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Uniqueness criteria for ordinary differential equations with a generalized transversality condition at the initial condition

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Abstract. In this paper, we present some uniqueness results for systems of ordinary differential equations. All of them are linked by a weak transversality condition at the initial condition, which generalizes those in the previous literature. Several examples are also provided to illustrate our results.

Keywords: uniqueness, ordinary differential equation.

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1 Introduction

This paper considers local uniqueness of solutions for the initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

where $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous and U is a neighborhood of the point $(t_0, x_0) \in \mathbb{R}^{n+1}$.

Hoag proved in [11] the following result concerning unique solvability of (1.1) in the scalar case ($n = 1$).

Theorem 1.1. *For $(t_0, x_0) \in \mathbb{R}^2$ and positive numbers a and b, define*

$$U = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b].$$

Let $f : U \to \mathbb{R}$ be a continuous function satisfying the following three conditions:

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(i) *there are constants* $c > 0$ *and* $r \in (0, 1/2)$ *such that*

$$|f(t,x)| \geq c|x - x_0|^r \quad \textit{for all } (t,x) \in U;$$

(ii) $f(t, x_0)$ *is not identically zero on any interval* $(t_0 - \varepsilon, t_0)$ *or* $(t_0, t_0 + \varepsilon)$ *for* $0 < \varepsilon < a$;

(iii) *there is a number* $K \geq 0$ *such that for all* (t, x) *and* (s, x) *in* U,

$$|f(t,x) - f(s,x)| \leq K|t - s|.$$

Then there is a unique solution to the initial value problem (1.1) *in some interval* $(t_0 - \alpha, t_0 + \alpha)$ *with* $\alpha > 0$.

Basically, Theorem 1.1 replaces the transversality condition $f(t_0, x_0) \neq 0$, employed in previous papers together with the Lipschitz condition with respect to the first variable (iii), see for instance [6,7,13,14,17], by assumptions (i) and (ii). See also [5].

On the other hand, a generalized Lipschitz condition which measures the field differences in different directions to that given by the axis was studied and progressively improved in a series of papers [9,10,16,17]. The following result is the main uniqueness criterion in [16].

Theorem 1.2. *Let* D *be an open neighborhood of the point* $(t_0, x_0) \in \mathbb{R}^{n+1}$ *and* $f : D \to \mathbb{R}^n$ *be a continuous function. Let* $\mathcal{V} \subset \mathbb{R}^{n+1}$ *be a hyperspace and assume that*

(i) $(1, f(t_0, x_0)) \notin \mathcal{V}$,

(ii) f *is Lipschitz continuous along the hyperspace* $\mathcal{V}$ *on* D, *i.e., there exists* $L \geq 0$ *such that for all* $(t, x), (s, y) \in D$,

$$\|f(t,x) - f(s,y)\| \leq L\,\|(t,x) - (s,y)\| \quad \textit{if } (t,x) - (s,y) \in \mathcal{V}.$$

Then problem (1.1) *has a unique local solution.*

The aim of this paper is to extend conditions (i) and (ii) in Theorem 1.1 to the case of systems and to combine them with different hypotheses about f, such as the generalized Lipschitz notion in [16] or the perturbed Lipschitz assumption in [12], in order to obtain uniqueness for (1.1).

The paper is organized as follows: in Section 2, Theorem 1.1 is extended to the case of systems. Our result relies on the concept of *Lipschitz continuous function when fixing a variable*, which was introduced in [4].

The main aim of Section 3 is to adapt the arguments in Theorem 1.1 to functions which are *Lipschitz continuous along a hyperspace* $\mathcal{V}$, as defined in [16, Theorem 2.1 (A2)], and thus allowing the transversality condition $(1, f(t_0, x_0)) \notin \mathcal{V}$ to fail. Therefore, the results in [11] and [16] are simultaneously weakened.

In Section 4 we present a different uniqueness result which was inspired by a particular form of expressing the function f as certain composition of functions due to Bressan and Shen [3]. The weak Lipschitz-type condition required on f is similar to that in the classical uniqueness criterion in [15]. It allows us to construct a set which contains all possible solutions of problem (1.1) and where f is Lipschitz with respect to x.

We point out that all uniqueness results in this paper are connected since they require a relaxed transversality condition at the initial point.

2 Uniqueness via a Lipschitz condition when fixing a variable

Here we prove a relaxed version of Theorem 1.1 in the case of systems. In the sequel we use the notation $\overline{B}_a(x)$ for the closed ball with radius $a > 0$ centered at $x \in \mathbb{R}^p$, with the metric defined by the maximum norm $\|(y_1, y_2, \dots, y_p)\| = \max\{|y_1|, |y_2|, \dots, |y_p|\}$.

Theorem 2.1. *For $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, $x_0 = (x_{0,1}, x_{0,2}, \dots, x_{0,n})$, and positive numbers a and b, define*

$$U = [t_0 - a, t_0 + a] \times \overline{B}_b(x_0).$$

Let $f : U \to \mathbb{R}^n$ be a continuous function satisfying the following three conditions:

(1) *there is a continuous function $M : [x_{0,n} - b, x_{0,n} + b] \to (0, +\infty)$ such that $1/M^2 \in L^1(x_{0,n} - b, x_{0,n} + b)$ and for all $z \in [x_{0,n} - b, x_{0,n} + b]$, $z \neq x_{0,n}$, we have*

$$|f_n(t, x_1, x_2, \dots, x_{n-1}, z)| \geq M(z) > 0$$

for all $(x_1, \dots, x_{n-1}) \in [x_{0,1} - b, x_{0,1} + b] \times \cdots \times [x_{0,n-1} - b, x_{0,n-1} + b]$ and all $t \in [t_0 - a, t_0 + a]$;

(2) *$f_n(t, y_1(t), \dots, y_{n-1}(t), x_{0,n})$ is not identically zero on any interval $(t_0 - \varepsilon, t_0)$ or $(t_0, t_0 + \varepsilon)$ for $0 < \varepsilon < a$ whenever $(y_1, \dots, y_{n-1}) \in \mathcal{C}(t_0 - \varepsilon, t_0 + \varepsilon)$;*

(3) *f is Lipschitz continuous when fixing the variable x_n, i.e., there exists $K \geq 0$ such that*

$$\|f(t_1, x_1, \dots, x_{n-1}, z) - f(t_2, y_1, \dots, y_{n-1}, z)\| \leq K \|(t_1, x_1, \dots, x_{n-1}) - (t_2, y_1, \dots, y_{n-1})\|$$

for all $(t_1, x_1, \dots, x_{n-1}, z), (t_2, y_1, \dots, y_{n-1}, z) \in U$, $z \neq x_{0,n}$.

Then there is a unique solution to the initial value problem (1.1) in some interval $(t_0 - \alpha, t_0 + \alpha)$ with $\alpha > 0$.

Proof. Firstly, notice that, since f is continuous, there exists $L > 0$ such that $\|f(t, x)\| \leq L$ for all $(t, x) \in U$. Furthermore, condition (1) implies that $f_n(t, x) \neq 0$ for any $(t, x) \in U$ such that $x_n \neq x_{0,n}$. Then f_n has constant sign on the connected sets

$$U^+ := \{(t, x) \in U : x_n > x_{0,n}\} \quad \text{and} \quad U^- := \{(t, x) \in U : x_n < x_{0,n}\}.$$

Now, from (2) it follows that the sign of f_n on both U^+ and U^- must be the same, so in particular f_n does not change sign on U (that is, either $f_n(t, x) \geq 0$ for all $(t, x) \in U$ or $f_n(t, x) \leq 0$ for all $(t, x) \in U$).

Let $x = (x_1, x_2, \dots, x_n)$ be a solution of (1.1) defined on an interval $[t_0 - a_1, t_0 + a_1]$, with $0 < a_1 < a$. We will show that x_n is strictly monotone on a neighborhood of t_0. First, observe that we either have $x_n'(t) \geq 0$ for all $t \in I = [t_0 - a_1, t_0 + a_1]$ or $x_n'(t) \leq 0$ for all $t \in I$, hence x_n is monotone on I.

Let us prove that $x_n'(t) \neq 0$ for all $t \in I$, $t \neq t_0$. Reasoning by contradiction, assume without loss of generality that for some $t^* \in I$, $t^* > t_0$, we have $0 = x_n'(t^*) = f_n(t^*, x(t^*))$. Then we deduce from condition (1) that $x_n(t^*) = x_{0,n}$. Since x_n is monotone and $x_n(t_0) = x_{0,n} = x_n(t^*)$, we deduce that x_n is constant between t_0 and t^*, hence $0 = x_n'(t) = f_n(t, x(t))$ for all $t \in (t_0, t^*)$, but this is impossible due to condition (2).

Summing up, x_n is strictly monotone on I, with nonzero derivative everywhere on $[t_0 - a_1, t_0)$ and on $(t_0, t_0 + a_1]$. Therefore, the function

$$y : J = x_n(I) \to [x_{0,1} - b, x_{0,1} + b] \times \cdots \times [x_{0,n-1} - b, x_{0,n-1} + b] \times I$$

given by $y = (x_1 \circ x_n^{-1}, \dots, x_{n-1} \circ x_n^{-1}, x_n^{-1})$ solves the problem

$$y'(r) = \tilde{f}(r, y(r)) \quad \text{for } r \in J \setminus \{x_{0,n}\},\ y(x_{0,n}) = y_0, \tag{2.1}$$

where $y_0 = (x_{0,1}, \dots, x_{0,n-1}, t_0)$ and $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n)$ with

$$\tilde{f}_i(r, y_1, \dots, y_{n-1}, y_n) = \frac{f_i(y_n, y_1, \dots, y_{n-1}, r)}{f_n(y_n, y_1, \dots, y_{n-1}, r)} \quad \text{if } i \in \{1, 2, \dots, n-1\}$$

and

$$\tilde{f}_n(r, y_1, \dots, y_{n-1}, y_n) = \frac{1}{f_n(y_n, y_1, \dots, y_{n-1}, r)}.$$

Indeed, this is a straightforward consequence of the chain rule, the formula for the derivative of the inverse and the fact that x is a solution to (1.1).

Now, it only remains to prove that the problem (2.1) has at most one local solution. Indeed, for all $(r, y_1, \dots, y_n), (r, z_1, \dots, z_n) \in U$, with $r \neq x_{0,n}$, and for all $i \in \{1, 2, \dots, n-1\}$, we have that

$$\begin{aligned} \left|\tilde{f}_i(r, y_1, \dots, y_n) - \tilde{f}_i(r, z_1, \dots, z_n)\right| &= \left|\frac{f_i(y_n, y_1, \dots, y_{n-1}, r)}{f_n(y_n, y_1, \dots, y_{n-1}, r)} - \frac{f_i(z_n, z_1, \dots, z_{n-1}, r)}{f_n(z_n, z_1, \dots, z_{n-1}, r)}\right| \\ &\leq \frac{2L\,K}{M^2(r)} \left\|(y_1, \dots, y_n) - (z_1, \dots, z_n)\right\|, \end{aligned}$$

and

$$\begin{aligned} \left|\tilde{f}_n(r, y_1, \dots, y_n) - \tilde{f}_n(r, z_1, \dots, z_n)\right| &= \left|\frac{1}{f_n(y_n, y_1, \dots, y_{n-1}, r)} - \frac{1}{f_n(z_n, z_1, \dots, z_{n-1}, r)}\right| \\ &\leq \frac{K}{M^2(r)} \left\|(y_1, \dots, y_n) - (z_1, \dots, z_n)\right\|. \end{aligned}$$

Hence, $\tilde{f}$ satisfies Montel–Tonelli's uniqueness theorem, [1], so it follows the existence of a constant $\alpha > 0$ such that problem (2.1) has at most one solution in the interval $[x_{0,n} - \alpha, x_{0,n} + \alpha]$. □

Remark 2.2. For simplicity, in Theorem 2.1 the function f is assumed to be Lipschitz continuous when fixing the last variable. However, local uniqueness for problem (1.1) is also derived if f is Lipschitz continuous when fixing another variable $i_0 \in \{1, 2, \dots, n-1\}$ and conditions (1) and (2) are given for f_{i_0} instead of f_n.

Remark 2.3. Condition (2) in Theorem 2.1 is satisfied if there exists a function

$$h : [t_0 - a, t_0 + a] \to [0, \infty)$$

such that for all $t \in [t_0 - a, t_0 + a]$ and all $(x_1, \dots, x_{n-1}) \in [x_{0,1} - b, x_{0,1} + b] \times \cdots \times [x_{0,n-1} - b, x_{0,n-1} + b]$, we have

$$|f_n(t, x_1, \dots, x_{n-1}, x_{0,n})| \geq h(t),$$

and $h(t)$ is not identically zero on any interval $(t_0 - \varepsilon, t_0)$ or $(t_0, t_0 + \varepsilon)$ for $0 < \varepsilon < a$.

Moreover, we emphasize that, in the scalar case, this condition (2) means that $f(t, x_0)$ is not identically zero on any interval $(t_0 - \varepsilon, t_0)$ or $(t_0, t_0 + \varepsilon)$, which is exactly condition (ii) in Theorem 1.1.

Remark 2.4. Note that the conclusion of Theorem 2.1 and its proof remain valid if the Lipschitz type condition (3) is replaced by the more general Montel–Tonelli condition, see [1]:

$\overline{(3)}$ There exist functions $\psi : [0,+\infty) \to [0,+\infty)$ continuous and $p : [x_{0,n} - b, x_{0,n} + b] \to [0,+\infty)$ such that for all $(t_1, x_1, \dots, x_{n-1}, z), (t_2, y_1, \dots, y_{n-1}, z) \in U$, $z \neq x_{0,n}$,

$$\begin{aligned} &\|f(t_1, x_1, \dots, x_{n-1}, z) - f(t_2, y_1, \dots, y_{n-1}, z)\| \\ &\qquad \leq p(z)\psi\left(\|(t_1, x_1, \dots, x_{n-1}) - (t_2, y_1, \dots, y_{n-1})\|\right) \end{aligned}$$

where $\psi(\tau) > 0$ when $\tau > 0$, $\int_{0^+} \frac{d\tau}{\psi(\tau)} = +\infty$ and $p/M^2 \in L^1(x_{0,n} - b, x_{0,n} + b)$ being M the function in condition (1) of Theorem 2.1.

Notice that, under $\overline{(3)}$, the condition $1/M^2 \in L^1(x_{0,n} - b, x_{0,n} + b)$ in (1) is not longer required and is replaced by $p/M^2 \in L^1(x_{0,n} - b, x_{0,n} + b)$.

Theorem 2.1 increases the applicability of the main result in [4] in case that $f(t_0, x_0) = 0$, as shown by the following example.

Example 2.5. The function $f : [-1,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(t, x_1, x_2) = \left(\sin(x_1 x_2), |t| + x_1^2 + \sqrt[4]{|x_2|}\right)$$

is not Lipschitz continuous with respect to $x = (x_1, x_2)$ in any neighborhood of $(0,0)$ and $f(0,0,0) = (0,0)$.

However, the initial value problem

$$\begin{cases} x_1' = \sin(x_1 x_2), & x_1(0) = 0, \\ x_2' = |t| + x_1^2 + \sqrt[4]{|x_2|}, & x_2(0) = 0, \end{cases} \tag{2.2}$$

has a unique local solution, since f restricted to $U = [-1,1]^3$ is Lipschitz continuous when fixing the last variable, assumption (1) in Theorem 2.1 holds with $M(z) = \sqrt[4]{|z|}$ and, moreover, the inequality

$$f_2(t, x_1, 0) \geq |t| \quad \text{for all } (t, x_1) \in [-1,1] \times \mathbb{R}$$

implies that f_2 satisfies condition (2). Therefore, Theorem 2.1 ensures local uniqueness for (2.2).

3 Uniqueness via a Lipschitz condition along a hyperspace

The following result is a straightforward consequence of the chain rule for functions of several variables.

Lemma 3.1. *Let $U, V \subset \mathbb{R}^{n+1}$ be open sets, $p_0 \in U$, $F : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ a continuous function and $\Phi : V \to U$ a diffeomorphism.*

Then $x : (t_0 - \varepsilon, t_0 + \varepsilon) \to U$ is a solution of the autonomous system

$$x'(t) = F(x(t)), \quad x(t_0) = p_0, \tag{3.1}$$

if and only if $y := \Phi^{-1}(x) : (t_0 - \varepsilon, t_0 + \varepsilon) \to V$ is a solution of the problem

$$y'(t) = G(y(t)), \quad y(t_0) = \Phi^{-1}(p_0), \tag{3.2}$$

where

$$G(y) = \Phi'(y)^{-1} F(\Phi(y)).$$

Next, an extension of the main theorem in [16] is established as a consequence of Theorem 2.1 and Lemma 3.1 applied to a linear diffeomorphism. Basically, we assume a weak transversality condition at the initial point and a *Lipschitz condition along a hyperspace* for f.

Let $\mathcal{V} \subset \mathbb{R}^{n+1}$ be a hyperspace and $a_0 \in \mathbb{R}^{n+1}$ be a unit vector such that $\mathcal{V} = a_0^{\perp}$. Note that $\mathbb{R}^{n+1} = \mathcal{V} \oplus \langle a_0 \rangle$, where $\langle a_0 \rangle = \{a_0 s \in \mathbb{R}^{n+1} : s \in \mathbb{R}\}$, and so there exist unique $v_0 \in \mathcal{V}$ and $s_0 \in \mathbb{R}$ such that $(t_0, x_0) = v_0 + a_0 s_0$.

Theorem 3.2. *Let U be a neighborhood of $(t_0, x_0) \in \mathbb{R}^{n+1}$ and $f : U \to \mathbb{R}^n$ be a continuous function satisfying the following conditions:*

(*a*) *there exist constants $a, b > 0$ and a continuous function $M : [s_0 - a, s_0 + a] \to [0, +\infty)$ such that $1/M^2 \in L^1(s_0 - a, s_0 + a)$ and for all $s \in [s_0 - a, s_0 + a]$, $s \neq s_0$, we have*
$$|a_0 \cdot (1, f(v + a_0 s))| \geq M(s) > 0 \quad \textit{for all } v \in \mathcal{V} \cap \overline{B}_b(v_0);$$

(*b*) *$a_0 \cdot (1, f(v(s) + a_0 s))$ is not identically zero on any interval $(s_0 - \varepsilon, s_0)$ or $(s_0, s_0 + \varepsilon)$ for $0 < \varepsilon < a$ whenever $v \in \mathcal{C}((s_0 - \varepsilon, s_0 + \varepsilon); \mathcal{V})$;*

(*c*) *f is Lipschitz continuous along the hyperspace $\mathcal{V}$ on U, i.e., there exists $L \geq 0$ such that for all $(t, x), (s, y) \in U$,*
$$\|f(t, x) - f(s, y)\| \leq L \, \|(t, x) - (s, y)\| \quad \textit{if } (t, x) - (s, y) \in \mathcal{V}.$$

Then there is a unique local solution to the initial value problem (1.1).

Proof. Since $\mathcal{V}$ is a hyperspace in $\mathbb{R}^{n+1}$, there exists an orthonormal set of vectors $v_1, v_2, \dots, v_n \in \mathbb{R}^{n+1}$ such that $\mathcal{V} = \operatorname{span}\{v_1, \dots, v_n\}$. Let us consider the full rank matrix
$$A := (v_1 \,|\, v_2 \,|\, \cdots \,|\, v_n \,|\, a_0),$$
which gives the change-of-basis matrix from $\{v_1, \dots, v_n, a_0\}$ to the standard Euclidean basis. Notice that A is an orthogonal matrix, so $A^{-1} = A^T$.

Define the linear diffeomorphism
$$\Phi(y) := Ay$$
and consider the map given by $F(x_1, \dots, x_{n+1}) := (1, f(x_1, \dots, x_{n+1}))$.

Let us show that the following autonomous initial value problem
$$y' = G(y), \quad y(t_0) = p_0, \tag{3.3}$$
where $G(y) = A^{-1} F(Ay)$ and $p_0 = A^T (t_0, x_0)^T$, is uniquely locally solvable. Indeed, we shall prove that G satisfies assumptions (1)–(3) in Theorem 2.1. Note that
$$G_{n+1}(y) = a_0 \cdot F(Ay) = a_0 \cdot (1, f(Ay)).$$

Moreover, for each $y \in \mathbb{R}^{n+1}$ we can express $Ay = v + a_0 s$ in a unique way with $v \in \mathcal{V}$ and $s \in \mathbb{R}$, explicitly
$$v = A \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ 0 \end{pmatrix} \quad \text{and} \quad s = y_{n+1}.$$

Hence,

$$G_{n+1}(y) = a_0 \cdot (1, f(v + a_0 s)),$$

and thus conditions (1) and (2) in Theorem 2.1 are directly deduced for G from assumptions (a) and (b). By assumption (c), we can deduce that G is Lipschitz continuous when fixing the last variable: for $(y,z), (\overline{y}, z) \in V$, with V a sufficiently small neighborhood of p_0, we have that

$$\begin{aligned}
\|G(y,z) - G(\overline{y},z)\| &= \left\| A^{-1} \left[F(A(y,0)^T + a_0 z) - F(A(\overline{y},0)^T + a_0 z) \right] \right\| \\
&= \left\| f(A(y,0)^T + a_0 z) - f(A(\overline{y},0)^T + a_0 z) \right\| \\
&\le L \left\| A(y,0)^T - A(\overline{y},0)^T) \right\| = L \left\| A(y - \overline{y}, 0)^T \right\| = L \left\| y - \overline{y} \right\| .
\end{aligned}$$

Therefore, Theorem 2.1 implies that the initial value problem (3.3) has a unique local solution. Finally, Lemma 3.1 ensures that (1.1) is uniquely locally solvable. □

Remark 3.3. Obviously, conditions (a) and (b) in Theorem 3.2 hold if the transversality condition

$$(1, f(t_0, x_0)) \notin \mathcal{V} \quad (\text{equivalently, } a_0 \cdot (1, f(v_0 + a_0 s_0)) \neq 0)$$

is satisfied, cf. [16, Theorem 2.1].

Notice that a more general version of the previous result can be obtained with a non necessarily linear diffeomorphism. The interested reader is referred to [9] for a general approach based on this idea, which we omit here for the sake of simplicity. However, a nonlinear diffeomorphism will be employed in the scalar case ($n = 1$) in order to provide a relaxed version of the main uniqueness criterion in [10].

Indeed, by using the diffeomorphism

$$\Phi(x_1, x_2) := \begin{pmatrix} x_1 \\ \varphi(x_1) + x_2 \end{pmatrix},$$

for a continuously differentiable function φ, we obtain from Lemma 3.1 the following local equivalence between two scalar initial value problems.

Corollary 3.4. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ a continuously differentiable function. Then, x is a solution of the problem (1.1) if and only if $y(t) = -\varphi(t) + x(t)$ is a solution of problem*

$$y'(t) = g(t, y(t)), \quad y(t_0) = y_0, \tag{3.4}$$

where $y_0 = x_0 - \varphi(t_0)$ and $g : V \to \mathbb{R}$ is defined as

$$g(t,y) = -\varphi'(t) + f\left(t, \varphi(t) + y\right)$$

in a neighborhood V of the point (t_0, y_0).

The use of the previous change of variables is standard for instance to translate a given periodic solution $\varphi(t)$ to the origin and then analyze its stability as an equilibrium, [2]. However, its application to derive new uniqueness criteria as in the following result seems to have being unnoticed.

Theorem 3.5. *Let U be a neighborhood of $(t_0, x_0) \in \mathbb{R}^2$, $f : U \to \mathbb{R}$ continuous and $\varphi : \mathbb{R} \to \mathbb{R}$ a continuously differentiable function with Lipschitz derivative. Moreover, assume that f satisfies the following conditions:*

(a) f is Lipschitz along φ, that is, there exists $L \geq 0$ such that if $(t, x + \varphi(t))$, $(t + k, x + \varphi(t + k)) \in U$ for some $k \in \mathbb{R}$, then

$$|f(t, x + \varphi(t)) - f(t + k, x + \varphi(t + k))| \leq L\,|k|\,.$$

(b) For $y_0 := x_0 - \varphi(t_0)$ and $r > 0$ such that $x - \varphi(t) \in [y_0 - r, y_0 + r]$ for all $(t, x) \in U$, there exists a continuous function $M : [y_0 - r, y_0 + r] \to [0, +\infty)$ such that

$$\left|-\varphi'(t) + f(t, x)\right| \geq M(x - \varphi(t)) > 0$$

for all $(t, x) \in U \setminus \{(t, x) \in \mathbb{R}^2 : t = t_0 \text{ or } x - \varphi(t) = y_0\}$ and $1/M^2 \in L^1(y_0 - r, y_0 + r)$.

(c) $-\varphi'(t) + f(t, y_0 + \varphi(t))$ is not identically zero on any interval $(t_0 - \varepsilon, t_0)$ or $(t_0, t_0 + \varepsilon)$ for $\varepsilon > 0$.

Then the scalar problem (1.1) has a unique local solution.

Proof. Let us check that the initial value problem (3.4) is under the assumptions of Theorem 2.1 (with $n = 1$) and, therefore, it has a unique local solution. Then Corollary 3.4 will provide also the uniqueness of local solution for problem (1.1).

First, from assumption (b), we have

$$|g(t, y)| \geq M\,(-\varphi(t) + y + \varphi(t)) = M(y) > 0$$

for all $(t, y) \in V \setminus \{(t, y) \in \mathbb{R}^2 : t = t_0 \text{ or } y = y_0\}$.

Similarly, from condition (c), it is immediate to obtain that $g(t, y_0)$ is not identically zero on any interval $(t_0 - \varepsilon, t_0)$ or $(t_0, t_0 + \varepsilon)$.

In addition, g is Lipschitz with respect to the first argument. Indeed, for $(t, y), (t + k, y) \in V$, we have that

$$\begin{aligned}
|g(t, y) - g(t + k, y)| &= \left|-\varphi'(t) + f(t, y + \varphi(t)) + \varphi'(t + k) - f(t + k, y + \varphi(t + k))\right| \\
&\leq \left|\varphi'(t + k) - \varphi'(t)\right| + |f(t, y + \varphi(t)) - f(t + k, y + \varphi(t + k))| \\
&\leq C\,|k| + L\,|k| = (C + L)\,|k|\,,
\end{aligned}$$

as a consequence of the fact that f is Lipschitz along φ and φ' is a Lipschitz continuous function. So condition (3) in Theorem 2.1 is clearly satisfied for $K = C + L$. □

Remark 3.6. Note that if a function φ is under the hypotheses of Theorem 3.5 and $\varphi(t_0) = c \neq 0$, then the function $\tilde{\varphi}(t) = \varphi(t) - c$ is also under the hypotheses of Theorem 3.5. Hence we can just consider functions φ with $\varphi(t_0) = 0$.

Remark 3.7. Observe that for $(v_1, v_2) \in \mathbb{R}^2$, $v_1 \neq 0$, and the function $\varphi(t) = \frac{v_2}{v_1} t$, condition (a) in Theorem 3.5 is equal to the Lipschitz condition in the direction of the vector $v = (1, v_2/v_1)$ or, equivalently, in the direction of (v_1, v_2), see [10]. For this choice of φ, Theorem 3.5 is just the scalar case of Theorem 3.2.

The applicability of Theorem 3.5 is shown by the following example.

Example 3.8. Consider the initial value problem

$$x' = \sqrt[4]{|x - \alpha\, t - \beta\, t^2|} + \alpha + \gamma\, |t| \,, \quad x(0) = 0,$$

where α, β and γ are constants with $\alpha, \beta, \gamma \in \mathbb{R}$ and $\gamma > 2\,|\beta|$.

First, observe that the function $f(t,x) = \sqrt[4]{|x - \alpha\, t - \beta\, t^2|} + \alpha + \gamma\, |t|$ is continuous and Lipschitz along the function $\varphi(t) = \alpha\, t + \beta\, t^2$ for any $\alpha, \beta \in \mathbb{R}$ and $\gamma > 0$. Indeed, for $(t,x) \in \mathbb{R}^2$ and $k \in \mathbb{R}$,

$$|f(t, x + \varphi(t)) - f(t + k, x + \varphi(t + k))| = \gamma\, ||t + k| - |t|| \leq \gamma\, |k| \,,$$

and thus condition *(a)* in Theorem 3.5 is satisfied with $L = \gamma$.

Notice also that $\varphi'(0) = f(0,0)$ and so the transversality condition asked in [9] is not satisfied. Nevertheless, assumptions (b) and (c) in Theorem 3.5 hold. Condition (b) can be easily verified by using the continuous function $M(y) = \sqrt[4]{|y|}$, which satisfies that $1/M^2 \in L^1(-\varepsilon, \varepsilon)$, together with the inequality $\gamma > 2\,|\beta|$, and condition (c) follows from the fact that

$$-\alpha - 2\,\beta\, t + f(t, \alpha\, t + \beta\, t^2) = -2\,\beta\, t + \gamma\, |t| > 0 \quad \text{for every } t \in \mathbb{R} \setminus \{0\},$$

which therefore it is not identically zero in any neighborhood of 0.

In conclusion, Theorem 3.5 ensures the uniqueness of a local solution for any $\alpha, \beta \in \mathbb{R}$ and $\gamma > 2\,|\beta|$. Finally, observe that if $\alpha = 1$ and $\beta = \gamma = 0$, then $x_1(t) = t$ and $x_2(t) = \left(\frac{3}{4}t\right)^{\frac{4}{3}} + t$ if $t \geq 0$ and $x_2(t) = -\left(\frac{3}{4}t\right)^{\frac{4}{3}} + t$ if $t < 0$ are two local solutions. In this case, condition (c) is no longer true.

4 Uniqueness via perturbed Lipschitz conditions

Our next result is another local uniqueness criterion for problem (1.1) which was inspired on a specific form of the function f considered in [3]. Basically, we shall assume that the function $f(t,x)$ satisfies a perturbed Lipschitz condition with respect to x outside some hypersurfaces $\tau_i(t,x) = 0$ $(i = 1, 2, \ldots, N)$ which satisfy a weak transversality condition around (t_0, x_0). Our result is closely related to the uniqueness theorem proven in [15] but ours is more general inasmuch our transversality conditions need not be satisfied at the point (t_0, x_0).

Let us state and prove the main result of this section.

Theorem 4.1. *Let U be a neighborhood of $(t_0, x_0) \in \mathbb{R}^{n+1}$, $f : U \to \mathbb{R}^n$ a continuous function and assume that:*

(i) There exist a constant $K > 0$ and functions $g_i : \mathbb{R} \to \mathbb{R}$ and $\tau_i : U \to \mathbb{R}$, for $i = 1, 2, \ldots, N$, such that

$$\|f(t,x) - f(t,y)\| \leq K\, \|x - y\| + K \max_{1 \leq i \leq N} |g_i(\tau_i(t,x)) - g_i(\tau_i(t,y))|,$$

for all (t,x), $(t,y) \in U$.

(ii) Each $\tau_i : U \to \mathbb{R}$ is continuously differentiable and $\tau_i(t_0, x_0) = 0$.

(iii) (Transversality) There exists a continuous function $M : \mathbb{R} \to \mathbb{R}$ such that for all $(t,x) \in U$ with $t \neq t_0$ we have

$$|\nabla \tau_i(t,x) \cdot (1, f(t,x))| \geq M(t) > 0 \quad \textit{for every } i \in \{1, 2, \ldots, N\}.$$

(*iv*) *Each* $g_i : \mathbb{R} \to \mathbb{R}$ *is continuous on* $\mathbb{R}$*, differentiable on* $\mathbb{R} \setminus \{0\}$*, and there exist* $\rho_1, \rho_2 > 0$ *and* $\psi : (0, \rho_1) \to \mathbb{R}$ *a decreasing function such that* $|g_i'(t)| \le \psi(|t|)$ *for all* $t \in (-\rho_1, \rho_1) \setminus \{0\}$ *and* $\psi(|\int_{t_0}^{\cdot} M(s)ds|) \in L^1(t_0 - \rho_2, t_0 + \rho_2)$.

Then the initial value problem (1.1) *has a unique local solution.*

Proof. Local existence follows from Peano's theorem. So, let us prove local uniqueness to the right, that is in an interval of the form $[t_0, t_0 + \alpha]$ (the proof of the local uniqueness to the left is similar).

Take $b, \delta > 0$ such that $[t_0, t_0 + \delta] \times \overline{B}_b(x_0) \subset U$ and $\|z(t) - x_0\| \le b$ for all $t \in [t_0, t_0 + \delta]$ and for any solution $z(t)$ of (1.1).

Now, by (*iii*), for $i = 1, 2, \ldots, N$ and $t \neq t_0$ we have

$$\left| \frac{d}{dt} \tau_i(t, z(t)) \right| = |\nabla \tau_i(t, z(t)) \cdot (1, f(t, z(t)))| \ge M(t),$$

and thus

$$|\tau_i(t, z(t))| \ge \left| \int_{t_0}^{t} M(s)\, ds \right| > 0,$$

for all $t \in [t_0, t_0 + \delta]$, $t \neq t_0$, and $i = 1, 2, \ldots, N$.

Define the compact set

$$D = \left\{ (t, x) \in [t_0, t_0 + \delta] \times \overline{B}_b(x_0) \; : \; |\tau_i(t, x)| \ge \left| \int_{t_0}^{t} M(s)\, ds \right| \text{ for } i = 1, 2, \ldots, N \right\}.$$

Notice that condition (*iii*) implies that for each $i \in \{1, 2, \ldots, N\}$

$$\nabla \tau_i(s, y) \cdot (1, f(s, y)) > 0 \quad \text{for } s \in (t_0, t_0 + \delta) \text{ and } y \in B_b(x_0), \tag{4.1}$$

or

$$\nabla \tau_i(s, y) \cdot (1, f(s, y)) < 0 \quad \text{for } s \in (t_0, t_0 + \delta) \text{ and } y \in B_b(x_0). \tag{4.2}$$

Assume that condition (4.1) holds for all $i \in \{1, 2, \ldots, N\}$ (if not, simply replace τ_i with $-\tau_i$ and $g_i(x)$ with $g_i(-x)$ so that (*i*) holds) and let $z(t)$ be a solution of problem (1.1). Then, by the chain rule, we deduce that

$$\frac{d}{dt} \tau_i(t, z(t)) = \nabla \tau_i(t, z(t)) \cdot (1, f(t, z(t))) > 0, \quad \text{for all } t \in (t_0, t_0 + \delta).$$

So $t \mapsto \tau_i(t, z(t))$ is increasing on $(t_0, t_0 + \delta)$ and then

$$\tau_i(t_0, z(t_0)) = 0 < \tau_i(t, z(t)) \quad \text{for all } t \in (t_0, t_0 + \delta).$$

In conclusion, $(t, z(t)) \in \tau_i^{-1}(0, \infty) \cap D$ on $(t_0, t_0 + \delta)$ for all $i \in \{1, 2, \ldots, N\}$.

Now, for (t, x), $(t, y) \in \cap_{i=1}^{N} \tau_i^{-1}(0, \infty) \cap D$, we deduce from (*i*), (*iv*) and the Mean Value Theorem that

$$\|f(t, x) - f(t, y)\| \le K \|x - y\| + K \max_{1 \le i \le N} |g_i'(\tau_{i,t})| |\tau_i(t, x) - \tau_i(t, y)|, \tag{4.3}$$

where the last identity is valid for some real numbers $\tau_{i,t}$ located between $\tau_i(t, x)$ and $\tau_i(t, y)$. In particular, $0 < |\int_{t_0}^{t} M(s)\, ds| \le |\tau_{i,t}|$, and then (*iv*) implies that

$$|g_i'(\tau_{i,t})| \le \psi(|\tau_{i,t}|) \le \psi \left(\left| \int_{t_0}^{t} M(s)\, ds \right| \right).$$

Therefore, for $(t,x),\ (t,y) \in \cap_{i=1}^N \tau_i^{-1}(0,\infty) \cap D$, we have

$$\begin{aligned} \|f(t,x)-f(t,y)\| &\le K\,\|x-y\| + K\psi\left(\int_{t_0}^t M(s)\,ds\right) \max_{1\le i\le N} |\tau_i(t,x)-\tau_i(t,y)| \\ &\le c_1(t)\,\|x-y\|, \end{aligned} \tag{4.4}$$

for some $c_1 \in L^1(t_0, t_0+\alpha)$ with $0 < \alpha \le \delta$, because the τ_i's are Lipschitz continuous with respect to x on the compact set D and assumption (iv).

Finally, let $x(t)$ and $y(t)$ be solutions of (1.1); for $t \in (t_0, t_0+\alpha)$, the previous computations ensure that $(t,x(t)),\ (t,y(t)) \in \cap_{i=1}^N \tau_i^{-1}(0,\infty) \cap D$ and thus, by (4.4), we have that

$$\|x(t)-y(t)\| \le \int_{t_0}^t \|f(s,x(s))-f(s,y(s))\|\,ds \le \int_{t_0}^t c_1(s)\,\|x(s)-y(s)\|\,ds, \quad t \in (t_0, t_0+\alpha),$$

so we deduce from Gronwall's inequality that $\|x(t)-y(t)\| = 0$ on $(t_0, t_0+\alpha)$. □

Remark 4.2. A particular case of Theorem 4.1, assuming the transversality condition

$$\nabla\tau_i(t_0,x_0)\cdot(1, f(t_0,x_0)) \ne 0, \quad \text{for all } i = 1,2,\ldots,N, \tag{4.5}$$

instead of condition (iii), was proven in [12, Corollary 4.4]. Notice that condition (i) in Theorem 4.1 is satisfied in particular if f can be expressed as the composition

$$f(t,x) = F(t,x,g_1(\tau_1(t,x)), g_2(\tau_2(t,x)),\ldots,g_N(\tau_N(t,x))) \text{ for some } N \in \mathbb{N}, \tag{4.6}$$

where $F : U \times V \subset \mathbb{R}^{n+1}\times\mathbb{R}^N \to \mathbb{R}^n$ satisfies:

$\overline{(i)}$ There exists $K > 0$ such that for every $(t,x,\xi),(t,y,\eta) \in U \times V$ we have

$$\|F(t,x,\xi) - F(t,y,\eta)\| \le K\,\|x-y\| + K\|\xi-\eta\|.$$

The particular form of f given by (4.6) was considered in [3]. On the other hand, the same condition (4.5) is also a key assumption in the uniqueness result obtained in [15] which was motivated by a certain n–body problem of classical electrodynamics.

Example 4.3. The following modification of the Example 2 in [15] illustrates how our Theorem 4.1 can improve the applicability of previous results. Let us consider the initial value problem

$$x' = (|t|^\alpha + |x|^{5/3})^{1/3}, \quad x(0) = 0, \quad \alpha \ge 1.$$

Note that $f(t,x) = (|t|^\alpha + |x|^{5/3})^{1/3}$ can be expressed as $f(t,x) = g(\tau(t,x))$ with $g(\tau) = \tau^{1/3}$ and $\tau(t,x) = |t|^\alpha + |x|^{5/3}$. Hence, it is easy to check that conditions (i)–(iv) in Theorem 4.1 are satisfied for $1 < \alpha < 3/2$.

However, condition (4.5) does not hold for any $1 < \alpha < 3/2$, so neither [12, Corollary 4.4] nor [15] are applicable.

Now we will show that Theorem 4.1 enables us to obtain also alternative uniqueness arguments for the examples provided in [11].

Example 4.4. Consider the initial value problem

$$x'(t) = p(t) + q(t)\,|x(t)|^r, \quad x(0) = 0, \tag{4.7}$$

where $0 < r < 1$, p, q are non-negative continuous functions and there exists $\varepsilon > 0$ such that $p(t) > 0$ for all $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$ and

$$\left| \int_0^t p(s) ds \right|^{r-1} \in L^1(-\varepsilon, \varepsilon). \tag{4.8}$$

Note that the function $f(t, x) = p(t) + q(t) |x|^r$ can be expressed in the form $f(t, x) = F(t, x, g(\tau(t, x)))$ with $\tau(t, x) = x$, $g(\tau) = |\tau|^r$ and $F(t, x, \xi) = p(t) + q(t)\xi$. One may easily verify conditions (i)–(ii) in Theorem 4.1 (see also Remark 4.2). In addition,

$$|\nabla\tau(t, x) \cdot (1, f(t, x))| = f(t, x) \geq M(t) = p(t),$$

so conditions (iii) and (iv) hold. Therefore, Theorem 4.1 implies local uniqueness for (4.7).

To end the example, observe that Theorem 4.1 is not applicable to problem (4.7) with $p(t) = t^2$, $q(t) = 1$ and $r = 1/4$ since condition (4.8) does not hold, but uniqueness still can be directly deduced from Theorem 2.1.

Finally, we provide an example for which uniqueness is guaranteed by Theorem 4.1, whereas the criteria in Sections 2 and 3 are not applicable.

Example 4.5. Consider the initial value problem

$$x' = \sqrt{|x - t|} + \sqrt{|t|} + 1, \qquad x(0) = 0,$$

which can be expressed in the form $f(t, x) = F(t, x, g(\tau(t, x)))$, $F(t, x, \xi) = \sqrt{|t|} + \xi + 1$, $g(r) = \sqrt{|r|}$ and $\tau(t, x) = x - t$.

Observe that the functions F, g and τ are under the hypotheses of Theorem 4.1. Moreover, for $t \neq 0$ we have

$$\nabla\tau(t, x) \cdot (1, f(t, x)) = -1 + \sqrt{|x - t|} + \sqrt{|t|} + 1 \geq M(t) = \sqrt{|t|} > 0,$$

so Theorem 4.1 implies the existence and uniqueness of a local solution for the initial value problem.

We highlight that the transversality condition (4.5) is not satisfied at the initial condition $(t_0, x_0) = (0, 0)$. In addition, f is not Lipschitz along any function φ with $\varphi(0) = 0$ since

$$|f(0, 0) - f(k, \varphi(k))| = \sqrt{|\varphi(k) - k|} + \sqrt{|k|},$$

which is not smaller than $L |k|$ for $k > 0$ small enough. Hence, in virtue of Remark 3.6, Theorem 3.5 cannot be applied here.

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