *holds, where* $\overline{Q}$ *is given by* (4.5)*.*

$(\overline{F}_1)$ *There exist* $0 < r_1 < r_2$ *with* $r_1 < r_2\overline{\gamma}$*, such that*

$(\overline{F}_1 a)$ *for* $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0, r_1] \times [-r_1, r_1] \times [-r_1, 0] \times [-\overline{Q}(r_1), \overline{Q}(r_1)]$*,*

$$\overline{f}(t, x_0, x_1, x_2, x_3) < \overline{m} r_1; \tag{4.7}$$

$(\overline{F}_1 b)$ *for* $(t, x_0, x_1, x_2, x_3) \in \overline{W}_1 := \overline{W}_{1,0} \cup \overline{W}_{1,1}$*,*

$$\overline{f}(t, x_0, x_1, x_2, x_3) > \overline{M} r_2, \tag{4.8}$$

where

$$\overline{W}_{1,0} = [a,b] \times [r_2\overline{\gamma}, r_2] \times [-r_2, r_2] \times [-r_2, 0] \times [-\overline{Q}(r_2), \overline{Q}(r_2)],$$
$$\overline{W}_{1,1} = [a,b] \times [0, r_2] \times [-r_2, r_2] \times [-r_2, -r_2\overline{\gamma}] \times [-\overline{Q}(r_2), \overline{Q}(r_2)].$$

$(\overline{F}_2)$ *There exist* $0 < r_1 < r_2$ *with* $\overline{M} r_1 < \overline{m} r_2$*, such that*

$(\overline{F}_2 a)$ *for* $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0, r_2] \times [-r_2, r_2] \times [-r_2, 0] \times [-\overline{Q}(r_2), \overline{Q}(r_2)]$*,*

$$\overline{f}(t, x_0, x_1, x_2, x_3) < \overline{m} r_2; \tag{4.9}$$

$(\overline{F}_2 b)$ *for* $(t, x_0, x_1, x_2, x_3) \in \overline{W}_2 := \overline{W}_{2,0} \cup \overline{W}_{2,1}$*,*

$$\overline{f}(t, x_0, x_1, x_2, x_3) > \overline{M} r_1, \tag{4.10}$$

where

$$\overline{W}_{2,0} = [a,b] \times [r_1\overline{\gamma}, r_1] \times [-r_1, r_1] \times [-r_1, 0] \times [-\overline{Q}(r_1), \overline{Q}(r_1)],$$
$$\overline{W}_{2,1} = [a,b] \times [0, r_1] \times [-r_1, r_1] \times [-r_1, -r_1\overline{\gamma}] \times [-\overline{Q}(r_1), \overline{Q}(r_1)].$$

Proof. Suppose that $(\overline{F}_1)$ holds.

Define an open (relative to $\overline{K}$) set

$$U_{r_1} := \left\{ u \in \overline{K} : \|u\|_C < r_1, \|u''\|_C < r_1, \|u'''\|_C < \overline{Q}(r_1) + 1 \right\}.$$

If $u \in U_{r_1}$, it follows from $u(0) = u(1)$ that there is $\zeta \in (0,1)$ such that $u'(\zeta) = 0$, and $|u'(t)| = \left| \int_\zeta^t u''(s) ds \right| \le \|u''\|_C$ for all $t \in [0,1]$ which implies that $\|u'\|_C < r_1$. Thus U_{r_1} is bounded. Similar to the proof of Theorem 2.5, we have that the fixed point index $i(\overline{S}, U_{r_1}, \overline{K}) = 1$ by Lemma 1.1.

Define an open (relative to $\overline{K}$) set

$$V_{r_2} := \left\{ u \in \overline{K} : \min_{t \in [a,b]} u(t) < r_2\overline{\gamma},\ \min_{t \in [a,b]} (-u''(t)) < r_2\overline{\gamma},\ \|u'''\|_C < Q(r_2) + 1 \right\}.$$

If $u \in V_{r_2}$, it follows from (4.3) that $\|u\|_C < r_2$ and $\|u''\|_C < r_2$. Since $u(0) = u(1)$, there is $\tau \in (0,1)$ such that $u'(\tau) = 0$, and $|u'(t)| = \left| \int_\tau^t u''(s) ds \right| \le \|u''\|_C$ for all $t \in [0,1]$ which implies that $\|u'\|_C < r_2$. Thus V_{r_2} is bounded. Again similar to the proof of Theorem 2.5, we have that the fixed point index $i(\overline{S}, V_{r_2}, \overline{K}) = 0$ by Lemma 1.2.

It is obvious from $r_1 < r_2\overline{\gamma}$ that $\overline{U}_{r_1} \subset V_{r_2}$. So there is a fixed point of $\overline{S}$ in the set $V_{r_2} \setminus \overline{U}_{r_1}$ which is clearly nonzero and the positive solutions to BVP (1.3) by Lemma 4.2.

The other case is proved similarly. □

Example 4.5. Consider

$$\begin{cases} u^{(4)}(t)=\overline{f}(t,u(t),u'(t),u''(t),u'''(t)), \quad t\in[0,1],\\ u(0)=u(1)=\frac{1}{2}u(\frac{1}{4})-\frac{1}{8}u(\frac{1}{2}),\\ u''(0)+\int_0^1 u(t)(t-\frac{1}{4})dt=0,\ u''(1)+\int_0^1 u(t)(t-\frac{1}{4})dt=0, \end{cases} \tag{4.11}$$

thus $\eta_1[u]=\frac{1}{2}u(\frac{1}{4})-\frac{1}{8}u(\frac{1}{2})$, $\eta_2[u]=\int_0^1 u(t)(t-\frac{1}{4})dt$. Then

$$\begin{aligned} 0\le \overline{\mathcal{K}}_1(s) &= \tfrac{1}{2}\overline{G}_0\left(\tfrac{1}{4},s\right)-\tfrac{1}{8}\overline{G}_0\left(\tfrac{1}{2},s\right)\\ &= \begin{cases} -\frac{5}{96}s^3+\frac{5}{256}s, & 0\le s\le \frac{1}{4},\\ \frac{1}{32}s^3-\frac{1}{16}s^2+\frac{9}{256}s-\frac{1}{768}, & \frac{1}{4}<s\le\frac{1}{2},\\ \frac{1}{96}s^3-\frac{1}{32}s^2+\frac{5}{256}s+\frac{1}{768}, & \frac{1}{2}<s\le 1, \end{cases} \end{aligned}$$

$$\overline{\mathcal{K}}_2(s)=\int_0^1 \overline{G}_0(t,s)\left(t-\frac{1}{4}\right)dt=\frac{1}{120}s^5-\frac{1}{96}s^4-\frac{1}{144}s^3+\frac{13}{1440}s\ge 0 \qquad (0\le s\le 1),$$

the 2×2 matrix

$$[H]=\begin{pmatrix} \eta_1[\xi_1] & \eta_1[\xi_2]\\ \eta_2[\xi_1] & \eta_2[\xi_2] \end{pmatrix}=\begin{pmatrix} \frac{3}{8} & \frac{1}{32}\\ \frac{1}{4} & \frac{1}{48} \end{pmatrix}$$

and its spectral radius $r([H])=\frac{19}{48}<1$. Therefore, $(\overline{C}_2)$ and $(\overline{C}_3)$ hold. We choose $[a,b]=[1/4,3/4]$ and note that $\overline{\gamma}=1/128$,

$$\overline{\kappa}_1(s)=\begin{cases} \frac{s(3577-9440s^2-60s^3+48s^4)}{111360}, & 0\le s\le\frac{1}{4},\\ \frac{-235+6397s-11280s^2+5600s^3-60s^4+48s^5}{111360}, & \frac{1}{4}<s\le\frac{1}{2},\\ \frac{235+3577s-5640s^2+1840s^3-60s^4+48s^5}{111360}, & \frac{1}{2}<s\le 1, \end{cases}$$

$$\overline{\kappa}_2(s)=\begin{cases} \frac{s(97-160s^2-60s^3+48s^4)}{5568}, & 0\le s\le\frac{1}{4},\\ \frac{-3+133s-144s^2+32s^3-60s^4+48s^5}{5568}, & \frac{1}{4}<s\le\frac{1}{2},\\ \frac{3+97s-72s^2-16s^3-60s^4+48s^5}{5568}, & \frac{1}{2}<s\le 1, \end{cases}$$

and hence

$$\int_0^1 \overline{\Phi}_0(s)ds=-\frac{5051}{712704}+\frac{2}{45\sqrt{3}}, \qquad \int_0^1 \overline{\Phi}_1(s)ds=\frac{15151}{89088},$$

$$\frac{1}{\overline{m}}=\max\left\{\int_0^1 \overline{\Phi}_0(s)ds,\int_0^1 \overline{\Phi}_1(s)ds\right\}=\frac{15151}{89088},$$

$$\int_{1/4}^{3/4} \overline{\Phi}_0(s)ds=-\frac{248861}{28508160}+\frac{5\sqrt{5}}{512}, \qquad \int_{1/4}^{3/4} \overline{\Phi}_1(s)ds=\frac{83365}{712704},$$

$$\frac{1}{\overline{M}}=\min\left\{\int_{1/4}^{3/4} \overline{\Phi}_0(s)ds,\int_{1/4}^{3/4} \overline{\Phi}_1(s)ds\right\}=-\frac{248861}{28508160}+\frac{5\sqrt{5}}{512},$$

$\overline{m}\approx 5.8800, \overline{M}\approx 76.2943$.

Let $\overline{f}(t,x_0,x_1,x_2,x_3)=\overline{d}\big(x_0^{k_0}+x_1^4+(-x_2)^{k_1}+x_3^2\big)$, $(t,x_0,x_1,x_2,x_3)\in[0,1]\times[0,\infty)\times(-\infty,\infty)\times(-\infty,0]\times(-\infty,\infty)$, here $k_i>1$ $(i=0,1)$, and $\overline{d}>0$ is a constant which is determined by the next step. Clearly $(\overline{C}_1)$ holds. For a given $r_1 > 0$, choosing $\overline{d}_0 > 0$ and $\overline{d}$ sufficiently small such that

$$\overline{d}\left(r_1^{k_0}+r_1^4+r_1^{k_1}+\left(\left(\overline{d}_0+\left(r_1^{k_0}+r_1^4+r_1^{k_1}\right)\overline{d}\right)\exp(\overline{d}r_1)\right)^2\right)<\overline{m}r_1,$$

we have that (4.4) and (4.7) are satisfied with $\overline{g}(x_0,x_1,x_2)=x_0^{k_0}+x_1^4+(-x_2)^{k_1}$. Choosing r_2 large enough such that $r_2 > r_1/\overline{\gamma}$ and $r_2^{k_i-1} > \overline{M}\overline{d}^{-1}\overline{\gamma}^{-k_i}$ $(i=0,1)$, we have that for $(t,x_0,x_1,x_2,x_3)\in\overline{W}_{1,i}$,

$$\overline{f}(t,x_0,x_1,x_2,x_3)\geq\overline{d}\,(r_2\overline{\gamma})^{k_i}>\overline{M}r_2,$$

i.e., (4.8) is satisfied. By Theorem 4.4 the BVP (4.11) has at least one positive solution. Of course 0 is also a solution of this problem. Especially, if $r_1=0.01, \overline{d}_0=0.01$ and $k_0=k_1=2$, we may take $\overline{d}=48$.

Example 4.6. Consider BVP (4.11) with $\overline{f}(t,x_0,x_1,x_2,x_3)=\overline{d}(x_0^{k_0}+x_1^4+(-x_2)^{k_1}+x_3^2)$ for $(t,x_0,x_1,x_2,x_3)\in[0,1]\times[0,\infty)\times(-\infty,\infty)\times(-\infty,0]\times(-\infty,\infty)$, here $k_i\in(0,1)$ $(i=0,1)$, and $\overline{d}>0$ is a constant which is determined by the next step. Clearly $(\overline{C}_1)$ holds. For a given $r_2>0$, choosing $\overline{d}_0>0$ and $\overline{d}$ sufficiently small such that

$$\overline{d}\left(r_2^{k_0}+r_2^4+r_2^{k_1}+\left(\left(\overline{d}_0+\left(r_2^{k_0}+r_2^4+r_2^{k_1}\right)\overline{d}\right)\exp(\overline{d}r_2)\right)^2\right)<\overline{m}r_2,$$

we have that (4.4) and (4.9) are satisfied with $g(x_0,x_1,x_2)=x_0^{k_0}+x_1^4+(-x_2)^{k_1}$. Choosing r_1 small enough such that $r_1<\overline{m}r_2\overline{M}^{-1}$ and $r_1^{1-k_i}<\overline{d}\overline{\gamma}^{k_i}\overline{M}^{-1}$ $(i=0,1)$, we have that for $(t,x_0,x_1,x_2,x_3)\in\overline{W}_{2,i}$,

$$f(t,x_0,x_1,x_2,x_3)\geq\overline{d}\,(r_1\overline{\gamma})^{k_i}>\overline{M}r_1\qquad(i=0,1),$$

i.e., (4.10) is satisfied. By Theorem 4.4 the BVP (4.11) has at least one positive solution. Of course 0 is also a solution of this problem. Especially, if $r_2=1, \overline{d}_0=0.01$ and $k_0=k_1=1/2$, we may take $\overline{d}=1/2$

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