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Existence and multiplicity of nontrivial solutions to the modified Kirchhoff equation without the growth and Ambrosetti–Rabinowitz conditions

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Abstract. The paper focuses on the modified Kirchhoff equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u-u\Delta(u^2)+V(x)u=\lambda f(u),\quad x\in\mathbb{R}^N,$$

where $a,b>0$, $V(x)\in C(\mathbb{R}^N,\mathbb{R})$ and $\lambda<1$ is a positive parameter. We just assume that the nonlinearity $f(t)$ is continuous and superlinear in a neighborhood of $t=0$ and at infinity. By applying the perturbation method and using the cutoff function, we get existence and multiplicity of nontrivial solutions to the revised equation. Then we use the Moser iteration to obtain existence and multiplicity of nontrivial solutions to the above original Kirchhoff equation. Moreover, the nonlinearity $f(t)$ may be supercritical.

Keywords: modified Kirchhoff-type equation, cutoff function, perturbation approach.

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1 Introduction

In this paper, we are devoted to studying the following modified Kirchhoff equation:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u-u\Delta(u^2)+V(x)u=\lambda f(u),\quad x\in\mathbb{R}^N,\tag{1.1}$$

where $a,b>0$, $V(x)\in C(\mathbb{R}^N,\mathbb{R})$, $\lambda<1$ is a positive parameter and f is continuous in $\mathbb{R}$. The equation (1.1) is the Euler–Lagrange equation of the energy functional

$$I_\lambda(u)=\frac{a}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx+\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^2+\frac{1}{2}\int_{\mathbb{R}^N}\left(V(x)u^2+2u^2|\nabla u|^2\right)dx-\lambda\int_{\mathbb{R}^N}F(u)dx,$$

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where $F(t)=\int_0^t f(s)\mathrm{d}s$.

Kirchhoff's model is a general version of the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

which was first proposed by Kirchhoff in [6] for extending the classical D'Alembert's wave equations for free vibration of elastic strings. Kirchhoff's model takes into account the changes in string length produced by transverse vibration. In (1.2), L is the length of the string, h is the area of cross section, E denotes the Young modulus of the material, ρ is the mass density and P_0 denotes the initial tension. In addition, we have to point out that nonlocal problems also appear in other fields as biological systems, where u describes a process which depends on the average of itself (for example, population density). Some early classical studies of Kirchhoff equations can be found in Bernstein [1] and Pohožaev [14]. Much attention was received after Lions [9] introducing an abstract functional framework to this problem. For more relevant mathematical and physical background, we refer readers to papers [8, 13, 21], and the references therein.

Especially, in recent paper [19], Wu studied the following problem:

$$-\left(a+b\int_{R^N}|\nabla u|^2dx\right)\Delta u+V(x)u=g(x,u),\quad x\in\mathbb{R}^N \tag{1.3}$$

and obtained four new existence results of nontrivial solutions and a sequence of high energy solutions for equation (1.3).

When $a=1$ and $b=0$, (1.3) is reduced to the well known quasilinear Schrödinger equation

$$-\Delta u+V(x)u-\Delta\left(u^2\right)u=g(x,u),\quad x\in\mathbb{R}^N. \tag{1.4}$$

Several methods can be used to solve the equation (1.4), such as, the existence of a positive ground state solution has been studied in [10, 15] by using a constrained minimization argument; the problem is transformed to a semilinear one in [2, 11] by a change of variables (dual approach); Nehari method is used to get the existence results of ground state solutions in [12, 17]. Especially, in [7], the existence of positive solutions, negative solutions and sequence of high energy solutions for the following problem

$$-\Delta u+V(x)u-\Delta\left(|u|^{2\alpha}\right)|u|^{2\alpha-2}u=g(x,\psi),\quad x\in R^N$$

was studied via a perturbation method, where $\alpha>\frac{3}{4}$, $V\in C\left(R^N,R\right)$ and $g\in C\left(R^N\times R,R\right)$.

Recently, Feng et al. [3] studied the following modified Kirchhoff type equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u-u\Delta(u^2)+V(x)u=h(x,u),\quad x\in\mathbb{R}^N, \tag{1.5}$$

where $a>0, b\geq 0, h\in C\left(\mathbb{R}^N\times\mathbb{R},\mathbb{R}\right)$ and $V\in C\left(\mathbb{R}^N,\mathbb{R}\right)$. Under appropriate assumptions on $V(x)$ and $h(x,u)$, some existence results for positive solutions, negative solutions and sequence of high energy solutions were obtained via a perturbation method. Subsequently, in 2015, Wu [20] studied the existence of infinitely many small energy solutions for equation (1.5) by applying Clark's Theorem to a perturbation functional. And in the same year, He [4] proved the existence of infinitely many solutions for equation (1.5) by the dual method and the non-smooth critical point theory. Last year, Huang and Jia [5] obtained the existence of infinitely many sign-changing solutions for equation (1.5) with $a = 1$ and $h(x,u) = h(u)$ by genus theory.

In the present paper, we assume that $f \in C(\mathbb{R})$ and $V \in C(\mathbb{R}^N)$ satisfy the following conditions

(f_1) $\lim_{t\to 0} \frac{f(t)}{t} = 0$;

(f_2) $\lim_{t\to +\infty} \frac{f(t)}{t} = +\infty$;

(V) $V(x)$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$, and $\lim_{|x|\to\infty} V(x) = +\infty$.

Moreover, f may be supercritical. But we do not assume the Ambrosetti–Rabinowitz condition or increasing condition.

Next, we give our main results.

Theorem 1.1. *Assume that $(V), (f_1), (f_2)$ hold. Then equation (1.1) has a positive and a negative weak solutions for all λ small enough.*

Theorem 1.2. *If $(V), (f_1), (f_2)$ hold and $f(t)$ is odd, then the equation (1.1) has a sequence $\{u_n\}$ of solutions such that $I_\lambda(u_n) \to +\infty$ for all λ small enough.*

This paper is organized as follows. In Section 2, we present the variational framework and some lemmas, which are bases of Section 3. In Section 3, we give the proof of Theorems 1.1 and 1.2.

In what follows, C_0, C, c_i and $C_i (i = 1, 2, \dots)$ denote positive generic constants.

2 Preliminaries and revised functional

In this section, we give work space, the revised functional and some lemmas.

Let $C_0^\infty(\mathbb{R}^N)$ be the collection of smooth functions with compact supports. Let

$$H^1\left(\mathbb{R}^N\right) := \left\{u \in L^2\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} |\nabla u|^2 dx < +\infty\right\}$$

with the inner product

$$\langle u, v\rangle_{H^1} = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv)\, dx$$

and the norm

$$\|u\|_{H^1} = \langle u, u\rangle_{H^1}^{1/2}.$$

Set

$$H_V^1\left(\mathbb{R}^N\right) := \left\{u \in H^1\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty\right\}$$

with the inner product

$$\langle u, v\rangle_{H_V^1} = \int_{\mathbb{R}^N} [\nabla u \cdot \nabla v + V(x)uv]\, dx$$

and the norm

$$\|u\|_{H_V^1} = \langle u, u\rangle_{H_V^1}^{1/2}.$$

Then both $H^1(\mathbb{R}^N)$ and $H_V^1(\mathbb{R}^N)$ are Hilbert spaces. Set $E = H_V^1(\mathbb{R}^N) \cap W^{1,4}(\mathbb{R}^N)$ with the norm $\|u\|_E = \|u\|_{H_V^1} + \|u\|_{W^{1,4}}$. Then E is a reflexive Banach space.

Notice that there is no growth condition $|f(t)| \leqslant C|t| + C|t|^{q-1}$ and no Ambrosetti–Rabinowitz condition $tf(t) - 4F(t) \geqslant 0$. So we need the cutoff function.

By (f_2), there exists $M > 0$ large such that $f(M) > 0$. And then given $M > 0$, let

$$h_M(t) = \begin{cases} f(t), & 0 < t \leqslant M \\ C_M t^{p-1}, & t > M \\ 0, & t \leqslant 0, \end{cases}$$

where $C_M = f(M)/M^{p-1}$ and $4 < p < 22^*$. The continuity of f implies the continuity of h_M. Moreover, by (f_1) and (f_2), h_M satisfies that

(h_1) There exists $4 < p < 22^*$ if $N \geq 3$ and $4 < p < \infty$ if $N = 1, 2$ such that

$$|h_M(t)| \leqslant C'_M|t| + C_M|t|^{p-1} \leqslant C(M)\left(|t| + |t|^{p-1}\right), \quad \forall t \in \mathbb{R},$$

where $C'_M = \max_{t\in[0,M]} |f(t)|/t$ and $C(M) = \max\{C'_M, C_M\}$;

(h_2) $\lim_{t\to 0} \frac{h_M(t)}{t} = 0$;

(h_3) There exists $\mu > 4$ and $r > M$ such that

$$\inf_{|t|=r} H_M(t) > 0$$

and

$$\mu H_M(t) \leq h_M(t)t$$

for $|t| \geq r$, where $H_M(t) = \int_0^t h_M(s)\mathrm{d}s$.

By [22, Lemma 3.4] and the condition (V), we get that the embedding $H^1_V(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ is compact for each $2 \leq s < 2^*$.

In what follows, we consider the revised problem

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u - u\Delta(u^2) + V(x)u = \lambda h_M(u), \quad x \in \mathbb{R}^N. \tag{2.1}$$

Equation (2.1) is the Euler–Lagrange equation associated of the natural energy functional $J_\lambda(u) : E \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^N} \left(V(x)u^2 + 2u^2|\nabla u|^2\right) dx \\ - \lambda \int_{\mathbb{R}^N} H_M(u) dx.$$

For $\theta \in (0,1]$, let $J_{\theta,\lambda}(u) = \frac{1}{4}\theta \int_{\mathbb{R}^N} \left(|\nabla u|^4 + u^4\right) dx + J_\lambda(u)$. Let $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. Set

$$J^\pm_\lambda(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^N} \left(V(x)u^2 + 2u^2|\nabla u|^2\right) dx \\ - \lambda \int_{\mathbb{R}^N} H_M(u^\pm) dx$$

and $J^{\pm}_{\theta,\lambda}(u) = \frac{1}{4}\theta \int_{\mathbb{R}^N} \left(|\nabla u|^4 + u^4\right) dx + J^{\pm}_{\lambda}(u)$.

A sequence $\{u_n\} \subset E$ is called a P. S. sequence of J_λ if $\{J_\lambda(u_n)\}$ is bounded and $J'_\lambda(u_n) \to 0$ in E^*. We say that J_λ satisfies the P. S. condition if every P. S. sequence possesses a convergent subsequence.

Our goal is to first prove that the critical point of $J_\lambda(u)$ can be obtained as limits of critical points of $J_{\theta,\lambda}(u)$. And then we need to prove that the nontrivial critical point u of $J_\lambda(u)$ satisfying $\|u\|_{L^\infty} \leq M$ is a nontrivial solution of (1.1).

Lemma 2.1. *Assume that $(V), (h_1)$ and (h_2) hold. Then the functionals J_λ and $J^{\pm}_{\theta,\lambda}$ are well defined in E and $J_\lambda, J^{\pm}_{\theta,\lambda} \in C^1(E, \mathbb{R})$.*

Proof. The proof is similar to [3, Lemma 2.1], we omit it here. □

Lemma 2.2. *Assume that $(V), (h_1)$ and (h_2) hold. Then every bounded P. S. sequence $\{u_n\} \subset E$ of $J_{\theta,\lambda}$ $\left(\text{respectively, } J^{\pm}_{\theta,\lambda}\right)$ possesses a convergent subsequence.*

Proof. The proof is analogous to [3, Lemma 2.2], we omit it here. □

Lemma 2.3. *Assume that (V) and (h_1)–(h_3) hold. Let $\{\theta_n\} \subset (0, 1]$ be such that $\theta_n \to 0$. Let $u_n \in E$ be a critical point of $J_{\theta_n,\lambda}$ with $J_{\theta_n,\lambda}(u_n) \leq c$ for some constant c independent of n. Then, passing to a subsequence, we have $u_n \to u$ in $H^1_V(\mathbb{R}^N)$, $u_n\nabla u_n \to u\nabla u$ in $L^2(\mathbb{R}^N)$, $\theta_n \int_{\mathbb{R}^N} \left(|\nabla u_n|^4 + u_n^4\right) dx \to 0$, $J_{\theta_n,\lambda}(u_n) \to J_\lambda(u)$ and u is a critical point of J_λ.*

Proof. Step 1: We need to prove that the sequences $\left\{\int_{R^N} u_n^2 |\nabla u_n|^2 dx\right\}, \left\{\theta_n \|u_n\|^4_{W^{1,4}}\right\}$ and $\left\{\|u_n\|^2_{H^1_V}\right\}$ are bounded.

By (h_2), for $0 < \varepsilon_0 < \frac{1}{4}\left(\frac{1}{2} - \frac{1}{\mu}\right)V_0$, there exists $\delta > 0$ such that

$$\left|\frac{1}{\mu} t h_M(t) - H_M(t)\right| \leq \varepsilon_0 t^2$$

for all $|t| \leq \delta$. By (h_1), for $\delta \leq |t| \leq r$ (r is the constant appearing in the condition (h_3)), one obtains

$$\left|\frac{1}{\mu} t h_M(t) - H_M(t)\right| \leq 2C(M)\left(1 + r^{p-2}\right) t^2,$$

where $C(M)$ is the constant appearing in the condition (h_1). Thus, we get

$$\left|\frac{1}{\mu} t h_M(t) - H_M(t)\right| \leq \varepsilon_0 t^2 + 2C(M)\left(1 + r^{p-2}\right) t^2, \quad \forall t \in [-r, r].$$

Since $\lim_{|x|\to\infty} V(x) = +\infty$, there exists $\rho_0 > 0$ such that

$$\frac{1}{4}\left(\frac{1}{2} - \frac{1}{\mu}\right) V(x) > 2\lambda C(M)\left(1 + r^{p-2}\right)$$

for all $|x| \geq \rho_0$. Thus,

$$\left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} V(x) u_n^2 dx + \lambda \int_{|u_n(x)| \leq r} \left[\frac{1}{\mu} u_n h_M(u_n) - H_M(u_n)\right] dx$$
$$\geq \left(\frac{1}{4} - \frac{1}{2\mu}\right) \int_{\mathbb{R}^N} V(x) u_n^2 dx - 2\lambda C(M)\left(1 + r^{p-2}\right) r^2 \left|B_{\rho_0}\right|, \quad (2.2)$$

where $B_{\rho_0} := \left\{x \in R^N : |x| < \rho_0\right\}, \left|B_{\rho_0}\right| := \text{meas}\left(B_{\rho_0}\right)$. Moreover, since $u_n \in E$ is a critical point of $J_{\theta_n,\lambda}$, for each $\phi \in E$, we have

$$\begin{aligned} 0 = \left\langle J'_{\theta_n,\lambda}\left(u_n\right), \phi\right\rangle &= \theta_n \int_{\mathbb{R}^N} \left[\left|\nabla u_n\right|^2 \nabla u_n \nabla\phi + \left|u_n\right|^2 u_n\phi\right] dx \\ &+ \left(a + b\int_{\mathbb{R}^N} \left|\nabla u_n\right|^2 dx\right) \int_{\mathbb{R}^N} \nabla u_n \nabla \phi dx + 2\int_{\mathbb{R}^N} \left(u_n^2 \nabla u_n \nabla\phi + \left|\nabla u_n\right|^2 u_n \phi\right) dx \\ &+ \int_{\mathbb{R}^N} V(x) u_n \phi dx - \lambda \int_{\mathbb{R}^N} h_M\left(u_n\right)\phi dx. \end{aligned} \tag{2.3}$$

Hence, it follows from (h_3) and (2.2) that

$$\begin{aligned} c &\geq J_{\theta_n,\lambda}\left(u_n\right) \\ &= J_{\theta_n,\lambda}\left(u_n\right) - \frac{1}{\mu}\left\langle J'_{\theta_n,\lambda}\left(u_n\right), u_n\right\rangle \\ &= \left(\frac{1}{4} - \frac{1}{\mu}\right)\theta_n \left\|u_n\right\|^4_{W^{1,4}} + \left(\frac{a}{2} - \frac{a}{\mu}\right)\int_{\mathbb{R}^N}\left|\nabla u_n\right|^2 dx + \left(\frac{b}{4} - \frac{b}{\mu}\right)\left(\int_{\mathbb{R}^N}\left|\nabla u_n\right|^2 dx\right)^2 \\ &\quad + \left(1 - \frac{4}{\mu}\right)\int_{\mathbb{R}^N}\left|\nabla u_n\right|^2 u_n^2 dx + \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x) u_n^2 dx \\ &\quad + \lambda\int_{\mathbb{R}^N}\left[\frac{1}{\mu} u_n h_M\left(u_n\right) - H_M\left(u_n\right)\right] dx \\ &\geq \left(\frac{1}{4} - \frac{1}{\mu}\right)\theta_n \left\|u_n\right\|^4_{W^{1,4}} + \left(\frac{a}{2} - \frac{a}{\mu}\right)\int_{\mathbb{R}^N}\left|\nabla u_n\right|^2 dx + \left(\frac{b}{4} - \frac{b}{\mu}\right)\left(\int_{\mathbb{R}^N}\left|\nabla u_n\right|^2 dx\right)^2 \\ &\quad + \left(1 - \frac{4}{\mu}\right)\int_{\mathbb{R}^N}\left|\nabla u_n\right|^2 u_n^2 dx + \left(\frac{1}{4} - \frac{1}{2\mu}\right)\int_{R^N} V(x) u_n^2 dx - 2\lambda C(M)\left(1 + r^{p-2}\right) r^2 \left|B_{\rho_0}\right| \\ &\geq \left(\frac{1}{4} - \frac{1}{\mu}\right)\theta_n \left\|u_n\right\|^4_{W^{1,4}} + c_1 \left\|u_n\right\|^2_{H^1_V} + c_2 \int_{\mathbb{R}^N} u_n^2 \left|\nabla u_n\right|^2 dx - \lambda C_1(M), \end{aligned}$$

where $C_1(M) = 2C(M)\left(1 + r^{p-2}\right) r^2 \left|B_{\rho_0}\right|$. Therefore, we get

$$\left(\frac{1}{4} - \frac{1}{\mu}\right)\theta_n \left\|u_n\right\|^4_{W^{1,4}} + c_1 \left\|u_n\right\|^2_{H^1_V} + c_2 \int_{\mathbb{R}^N} u_n^2 \left|\nabla u_n\right|^2 dx \leq C_0 + \lambda C_1(M). \tag{2.4}$$

By (2.4), going if necessary to a subsequence, we get $u_n \rightharpoonup u$ in $H^1_V\left(\mathbb{R}^N\right), u_n \nabla u_n \rightharpoonup u\nabla u$ in $L^2\left(\mathbb{R}^N\right), u_n \to u$ in $L^s\left(\mathbb{R}^N\right)$ for $s \in [2, 22^*)$ and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$. This completes the proof of Step 1.

Step 2: We claim that $u_n \in L^\infty\left(\mathbb{R}^N\right)$, $\|u_n\|_{L^\infty} \leq M$ and $\|u\|_{L^\infty} \leq M$, where the positive constant M is independent of n.

Depending on (2.4), we infer

$$\left\|u_n\right\|^4_{L^{22^*}} = \left\|u_n^2\right\|^2_{L^{2^*}} \leq C\left\|\nabla u_n^2\right\|^2_{L^2} \leq C_0 + \lambda C_1(M). \tag{2.5}$$

Set $T > 2, r > 0$ and $\tilde{u}_n^T = \gamma\left(u_n\right)$, where $\gamma : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $\gamma(t) = t$ for $|t| \leq T - 1, \gamma(-t) = -\gamma(t); \gamma'(t) = 0$ for $t \geq T$ and $\gamma'(t)$ is decreasing in $[T-1, T]$. This means that $\tilde{u}_n^T = u_n$ for $|u_n| \leq T - 1; \left|\tilde{u}_n^T\right| = \left|\gamma\left(u_n\right)\right| \leq |u_n|$ for $T - 1 \leq |u_n| \leq T; \left|\tilde{u}_n^T\right| = C_T > 0$ for $|u_n| \geq T$, where $T - 1 \leq C_T \leq T$.

Setting $\phi = u_n \left|\tilde{u}_n^T\right|^{2r}$, then we easily infer that $\phi \in E$. Therefore, it follows from (2.3) that

$$
\begin{aligned}
&\lambda \int_{\mathbb{R}^N} h_M(u_n)\phi dx - \int_{\mathbb{R}^N} V(x)u_n\phi dx \\
&\quad = \theta_n \int_{\mathbb{R}^N} \left[|\nabla u_n|^2 \nabla u_n \nabla \phi + |u_n|^2 u_n\phi\right] dx + \left(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N} \nabla u_n \nabla \phi dx \\
&\qquad + 2\int_{\mathbb{R}^N}\left(u_n^2 \nabla u_n \nabla \phi + |\nabla u_n|^2 u_n \phi\right) dx \\
&\quad \geq 2\int_{\mathbb{R}^N} u_n^2 \nabla u_n \nabla \phi dx \\
&\quad = 2\int_{|u_n|\geq T} |u_n|^2 |\nabla u_n|^2 \left|\tilde{u}_n^T\right|^{2r} dx + 2\int_{|u_n|\leq T-1} (1+2r)|u_n|^{2r+2}|\nabla u_n|^2 dx \\
&\qquad + 2\int_{T-1<|u_n|<T}\left[|\gamma(u_n)|^{2r} + 2ru_n\gamma(u_n)|\gamma(u_n)|^{2r-2}\gamma'(u_n)\right]|u_n|^2|\nabla u_n|^2 dx \\
&\quad \geq \frac{1}{2}\int_{|u_n|\geq T}\left|\nabla\left[|u_n|^2\left(\tilde{u}_n^T\right)^r\right]\right|^2 dx + \int_{|u_n|\leq T-1}|u_n|^{2r+2}|\nabla u_n|^2 dx \\
&\qquad + \frac{1}{2}\int_{T-1\leq|u_n|\leq T}\left|\left(\tilde{u}_n^T\right)^r \nabla\left(|u_n|^2\right)\right|^2 dx \\
&\qquad + 2r\int_{T-1\leq|u_n|\leq T}|u_n|^4\left|\tilde{u}_n^T\right|^{2r-2}\left(\gamma'(u_n)\right)^2|\nabla u_n|^2 dx \\
&\quad = \frac{1}{2}\int_{|u_n|\geq T}\left|\nabla\left[|u_n|^2\left(\tilde{u}_n^T\right)^r\right]\right|^2 dx + \int_{|u_n|\leq T-1}|u_n|^{2r+2}|\nabla u_n|^2 dx \\
&\qquad + \frac{1}{2}\int_{T-1\leq|u_n|\leq T}\left|\left(\tilde{u}_n^T\right)^r\nabla\left(|u_n|^2\right)\right|^2 dx + \frac{2}{r}\int_{T-1\leq|u_n|\leq T}\left||u_n|^2\nabla\left(\tilde{u}_n^T\right)^r\right|^2 dx \\
&\quad \geq \frac{2}{(r+2)^2}\int_{|u_n|\geq T}\left|\nabla\left[|u_n|^2\left(\tilde{u}_n^T\right)^r\right]\right|^2 dx + \frac{1}{(r+2)^2}\int_{|u_n|\leq T-1}\left|\nabla\left[|u_n|^2\left(\tilde{u}_n^T\right)^r\right]\right|^2 dx \\
&\qquad + \frac{2}{(r+2)^2}\int_{T-1\leq|u_n|\leq T}\left[\left|\left(\tilde{u}_n^T\right)^r\nabla\left(|u_n|^2\right)\right|^2 + \left||u_n|^2\nabla\left(\tilde{u}_n^T\right)^r\right|^2\right] dx \\
&\quad \geq \frac{1}{(r+2)^2}\int_{|u_n|\geq T}\left|\nabla\left[|u_n|^2\left(\tilde{u}_n^T\right)^r\right]\right|^2 dx + \frac{1}{(r+2)^2}\int_{|u_n|\leq T-1}\left|\nabla\left[|u_n|^2\left(\tilde{u}_n^T\right)^r\right]\right|^2 dx \\
&\qquad + \frac{1}{(r+2)^2}\int_{T-1\leq|u_n|\leq T}\left|\nabla\left[|u_n|^2\left(\tilde{u}_n^T\right)^r\right]\right|^2 dx \\
&\quad = \frac{1}{(r+2)^2}\int_{R^N}\left|\nabla\left[|u_n|^2\left(\tilde{u}_n^T\right)^r\right]\right|^2 dx.
\end{aligned}
\tag{2.6}
$$

Choosing $0 < \lambda \leq V_0/C'_M$, then it follows from (h_1) and (2.6) that

$$
\frac{1}{(r+2)^2}\int_{\mathbb{R}^N}\left|\nabla\left[u_n^2\left(\tilde{u}_n^T\right)^r\right]\right|^2 dx \leq \lambda C_M \int_{\mathbb{R}^N}|u_n|^p\left|\tilde{u}_n^T\right|^{2r} dx. \tag{2.7}
$$

By (2.5) and Hölder inequality, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N}|u_n|^p\left|\tilde{u}_n^T\right|^{2r} dx \\
&\quad = \int_{\mathbb{R}^N}|u_n|^{p-4}\left|\tilde{u}_n^T\right|^{2r}|u_n|^4 dx \\
&\quad \leq \left(\int_{\mathbb{R}^N}|u_n|^{(p-4)\frac{4N}{(p-4)(N-2)}} dx\right)^{\frac{(p-4)(N-2)}{4N}}\left(\int_{\mathbb{R}^N}\left(\left|\tilde{u}_n^T\right|^{2r} u_n^4\right)^{\frac{4N}{4N-(p-4)(N-2)}} dx\right)^{\frac{4N-(p-4)(N-2)}{4N}}
\end{aligned}
$$

$$= \left(\int_{\mathbb{R}^N} |u_n|^{22^*}\, dx\right)^{\frac{(p-4)(N-2)}{4N}} \left(\int_{\mathbb{R}^N} \left(\left|\tilde{u}_n^T\right|^r u_n^2\right)^{\frac{4N}{4N-(p-4)(N-2)}} dx\right)^{\frac{4N-(p-4)(N-2)}{4N}}$$

$$\leq \left(C_0 + \lambda C_1(M)\right)^{\frac{p-4}{4}} \left(\int_{\mathbb{R}^N} \left(\left|\tilde{u}_n^T\right|^r u_n^2\right)^{\frac{8N}{4N-(p-4)(N-2)}} dx\right)^{\frac{4N-(p-4)(N-2)}{4N}}. \tag{2.8}$$

Since $u_n^2 \left|\tilde{u}_n^T\right|^r \in D^{1,2}\left(\mathbb{R}^N\right)$, by the Sobolev embedding theorem, we infer

$$\left[\int_{\mathbb{R}^N} \left(u_n^2 \left|\tilde{u}_n^T\right|^r\right)^{2^*} dx\right]^{\frac{2}{2^*}} \leq C \int_{\mathbb{R}^N} \left|\nabla\left[u_n^2 \left(\tilde{u}_n^T\right)^r\right]\right|^2 dx. \tag{2.9}$$

Then by (2.7), (2.8) and (2.9), one has

$$\left[\int_{\mathbb{R}^N} \left(u_n^2 \left|\tilde{u}_n^T\right|^r\right)^{2^*} dx\right]^{\frac{2}{2^*}} \leq \lambda C_2(M)(r+2)^2 \left[\int_{\mathbb{R}^N} \left(\left|\tilde{u}_n^T\right|^r u_n^2\right)^{\frac{8N}{4N-(p-4)(N-2)}} dx\right]^{\frac{4N-(p-4)(N-2)}{4N}},$$

where the constant $C_2(M) > 0$ is dependent on M. Since $4 < p < 22^*$, $d := 2^*/q = \frac{2^*}{2} - \frac{p}{4} + 1 > 1$, where $q = \frac{8N}{4N-(p-4)(N-2)}$. Then

$$\left(\int_{\mathbb{R}^N} \left(u_n^2 \left|\tilde{u}_n^T\right|^r\right)^{qd} dx\right)^{\frac{1}{qd(r+2)}} \leq \left[\lambda C_2(M)(r+2)^2\right]^{\frac{1}{2(r+2)}} \left(\int_{\mathbb{R}^N} \left[u_n^2 \left|\tilde{u}_n^T\right|^r\right]^q dx\right)^{\frac{1}{q(r+2)}}. \tag{2.10}$$

Take $r = r_0$ be such that $(2+r_0)\, q = 22^*$. From $\left|\tilde{u}_n^T\right| = |\gamma(u_n)| \leq |u_n|$ and (2.5), one has

$$\int_{\mathbb{R}^N} \left[\left|\tilde{u}_n^T\right|^{r_0} u_n^2\right]^q dx \leq \int_{\mathbb{R}^N} |u_n|^{(2+r_0)q}\, dx < C_0 + \lambda C_1(M).$$

Takeing the limit $T \to \infty$ in (2.10) with $r = r_0$, we obtain

$$\left(\int_{\mathbb{R}^N} |u_n|^{(2+r_0)qd}\, dx\right)^{\frac{1}{qd(r_0+2)}} \leq \left[\lambda C_2(M)\left(r_0+2\right)^2\right]^{\frac{1}{2(r_0+2)}} \left(\int_{\mathbb{R}^N} |u_n|^{(2+r_0)q}\, dx\right)^{\frac{1}{q(r_0+2)}}.$$

Further, setting $2 + r_1 = d\,(2+r_0)$, we get

$$\left(\int_{\mathbb{R}^N} |u_n|^{(2+r_1)q}\, dx\right)^{\frac{1}{q(r_1+2)}} \leq \left[\lambda C_2(M)\left(r_0+2\right)^2\right]^{\frac{1}{2(r_0+2)}} \left(\int_{\mathbb{R}^N} |u_n|^{(2+r_0)q}\, dx\right)^{\frac{1}{q(r_0+2)}}.$$

Inductively, we have

$$\begin{aligned} \left(\int_{\mathbb{R}^N} |u_n|^{(2+r_{k+1})q}\, dx\right)^{\frac{1}{q(r_{k+1}+2)}} &\leq \left[\lambda C_2(M)\left(r_k+2\right)^2\right]^{\frac{1}{2(r_k+2)}} \left(\int_{\mathbb{R}^N} |u_n|^{(2+r_k)q}\, dx\right)^{\frac{1}{q(r_k+2)}} \\ &\leq \prod_{i=0}^{k} \left[\lambda C_2(M)\left(r_i+2\right)^2\right]^{\frac{1}{2(r_i+2)}} \left(\int_{\mathbb{R}^N} |u_n|^{(2+r_0)q}\, dx\right)^{\frac{1}{q(r_0+2)}}, \end{aligned}$$

where $(2+r_i) = d^i\,(2+r_0)$ $(i = 0, 1, \ldots, k)$. Moreover,

$$\begin{aligned} \prod_{i=0}^{k} \left[\lambda C_2(M)\left(r_i+2\right)^2\right]^{\frac{1}{2(r_i+2)}} &= \exp\left\{\sum_{i=0}^{k} \frac{\ln\sqrt{\lambda C_2(M)}d^i\left(r_0+2\right)}{d^i\left(r_0+2\right)}\right\} \\ &= \exp\left\{\sum_{i=0}^{k}\left[\frac{\ln\sqrt{\lambda C_2(M)}\left(r_0+2\right)}{d^i\left(r_0+2\right)} + \frac{i\ln d}{d^i\left(r_0+2\right)}\right]\right\} \end{aligned}$$

is convergent as $k \to \infty$. Let $C_k = \prod_{i=0}^{k} \left[\lambda C_2(M)\left(r_i+2\right)^2\right]^{\frac{1}{2(r_i+2)}}$. For C_k, we can choose $0 < \lambda_0 \leq C_0/C_1(M)$ small enough and $\frac{1}{2}\lambda_0 < \lambda < \lambda_0$ such that $C_k \to C_\infty > 0$ as $k \to \infty$ and $C_\infty \leq M/(2C_0^{\frac{1}{4}})$. Then we get

$$\|u_n\|_{L^{(2+r_0)qd^{k+1}}} \leq C_k \|u_n\|_{L^{22^*}}.$$

Let $k \to \infty$, for fixed constant M and $\frac{1}{2}\lambda_0 < \lambda < \lambda_0$, by (2.5) we have

$$\|u_n\|_{L^\infty} \leq C_\infty \|u_n\|_{L^{22^*}} \leq M, \quad \|u\|_{L^\infty} \leq M. \tag{2.11}$$

Step 3: We will show that u is a critical point of J_λ.

For any $\psi \in C_0^\infty(\mathbb{R}^N)$, there exists a bounded domain $\Omega \subset \mathbb{R}^N$ such that $\operatorname{supp}(\psi) \subset \Omega$. Thus, by (2.11), we know $\phi = \psi \exp(-Ku_n) \in E$ for any $\psi \geq 0$ and $K > 0$. Taking $\phi = \psi \exp(-Ku_n)$ as the test function in (2.3), we have

$$\begin{aligned}
0 = {} & \theta_n \int_{\mathbb{R}^N} \exp(-Ku_n)\left[|\nabla u_n|^2 \nabla u_n (\nabla\psi - K\psi\nabla u_n) + |u_n|^2 u_n\psi\right] dx \\
& + \left(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^N} \exp(-Ku_n) \nabla u_n (\nabla\psi - K\psi\nabla u_n)\, dx \\
& + 2\int_{\mathbb{R}^N} \left[\exp(-Ku_n) u_n^2 \nabla u_n(\nabla\psi - K\psi\nabla u_n) + \exp(-Ku_n)\psi |\nabla u_n|^2 u_n\right] dx \\
& + \int_{\mathbb{R}^N} V(x) u_n \psi \exp(-Ku_n)\, dx - \lambda \int_{\mathbb{R}^N} h_M(u_n)\psi \exp(-Ku_n)\, dx \\
\leq {} & \theta_n \int_{\mathbb{R}^N} \exp(-Ku_n)\left[|\nabla u_n|^2 \nabla u_n \nabla\psi + |u_n|^2 u_n\psi\right] dx \\
& + \left(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^N} \exp(-Ku_n) \nabla u_n \nabla\psi dx \\
& + 2\int_{\mathbb{R}^N} \exp(-Ku_n) u_n^2 \nabla u_n \nabla\psi dx \\
& - \int_{\mathbb{R}^N} \exp(-Ku_n)\psi |\nabla u_n|^2 \left[K\left(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + 2u_n^2\right) - 2u_n\right] dx \\
& + \int_{\mathbb{R}^N} V(x) u_n \psi \exp(-Ku_n)\, dx - \lambda \int_{\mathbb{R}^N} h_M(u_n)\psi \exp(-Ku_n)\, dx.
\end{aligned} \tag{2.12}$$

Choose large $K > 1$ be such that $Ka > 1$. Then, by

$$\int_{\mathbb{R}^N} \exp(-Ku_n)\psi |\nabla(u_n - u)|^2 \left[K\left(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + 2u_n^2\right) - 2u_n\right] dx \geq 0,$$

one has

$$\begin{aligned}
& \int_{\mathbb{R}^N} \exp(-Ku_n)\psi |\nabla u_n|^2 \left[K\left(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + 2u_n^2\right) - 2u_n\right] dx \\
& \quad \geq \int_{\mathbb{R}^N} \exp(-Ku_n)\psi \left(2\nabla u_n \nabla u - |\nabla u|^2\right) \left[K\left(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + 2u_n^2\right) - 2u_n\right] dx \\
& \quad \to \int_{\mathbb{R}^N} \exp(-Ku)\psi|\nabla u|^2 \left[K\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx + 2u^2\right) - 2u\right] dx.
\end{aligned}$$

Because $\theta_n \to 0$ and $\|u_n\|_\infty \leq M$, (2.4) implies

$$\theta_n \int_{\mathbb{R}^N} \exp(-Ku_n)\left[|\nabla u_n|^2 \nabla u_n \nabla\psi + |u_n|^2 u_n\psi\right] dx \to 0$$

as $n \to \infty$. By the weak convergence of u_n, the Hölder inequality and Lebesgue's dominated convergence theorem, we infer

$$\left(a+b\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx\right)\int_{\mathbb{R}^N}e^{(-Ku_n)}\nabla u_n\nabla\psi dx \to \left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\int_{\mathbb{R}^N}e^{(-Ku)}\nabla u\nabla\psi dx,$$
$$\int_{\mathbb{R}^N}\exp\left(-Ku_n\right)u_n^2\nabla u_n\nabla\psi dx \to \int_{\mathbb{R}^N}\exp(-Ku)u^2\nabla u\nabla\psi dx,$$
$$\int_{\mathbb{R}^N}V(x)u_n\psi\exp\left(-Ku_n\right)dx \to \int_{\mathbb{R}^N}V(x)u\psi\exp(-Ku)dx$$

and

$$\lambda\int_{\mathbb{R}^N}h_M\left(u_n\right)\psi\exp\left(-Ku_n\right)dx \to \lambda\int_{\mathbb{R}^N}h_M(u)\psi\exp(-Ku)dx.$$

Hence, these together with (2.12) can deduce that

$$\begin{aligned}0 \le &\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\int_{\mathbb{R}^N}\exp(-Ku)\nabla u\nabla\psi dx+2\int_{\mathbb{R}^N}\exp(-Ku)u^2\nabla u\nabla\psi dx\\ &-\int_{\mathbb{R}^N}\exp(-Ku)\psi|\nabla u|^2\left[K\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx+2u^2\right)-2u\right]dx\\ &+\int_{\mathbb{R}^N}V(x)u\psi\exp(-Ku)dx-\lambda\int_{\mathbb{R}^N}h_M(u)\psi\exp(-Ku)dx.\end{aligned} \tag{2.13}$$

For any $\varphi \in E$ with $\varphi \ge 0$, by (2.11), we know $v := \varphi\exp(Ku) \in E$. By applying [18, Theorem 2.8], there exists a sequence $\{\psi_n\} \subset C_0^\infty\left(\mathbb{R}^N\right)$ of functions such that $\psi_n \ge 0$, $\psi_n \to v$ in $H_V^1(\mathbb{R}^N)$ and $\psi_n(x) \to v(x)$ for a.e. $x \in \mathbb{R}^N$. Taking $\psi = \psi_n$ in (2.13) and letting $n \to \infty$, we have

$$\begin{aligned}0 \le &\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\int_{\mathbb{R}^N}\nabla u\nabla\varphi dx+2\int_{\mathbb{R}^N}u^2\nabla u\nabla\varphi dx\\ &+2\int_{\mathbb{R}^N}|\nabla u|^2u\varphi dx+\int_{\mathbb{R}^N}V(x)u\varphi dx-\lambda\int_{\mathbb{R}^N}h_M(u)\varphi dx.\end{aligned}$$

The opposite inequality can be obtained in a similar way. Therefore,

$$\begin{aligned}&\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\int_{\mathbb{R}^N}\nabla u\nabla\varphi dx+2\int_{\mathbb{R}^N}\left(u^2\nabla u\nabla\varphi+|\nabla u|^2u\varphi\right)dx\\ &\qquad+\int_{\mathbb{R}^N}V(x)u\varphi dx-\lambda\int_{\mathbb{R}^N}h_M(u)\varphi dx=0\end{aligned}$$

for all $\varphi \in E$. This shows that $u \in E$ is a critical point of J_λ and

$$\begin{aligned}&\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\int_{\mathbb{R}^N}|\nabla u|^2dx+4\int_{\mathbb{R}^N}u^2|\nabla u|^2dx\\ &\qquad+\int_{\mathbb{R}^N}V(x)u^2dx-\lambda\int_{\mathbb{R}^N}h_M(u)udx=0.\end{aligned} \tag{2.14}$$

Finally, taking $\phi = u_n$ as the test function in (2.3), one has

$$\begin{aligned}0 = &\,\theta_n\int_{\mathbb{R}^N}\left[|\nabla u_n|^4+|u_n|^4\right]dx+\left(a+b\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx\right)\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx\\ &+4\int_{\mathbb{R}^N}u_n^2\,|\nabla u_n|^2\,dx+\int_{\mathbb{R}^N}V(x)u_n^2dx-\lambda\int_{\mathbb{R}^N}h_M\left(u_n\right)u_ndx.\end{aligned}$$

Notice that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \geq 2\int_{\mathbb{R}^N} \nabla u_n \nabla u dx - \int_{\mathbb{R}^N} |\nabla u|^2 dx \longrightarrow \int_{\mathbb{R}^N} |\nabla u|^2 dx,\\
&\int_{\mathbb{R}^N} u_n^2 |\nabla u_n|^2 \, dx \geq 2\int_{\mathbb{R}^N} u_n^2 \nabla u_n \nabla u dx - \int_{\mathbb{R}^N} u_n^2 |\nabla u|^2 dx \longrightarrow \int_{\mathbb{R}^N} u^2 |\nabla u|^2 dx,\\
&\lambda \int_{\mathbb{R}^N} h_M(u_n)\, u_n dx \to \lambda \int_{\mathbb{R}^N} h_M(u) u dx
\end{aligned}
$$

and

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x) u_n^2 dx \geq \int_{\mathbb{R}^N} V(x) u^2 dx.$$

By (2.4) and (2.14), up to a subsequence, one has

$$\theta_n \|u_n\|_{W^{1,4}}^4 \to 0, \|u_n\|_{H_V^1} \to \|u\|_{H_V^1}, \int_{\mathbb{R}^N} u_n^2 |\nabla u_n|^2 \, dx \to \int_{\mathbb{R}^N} u^2 |\nabla u|^2 dx.$$

Hence, $J_{\theta_n,\lambda}(u_n) \to J_\lambda(u)$ and $u_n \to u$ in $H_V^1(\mathbb{R}^N)$. This completes the proof. □

3 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. First, we will show that for each $\theta \in (0,1]$, $J_{\theta,\lambda}$ and $J_{\theta,\lambda}^{\pm}$ satisfy the P. S. condition. Indeed, by Lemma 2.2, it is sufficient to prove that any P. S. sequence of $J_{\theta,\lambda}$ is bounded.

Let $\{u_n\} \subset E$ be an arbitrary P. S. sequence for $J_{\theta,\lambda}$. If $\{u_n\}$ is unbounded in E, we can assume $\|u_n\|_E \to +\infty$. By (2.2) and (h_3), we get

$$
\begin{aligned}
&J_{\theta,\lambda}(u_n) - \frac{1}{\mu}\langle J'_{\theta,\lambda}(u_n), u_n\rangle\\
&= \left(\frac{1}{4}-\frac{1}{\mu}\right)\theta \|u_n\|_{W^{1,4}}^4 + \left(\frac{a}{2}-\frac{a}{\mu}\right)\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \left(\frac{b}{4}-\frac{b}{\mu}\right)\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx\right)^2\\
&\quad + \left(1-\frac{4}{\mu}\right)\int_{\mathbb{R}^N} |\nabla u_n|^2 u_n^2 dx + \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x) u_n^2 dx\\
&\quad + \lambda \int_{\mathbb{R}^N} \left[\frac{1}{\mu} u_n h_M(u_n) - H_M(u_n)\right] dx\\
&\geq \left(\frac{a}{2}-\frac{a}{\mu}\right)\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N} V(x) u_n^2 dx\\
&\quad + \lambda \int_{\mathbb{R}^N} \left[\frac{1}{\mu} u_n h_M(u_n) - H_M(u_n)\right] dx\\
&\geq \left(\frac{a}{2}-\frac{a}{\mu}\right)\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \left(\frac{1}{4}-\frac{1}{2\mu}\right)\int_{\mathbb{R}^N} V(x) u_n^2 dx - \lambda C_1(M)\\
&\geq \min\left\{\frac{a}{2}-\frac{a}{\mu}, \frac{1}{4}-\frac{1}{2\mu}\right\}\|u_n\|_{H_V^1}^2 - \lambda C_1(M).
\end{aligned}
\tag{3.1}
$$

If $\{\|u_n\|_{W^{1,4}}\}$ is bounded, then $\frac{\|u_n\|_{H_V^1}}{\|u_n\|_E} \to 1$. Therefore, by (3.1), we infer

$$\frac{J_{\theta,\lambda}(u_n) - \frac{1}{\mu}\left\langle J'_{\theta,\lambda}(u_n), u_n\right\rangle}{\|u_n\|_E^2} \geq \min\left\{\frac{a}{2}-\frac{a}{\mu}, \frac{1}{4}-\frac{1}{2\mu}\right\}\frac{\|u_n\|_{H_V^1}^2}{\|u_n\|_E^2} - \frac{\lambda C_1(M)}{\|u_n\|_E^2},$$

which implies $0 \geq \min\left\{\frac{a}{2}-\frac{a}{\mu}, \frac{1}{4}-\frac{1}{2\mu}\right\} > 0$. That is to say, it is a contradiction. Hence, we can assume $\|u_n\|_{W^{1,4}} \to \infty$. For large n, it follows from (3.1) that

$$\begin{aligned}
&J_{\theta,\lambda}(u_n) - \frac{1}{\mu}\left\langle J'_{\theta,\lambda}(u_n), u_n\right\rangle \\
&\geq \left(\frac{1}{4}-\frac{1}{\mu}\right)\theta\|u_n\|^4_{W^{1,4}} + \min\left\{\frac{a}{2}-\frac{a}{\mu}, \frac{1}{4}-\frac{1}{2\mu}\right\}\|u_n\|^2_{H^1_V} - \lambda C_1(M) \\
&\geq \left(\frac{1}{4}-\frac{1}{\mu}\right)\theta\|u_n\|^2_{W^{1,4}} + \min\left\{\frac{a}{2}-\frac{a}{\mu}, \frac{1}{4}-\frac{1}{2\mu}\right\}\|u_n\|^2_{H^1_V} - \lambda C_1(M) \\
&\geq \frac{1}{2}\min\left\{\left(\frac{1}{4}-\frac{1}{\mu}\right)\theta, \frac{a}{2}-\frac{a}{\mu}, \frac{1}{4}-\frac{1}{2\mu}\right\}\|u_n\|^2_E - \lambda C_1(M).
\end{aligned}$$

This together with $\|u_n\|_{W^{1,4}} \to \infty$ implies $0 \geq \frac{1}{2}\min\left\{(\frac{1}{4}-\frac{1}{\mu})\theta, \frac{a}{2}-\frac{a}{\mu}, \frac{1}{4}-\frac{1}{2\mu}\right\} > 0$, a contradiction. This shows that $\{u_n\}$ is bounded in E.

Next, by (h_1) and (h_2), we get

$$|H_M(v)| \leq C'_M|v|^2 + C_M|v|^{22^*} \tag{3.2}$$

for all $v \in \mathbb{R}$. For small $0 < \rho \ll 1$, set

$$S_\rho = \{v \in E : \|v\|_E = \rho\}.$$

Then for $v \in S_\rho$ and $0 < \lambda \leq V_0/4C'_M$, by (3.2), we have

$$\begin{aligned}
J^+_{\theta,\lambda}(v) &= \frac{1}{4}\theta\int_{\mathbb{R}^N}\left(|\nabla v|^4 + v^4\right)dx + \frac{a}{2}\int_{\mathbb{R}^N}|\nabla v|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla v|^2 dx\right)^2 \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N}\left(V(x)v^2 + 2v^2|\nabla v|^2\right)dx - \lambda\int_{\mathbb{R}^N}H_M(v^+)\,dx \\
&\geq \frac{1}{4}\theta\|v\|^4_{W^{1,4}} + \frac{1}{4}\min\{2a,1\}\|v\|^2_{H^1_V} + \int_{\mathbb{R}^N}v^2|\nabla v|^2 dx - \lambda C_M\left(\int_{\mathbb{R}^N}v^2|\nabla v|^2 dx\right)^{\frac{2^*}{2}} \\
&\geq \frac{1}{4}\theta\|v\|^4_{W^{1,4}} + \frac{1}{4}\min\{2a,1\}\|v\|^2_{H^1_V} \\
&\geq \frac{1}{4}\min\{\theta,2a,1\}\left[\|v\|^4_{W^{1,4}} + \|v\|^2_{H^1_V}\right] \\
&\geq \frac{1}{64}\min\{\theta,2a,1\}\rho^4 := \delta > 0.
\end{aligned}$$

Moreover, for $|t| \geq r$, by (h_3), we can infer $H_M(v) \geq C|v|^\mu$. Thus, by (h_1) and (h_2), there is a constant $C_3(M) > 0$ that depends on M such that

$$H_M(v) \geq C|v|^\mu - C_3(M)v^2 \tag{3.3}$$

for all $v \in E$. For any finite-dimensional subspace $\tilde{E} \subset E$, by the equivalency of all norms in the finite-dimensional space, there is a constant $\beta > 0$ such that

$$\|v\|_\mu \geq \beta\|v\|_E \tag{3.4}$$

for all $v \in \tilde{E}$. Hence, by (3.3) and (3.4), one has

$$\begin{aligned} J_{\theta,\lambda}(v) &= \frac{1}{4}\theta\int_{\mathbb{R}^N}\left(|\nabla v|^4+v^4\right)dx+\frac{a}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^2 \\ &\quad +\frac{1}{2}\int_{\mathbb{R}^N}\left(V(x)v^2+2v^2|\nabla v|^2\right)dx-\lambda\int_{\mathbb{R}^N}H_M(v)dx \\ &\le \frac{1}{4}\theta\|v\|^4_{W^{1,4}}+\frac{1}{2}\max\{a,1\}\|v\|^2_{H^1_V}+\int_{\mathbb{R}^N}v^2|\nabla v|^2dx \\ &\quad +\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^2-\lambda\int_{\mathbb{R}^N}\left[C|v|^\mu-C_3(M)v^2\right]dx \\ &\le \frac{3}{4}\|v\|^4_{W^{1,4}}+\frac{b}{4}\|v\|^4_{H^1_V}+\frac{1}{2}\max\{a,1\}\|v\|^2_{H^1_V}-\lambda C\|v\|^\mu_\mu+\lambda C_3(M)\|v\|_2^2 \\ &\le \frac{1}{4}\max\{3,b\}\|v\|^4_E+\left(\lambda C_3(M)+\frac{1}{2}\max\{a,1\}\right)\|v\|^2_E-\lambda C\beta^\mu\|v\|^\mu_E \end{aligned} \tag{3.5}$$

for all $v \in \tilde{E}$ and $0<\theta\le 1$. Thus, there is a large $R>0$ such that $J_{\theta,\lambda}<0$ on $\tilde{E}\setminus B_R$, where $B_R := \{u \in E : \|u\|_E < R\}$. Set a fixed $e \in \tilde{E}$ with $e \ge 0$ and $\|e\|_E = 1$. For any fixed constant $T>0$, define the path $h_T : [0,1] \to \tilde{E} \subset E$ by $h_T(t) = tTe$. Then for large $T>1$ and $\mu>4$, by (3.5), we get

$$J^+_{\theta,\lambda}(h_T(1)) \le \frac{1}{4}\max\{3,b\}T^4+\left(\lambda C_3(M)+\frac{1}{2}\max\{a,1\}\right)T^2-\lambda C\beta^\mu T^\mu<0$$

with $\|h_T(1)\|_E > \rho$, and

$$\max_{t\in[0,1]} J^+_{\theta,\lambda}(h_T(t)) \le C.$$

Hence, by [16, Theorem 2.2], $J^+_{\theta,\lambda}$ possesses a critical value

$$c_\theta := \inf_{\eta\in\Gamma}\max_{t\in[0,1]} J^+_{\theta,\lambda}(\eta(t)) \ge \delta > 0$$

and

$$c_\theta \le \max_{t\in[0,1]} J^+_{\theta,\lambda}(h_T(t)) \le C,$$

where

$$\Gamma = \{\eta \in C([0,1],E) : \eta(0)=0, \eta(1)=h_T(1)\}.$$

Therefore, $J^+_{\theta,\lambda}$ possesses the Mountain Pass geometry. Further, by Lemma 2.3 and Mountain Pass Theorem, we know that the equation (2.1) has a positive weak solution. This together with (2.11) implies that (1.1) has a positive weak solution. Moreover, by a similar argument, we infer that the equation (1.1) has a negative weak solution. This completes the proof. □

Next, in order to prove Theorem 1.2, we need to revise the cutoff function. Let

$$\hat{h}_M(t) = \begin{cases} f(t), & 0<t\leqslant M \\ C_M t^{p-1}, & t>M \\ -\hat{h}_M(-t), & t\leqslant 0. \end{cases}$$

Then for the odd function $f(t)$, it is easy to know that $\hat{h}_M(t)$ satisfies (h_1)–(h_3) and the odd function property.

Hereinafter, we will concentrate on the following equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u-u\Delta(u^2)+V(x)u=\lambda\hat{h}_M(u),\quad x\in\mathbb{R}^N. \tag{3.6}$$

Here $\hat{J}_\lambda(u):E\to\mathbb{R}$ is the natural energy functional corresponding to (3.6)

$$\begin{aligned}\hat{J}_\lambda(u)=&\frac{a}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx+\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^2+\frac{1}{2}\int_{\mathbb{R}^N}\left(V(x)u^2+2u^2|\nabla u|^2\right)dx\\&-\lambda\int_{\mathbb{R}^N}\hat{H}_M(u)dx,\end{aligned}$$

where $\hat{H}_M(t)=\int_0^t\hat{h}_M(s)\mathrm{d}s$. For $\theta\in(0,1]$, let $\hat{J}_{\theta,\lambda}(u)=\frac{1}{4}\theta\int_{\mathbb{R}^N}\left(|\nabla u|^4+u^4\right)dx+\hat{J}_\lambda(u)$.

Lemma 3.1. *Assume that $(V),(f_1),(f_2)$ hold. If $f(t)$ is odd, then for all $\theta\in(0,1]$ fixed, $\hat{J}_{\theta,\lambda}$ has a sequence of critical points u_j such that there exist α_j,β_j both of which are independent of θ to satisfy $\alpha_j\to\infty$ as $j\to\infty$, $\alpha_j<\beta_j$ and $c_j(\theta)\in[\alpha_j,\beta_j]$ for all $\theta>0$.*

Proof. Consider the eigenvalue problem

$$\int_{\mathbb{R}^N}(\nabla u\cdot\nabla\varphi+V(x)u\varphi)dx=\xi\int_{\mathbb{R}^N}u\varphi dx,\quad\forall\varphi\in H_V^1(\mathbb{R}^N). \tag{3.7}$$

For real number ξ, if there exists $u\in H_V^1\left(\mathbb{R}^N\right)(u\neq 0)$ to satisfy (3.7), then ξ is called a eigenvalue of the operator $L=-\Delta+V$. Further, by the condition (V) and the compactness of the embedding $H_V^1\left(\mathbb{R}^N\right)\hookrightarrow L^2\left(\mathbb{R}^N\right)$, we infer that the spectrum $\sigma(L)=\{\xi_1,\xi_2,\dots,\xi_n,\dots\}$ of L satisfies

$$0<\xi_1<\xi_2<\cdots<\xi_n<\cdots$$

and $\xi_n\to+\infty$ as $n\to\infty$. Let ϕ_n be the eigenfunction corresponding to the eigenvalue ξ_n. By regularity argument, we know $\phi_n\in E$. Set $E_n=\operatorname{span}\{\phi_1,\phi_2,\dots,\phi_n\}$. Then we decompose the space E as a direct sum $E=E_n\oplus W_n$ for $n=1,2,\dots$, where W_n is orthogonal to E_n in $H_V^1\left(\mathbb{R}^N\right)$. For $\rho>0$, set

$$\mathcal{Z}_\rho=\left\{u\in E:\|u\|^2_{H_V^1}+\int_{\mathbb{R}^N}u^2|\nabla u|^2dx\le\rho^2\right\}.$$

By (3.5), there exists $r_n>0$ independent of θ such that

$$\hat{J}_{\theta,\lambda}(u)<0,\quad\forall u\in\overline{E_n\backslash\mathcal{Z}_{r_n}}. \tag{3.8}$$

Set

$$D_n=E_n\cap\mathcal{Z}_{r_n},\quad G_n=\left\{\varphi\in C\left(D_n,E\right):\varphi\text{ is odd and }\varphi|_{\partial\mathcal{Z}_{r_n}\cap E_n}=id\right\}$$

and

$$\Gamma_j=\left\{\varphi\left(\overline{D_n\backslash A}\right):\varphi\in G_n,n\ge j,A=-A\subset E_n\cap\mathcal{Z}_{r_n}\text{ is closed and }\gamma(A)\le n-j\right\},$$

where $\gamma(\cdot)$ is the genus. Let

$$c_j(\theta)=\inf_{B\in\Gamma_j}\sup_{u\in B}\hat{J}_{\theta,\lambda}(u),\quad j=1,2,\dots$$

We claim that $c_j(\theta)$ $(j=1,2,\dots)$ are critical values of $\hat{J}_{\theta,\lambda}$ and there exist $\beta_j>\alpha_j$ such that $c_j(\theta)\in[\alpha_j,\beta_j]$ and $\alpha_j\to\infty$ as $j\to\infty$.

Since $\hat{J}_{\theta,\lambda}$ is increasing with respect to θ, we have $c_j(\theta) \leq c_j(1) := \beta_j$ $(j = 1, 2, \dots)$. And then we will estimate the lower bound for $c_j(\theta)$. Depending on the following Lemma 3.2, we have an intersection property: If $\rho < r_n$ for all $n \geq j$, then for $B \in \Gamma_j$, we have $B \cap \partial \mathcal{Z}_\rho \cap W_{j-1} \neq \emptyset$. Therefore,

$$c_j(\theta) \geq \inf_{u \in \partial \mathcal{Z}_\rho \cap W_{j-1}} \hat{J}_{\theta,\lambda}(u) \geq \inf_{u \in \partial \mathcal{Z}_\rho \cap W_{j-1}} \hat{J}_\lambda(u).$$

For small $\varepsilon > 0$ and $u \in \partial \mathcal{Z}_\rho \cap W_{j-1}$, by (h_1), for $0 < \lambda \leq V_0/4C'_M$ one has

$$\begin{aligned}
\hat{J}_{\theta,\lambda}(u) &\geq \hat{J}_\lambda(u) \\
&\geq \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^N} \left(V(x)u^2 + 2u^2|\nabla u|^2\right) dx \\
&\quad - \lambda \int_{\mathbb{R}^N} \left(C'_M u^2 + C_M |u|^p\right) dx \\
&\geq \frac{1}{4}\min\{a, 1\}\|u\|^2_{H^1_V} + \int_{\mathbb{R}^N} u^2|\nabla u|^2 dx - \lambda C_M \int_{\mathbb{R}^N} |u|^p dx \\
&\geq \frac{1}{4}\min\{a, 1\}\rho^2 - \lambda C_M \|u\|_2^{(1-t)p} \|u\|_{22^*}^{tp} \\
&\geq \frac{1}{4}\min\{a, 1\}\rho^2 - \lambda C_M \xi_j^{-\frac{(1-t)p}{2}} \rho^{(1-t)p+\frac{t}{2}} \\
&= \rho^2\left(\frac{1}{4}\min\{a, 1\} - \lambda C_M \xi_j^{-\frac{(1-t)p}{2}} \rho^{(1-t)p+\frac{p}{2}-2}\right),
\end{aligned}$$

where $t \in (0, 1)$ satisfies $\frac{1}{p} = \frac{t}{22^*} + \frac{1-t}{2}$. Take $\rho = \rho_j$ be such that $\rho_j^{(1-t)p+\frac{tp}{2}-2} = \frac{\min\{a,1\}}{8\lambda C_M}\xi_j^{\frac{(1-t)p}{2}}$. Then choosing $r_n > \rho_n$, we infer $\hat{J}_{\theta,\lambda}(u) \geq \frac{\min\{a,1\}}{8}\rho_j^2 := \alpha_j \to +\infty$. Thus, $c_j(\theta) \in [\alpha_j, \beta_j]$ $(\alpha_j \to \infty$ as $j \to \infty)$.

Now we show that $c_j(\theta)$ $(j = 1, 2, \dots)$ are critical values of $\hat{J}_{\theta,\lambda}$. Indeed, if $c_j(\theta)$ is not a critical value of $\hat{J}_{\theta,\lambda}$, then by [16, Theorem A.4], we know that for given $0 < \bar{\varepsilon} < \min\{\alpha_j : j = 1, 2, \dots\}$, there exist $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0, 1] \times E, E)$ such that

- (a) $\eta(t, u) = u$ for all $t \in [0, 1]$ if $\hat{J}_{\theta,\lambda}(u) \notin [c_j(\theta) - \bar{\varepsilon}, c_j(\theta) + \bar{\varepsilon}]$.
- (b) $\eta(t, \cdot) : E \to E$ is a homeomorphism for each $t \in [0, 1]$.
- (c) $\eta(1, \hat{J}_{\theta,\lambda}^{c_j(\theta)+\varepsilon}) \subset \hat{J}_{\theta,\lambda}^{c_j(\theta)-\varepsilon}$, where $\hat{J}^\kappa_{\theta,\lambda} = \{u \in E : \hat{J}_{\theta,\lambda}(u) \leq \kappa\}$.
- (d) $\eta(t, u)$ is odd in u.

Set $\psi = \eta(1, \cdot)$. Then, by (3.8), $\psi = id$ on $\partial \mathcal{Z}_{r_n} \cap E_n$ for all n. By the definition of $c_j(\theta)$, there exists $B \in \Gamma_j$ such that

$$\sup_{u \in B} \hat{J}_{\theta,\lambda}(u) \leq c_j(\theta) + \varepsilon.$$

Notice that $A = \psi(B) \in \Gamma_j$. By (c), we know

$$c_j(\theta) \leq \sup_{u \in A} \hat{J}_{\theta,\lambda}(u) \leq c_j(\theta) - \varepsilon,$$

which is a contradiction. Hence, $c_j(\theta)$ $(j = 1, 2, \dots)$ are critical values of $\hat{J}_{\theta,\lambda}$. This completes the proof of Lemma 3.1. □

Lemma 3.2. *For $B \in \Gamma_j$, it follows that $B \cap \partial \mathcal{Z}_\rho \cap W_{j-1} \neq \emptyset$ provided $\rho < r_n$ for all $n \geq j$.*

Proof. Set $B = \varphi(\overline{D_n \backslash A})$ with $n \geq j$ and $\gamma(A) \leq n - j$. Let $\widetilde{\mathcal{X}} = \{u \in D_n : \varphi(u) \in \mathcal{Z}_\rho\}$. Then we can easily infer that 0 is an interior point of $\widetilde{\mathcal{X}}$. Let $\mathcal{X}$ be the connected component of $\widetilde{\mathcal{X}}$ containing 0 . Then $\mathcal{X}$ is a bounded symmetric neighborhood of 0 in E_n. Hence, by [16, Proposition 7.7], $\gamma(\partial\mathcal{X}) = n$. Since $\varphi|_{\partial \mathcal{Z}_{r_n} \cap E_n} = id$, we obtain

$$\|\varphi(u)\|^2_{H^1_V} + \int_{\mathbb{R}^N} \varphi^2(u)|\nabla \varphi(u)|^2 dx = r_n^2 > \rho^2, \quad \forall u \in \partial \mathcal{Z}_{r_n} \cap E_n. \tag{3.9}$$

Then we get $\varphi(\partial\mathcal{X}) \subset \partial\mathcal{Z}_\rho$. In fact, for each $u \in \partial\mathcal{X}$, because $\varphi(u) \in \mathcal{Z}_\rho$, (3.9) implies that $u \in \operatorname{int}(\mathcal{Z}_{r_n}) \cap E_n$. Hence, if $\varphi(u) \in \operatorname{int}(\mathcal{Z}_\rho)$, then the continuity of φ implies that there exists an open ball $B(u,r) \subset D_n$ centered at u with radius r such that $\varphi(B(u,r)) \subset \operatorname{int}(\mathcal{Z}_\rho)$. Since $B(u,r)$ is connected, $u \in \mathcal{X}$ and $B(u,r) \subset \mathcal{X}$, we know that u is an interior point of $\mathcal{X}$. It contradicts that $u \in \partial\mathcal{X}$. Hence, $\varphi(u) \in \partial\mathcal{Z}_\rho$. Set $W = \{u \in D_n : \varphi(u) \in \partial\mathcal{Z}_\rho\}$. Then $\partial\mathcal{X} \subset W, \gamma(W) = n$ and $\gamma(\overline{W\backslash A}) \geq n - (n-j) > j-1$. Hence [16, Proposition 7.5–2^0] implies $\gamma(\varphi(\overline{W\backslash A})) > j-1$. Notice that $\operatorname{codim}(W_{j-1}) = j-1$. Consequently, $\varphi(\overline{W\backslash A}) \cap W_{j-1} \neq \emptyset$, that is to say, $B \cap \partial\mathcal{Z}_\rho \cap W_{j-1} \supset \varphi(\overline{W\backslash A}) \cap W_{j-1} \neq \emptyset$. The proof is finished. □

Proof of Theorem 1.2. Depending on Lemma 2.3, Lemma 3.1 and Lemma 3.2, we get that the equation (2.1) has a sequence $\{u_n\}$ of solutions such that $\hat{J}_\lambda(u_n) \to +\infty$. Then for λ small enough and fixed M, it follows from (2.11) that the equation (1.1) has a sequence $\{u_n\}$ of solutions such that $I_\lambda(u_n) \to +\infty$. □

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