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Ground state solution of a semilinear Schrödinger system with local super-quadratic conditions

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Abstract. This paper is dedicated to studying the following semilinear Schrödinger system

$$\begin{cases} -\Delta u + V_1(x)u = F_u(x,u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v = F_v(x,u,v) & \text{in } \mathbb{R}^N, \\ u,\ v \in H^1(\mathbb{R}^N), \end{cases}$$

where the potential V_i are periodic in x, $i = 1,2$, the nonlinearity F is assumed to be super-quadratic at some $x \in \mathbb{R}^N$ and asymptotically quadratic otherwise. Under a local super-quadratic condition of F, an approximation argument and variational method are used to prove the existence of Nehari–Pankov type ground state solutions and the least energy solutions.

Keywords: Schrödinger system, local super-quadratic condition, ground state solution.

2020 Mathematics Subject Classification: 35J20, 35J61.

1 Introduction

We consider the following system of semilinear Schrödinger equations:

$$\begin{cases} -\Delta u + V_1(x)u = F_u(x,u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v = F_v(x,u,v) & \text{in } \mathbb{R}^N, \\ u,\ v \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $V_1,\ V_2 \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$, $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the following assumptions:

(V) $V_1,\ V_2 \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ are 1-periodic in $x_j, j = 1, 2, \dots, N$, and

$$\sup[\sigma(-\Delta + V_i) \cap (-\infty, 0)] =: \underline{\Lambda}_i < 0 < \overline{\Lambda}_i := \inf[\sigma(-\Delta + V_i) \cap (0, \infty)];$$

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(F1) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, [0,\infty))$ and there exist constants $p \in (2, 2^*)$, $C_1 > 0$ such that

$$|F_z(x,z)| \le C_1(1 + |z|^{p-1}), \quad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where $F_z := (F_u, F_v) = \nabla F$, $2^* := 2N/(N-2)$ if $N \ge 3$ and $2^* := +\infty$ if $N = 1$ or 2;

(F2) $|F_z(x,z)| = o(|z|)$ as $|z| \to 0$ uniformly in $x \in \mathbb{R}^N$.

From (V), (F1) and (F2), we can easily get that the critical points of functional Φ are the solutions of (1.1), here Φ is defined as:

$$\Phi(z) = \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla u|^2 + V_1(x)|u|^2 + |\nabla v|^2 + V_2(x)|v|^2]\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,z)\mathrm{d}x, \quad z = (u,v) \in E, \tag{1.2}$$

where $E = H_1 \times H_2$ is defined in Section 2.

There is a scalar case of the Schrödinger system:

$$\begin{cases} -\Delta u + V(x)u = \nabla F(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.3}$$

we can easily obtain that case when $V_1 = V_2$ and $u = v$. That equation has been widely studied in the literature, such as [2, 9, 15, 16, 30, 32].

Solution of (1.1) was related to the following system:

$$\begin{cases} -i\frac{\partial \Psi}{\partial t} = \Delta\Psi - V_1(x)\Psi + F_1(x,\Psi), & x \in \mathbb{R}^N, t \ge 0, \\ -i\frac{\partial \Phi}{\partial t} = \Delta\Phi - V_2(x)\Phi + F_2(x,\Phi), & x \in \mathbb{R}^N, t \ge 0, \end{cases}$$

where i denotes the imaginary unit, V_1 and V_2 are the relevant potentials, Φ and Ψ represent the condensate wave functions. This type of Schrödinger systems arise in nonlinear optics, and have extensively been applied in many areas, such as the investigation of pulse propagation, Bose–Einstein condensates, Hartree–Fock theory for a double condensate, gap solitons in photonic crystals and so on, see as [6, 10, 13, 14, 22, 31]. In recent years, many researchers were interested in such type of systems, we refer the readers to [1, 3–7, 17–20, 24, 25].

Manassés and João [29] investigated the existence of nontrivial solutions for the following strongly coupled system in $\mathbb{R}^2$:

$$\begin{cases} -\Delta u + V(x)u = g(x,v), & v > 0 \text{ in } \mathbb{R}^2, \\ -\Delta v + V(x)v = f(x,u), & u > 0 \text{ in } \mathbb{R}^2, \end{cases} \tag{1.4}$$

where $V : \mathbb{R}^2 \to \mathbb{R}$ may change sign and vanish, f, g are superlinear at infinity and satisfy critical or subcritical growth of Trudinger–Moser type. By using the linking geometry and a Trudinger–Moser type inequality, they obtained the boundedness of a Palais–Smale sequence, and proved there exists a subsequence that converges to a weak solution of (1.4). Finally, applying a Galerkin approximation procedure, they proved the existence of solutions in the subcritical case and critical case respectively.

Qin and Tang [23] established a nontrivial solution for the following elliptic system:

$$\begin{cases} -\Delta u + U_1(x)u = F_u(x,u,v) & \text{in } \mathbb{R}^N, \\ -\Delta v + U_2(x)v = F_v(x,u,v) & \text{in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{cases}$$

where $U_i(x) \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}), i = 1, 2$, $F \in \mathcal{C}^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ and $\nabla F = (F_u, F_v)$. In that paper, the authors distinguished two situations about U_i and F: periodic and asymptotically periodic case. For the periodic case, by using the diagonal method [32], the authors found a minimizing Cerami sequence outside the Nehari–Pankov manifold, then they proved the existence of the least energy solution and the ground state solution. For the latter case, by using a generalized linking theorem, they obtained a nontrivial solution. In that paper, F satisfies the following super-quadratic assumption:

(SQ) $\lim_{|z|\to\infty} \frac{F(x,z)}{|z|^2} = \infty$ uniformly in x.

By using (SQ), one can prove the linking geometry, mountain pass geometry and verify the boundedness of Cerami or Palais–Smale sequence. Moreover, it is standard to show that $\mathcal{N}^- \neq \emptyset$, where

$$\mathcal{N}^- := \{z \in E \setminus E^- : \langle \Phi'(z), z\rangle = \langle \Phi'(z), \zeta\rangle = 0,\ \forall \zeta \in E^-\}, \tag{1.5}$$

here E^- defined in (2.11). Introduced by Pankov [22], $\mathcal{N}^-$ is a natural constraint and contains all nontrivial critical points of the energy functional Φ, and every minimizer u of Φ on the manifold $\mathcal{N}^-$ is a solution which is called a ground state solution of Nehari–Pankov type. Also, the set $\mathcal{N}^-$ plays a crucial role in proving the existence of the ground state solution.

Later, Tang et al. [33] investigated the existence of the ground state solutions about (1.3) under the assumptions (V), (F1), (F2) and the following assumptions:

(F3) There exists a domain $G \subset \mathbb{R}^N$ such that $\lim_{|z|\to\infty} \frac{F(x,z)}{|z|^2} = \infty$ a.e. $x \in G$.

(F4) $z \mapsto \frac{F_z(x,z)}{|z|}$ is non-decreasing on $|z| \neq 0$.

(F5) $\mathcal{F}(x,z) := \frac{1}{2}F_z(x,z)\cdot z - F(x,z) \geq 0$, and there exist some constants $C_2 > 0$, $R_0 > 0$ and $\sigma \in (0,1)$, such that

$$\left(\frac{|F_z(x,z)|}{|z|^\sigma}\right)^\kappa \leq C_2\mathcal{F}(x,z), \quad \forall\ |z| \geq R_0$$

holds with $\kappa = \frac{2N}{2N-(1+\sigma)(N-2)}$ if $N \geq 3$, or with $\kappa \in \left(1, \frac{2}{1-\sigma}\right)$ if $N = 1, 2$.

Since they relaxed condition (SQ) to the above local version (F3), it is difficult to demonstrate $\mathcal{N}^- \neq \emptyset$ and prove the boundedness of Cerami or Palais–Smale sequences for the energy functional Φ. They use some new techniques to conquer the above difficulties. For the first one, by using linking geometry and verifying $\sup \Phi(z) < \infty$ for $z \in E^- \oplus \mathbb{R}^+\bar{e}^+$, they illustrate that Φ is weakly upper semi-continuous, hence, they can prove that $\mathcal{N}^- \neq \emptyset$. For the second, they consider an approximation argument to find a minimizing sequence satisfying the PS condition for the corresponding functional. Finally, by using the uniqueness of the continuous spectrum about the operator $\mathcal{A}_i = -\Delta + V_i$, they make a contradiction to get the boundedness of the above sequence.

Recently, Qin et al. [26] proved the existence of nontrivial solutions for (1.1) by using generalized linking theorem and variational methods. More precisely, they found a Cerami sequence for the corresponding energy functional, and then proved the boundedness of the Cerami sequence. By applying linking geometry, they proved there exists a ground state solution of (1.1) with assumptions (V), (F1)–(F3). Besides, they used the following assumption to prove the boundedness of Cerami sequences:

(F6') $\mathcal{F}(x,z) \geq 0$, and there exist some constants $\tilde{C}_1 > 0$, $\delta_0 \in (0, \Lambda_0)$ and $\sigma \in (0,1)$, such that

$$\frac{|F_z(x,z)|}{|z|} \geq \frac{\sqrt{2}}{2}\tau \implies \left(\frac{|F_z(x,z)|}{|z|^\sigma}\right)^\kappa \leq \tilde{C}_1 \mathcal{F}(x,z), \quad \forall\, (x,z) \in \mathbb{R}^N \times \mathbb{R}^2$$

holds with $\kappa = \frac{2N}{2N-(1+\sigma)(N-2)}$ if $N \geq 3$, or with $\kappa \in \left(1, \frac{2}{1-\sigma}\right)$ if $N = 1,2$, where

$$\tau := \Lambda_0 - \delta_0, \quad \Lambda_0 := \min\left\{-\underline{\Lambda}_1,\ \overline{\Lambda}_1,\ -\underline{\Lambda}_2,\ \overline{\Lambda}_2\right\}. \tag{1.6}$$

To the best of our knowledge, there is few result about the ground state solution of system (1.1). Motivated by [26, 33], we aim to prove the existence of ground state solutions about system (1.1) by using approximation argument and variational method. We try to obtain the ground state solutions of Nehari–Pankov type and least energy solutions under assumptions (V), (F1)–(F5) and the following conditions:

(F6) $F(x,z) \geq 0, \mathcal{F}(x,z) \geq 0$, and there exist constants $C_3 > 0$, $\delta_0 \in (0, \Lambda_0)$ and $\sigma \in (0,1)$, such that

$$\frac{|F_z(x,z)|}{|z|} \geq \tau \implies \left(\frac{|F_z(x,z)|}{|z|^\sigma}\right)^\kappa \leq C_3 \mathcal{F}(x,z), \quad \forall\, (x,z) \in \mathbb{R}^N \times \mathbb{R}^2$$

holds with $\kappa = \frac{2N}{2N-(1+\sigma)(N-2)}$ if $N \geq 3$, or with $\kappa \in \left(1, \frac{2}{1-\sigma}\right)$ if $N = 1,2$, note that τ is the same with (1.6).

Now, we state our results of this paper.

Theorem 1.1. *Let* (V), (F1)–(F5) *be satisfied. Then* (1.1) *has a Nehari–Pankov type ground state solution.*

Theorem 1.2. *Let* (V), (F1)–(F3) *and* (F6) *be satisfied. Then* (1.1) *has a least energy solution $\bar{z}$ in K, where $K := \{z \in E \setminus \{0\} : \Phi'(z) = 0\}$.*

There is an example to illustrate that the assumptions (F3)–(F6) can be satisfied.

Let $N \geq 3$ and $F(x,z) = \cos^2(2\pi x_1)|z|^2 \ln(1+|z|^2)$, it is easy to verify that

$$F_z(x,z) = 2z\cos^2(2\pi x_1)\left[\ln(1+|z|^2) + \frac{|z|^2}{1+|z|^2}\right]$$

and

$$\mathcal{F}(x,z) = \frac{\cos(2\pi x_1)|z|^4}{1+|z|^2} \geq 0.$$

It is clear that F satisfies (F1)–(F6) with $G = (-\frac{1}{8}, \frac{1}{8}) \times \mathbb{R}^{N-1}$, but does not satisfy (SQ).

Remark 1.3. Assume that (F1), (F2), (F4) and (F5) hold. Then (F6) holds also. See as [33, Lemma 3.8]. Moreover, (F6') implies (F6).

To prove the existence of ground state solutions about (1.1), at first, we show that $\mathcal{N}^- \neq \emptyset$. Inspired by Tang [33], we consider an approximation argument about the auxiliary functionals $I_\epsilon(z) = \Phi(z) - \epsilon \int_{\mathbb{R}^N} |z|^p dx$, which makes the corresponding problem superlinear in $\mathbb{R}^N$. Moreover, by demonstrating a key inequality (3.3) and using $\mathcal{N}^- \neq \emptyset$, we prove that $I_{\epsilon_n}(z_{\epsilon_n})$ is bounded and $I'_{\epsilon_n}(z_{\epsilon_n}) = 0$, here $\epsilon_n \to 0$ as $n \to \infty$. Finally, by using Sobolev embedding theorem and Lion's concentration compactness principle, we prove the sequence $\{z_{\epsilon_n}\}$ is bounded, then we can get that $\{z_{\epsilon_n}\}$ is convergent to a solution of (1.1).

The reminder of this paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we give the proof of Theorem 1.1 and Theorem 1.2. For convenience, let $C_0, \tilde{C}_0, C_1, \tilde{C}_1, \ldots$ denote different constants in different places.

2 Preliminaries

Let $\mathcal{A}_i = -\Delta + V_i$, here and in what follows $i = 1, 2$. Then $\mathcal{A}_i$ are self-adjoint in $L^2(\mathbb{R}^N)$ with domain $\mathfrak{D}(\mathcal{A}_i) = H^2(\mathbb{R}^N)$ (see [12, Theorem 4.26]). Let $\{\mathcal{E}_i(\lambda) : -\infty \le \lambda \le +\infty\}$ and $|\mathcal{A}_i|$ be the spectral family and the absolute value of $\mathcal{A}_i$, respectively, and $|\mathcal{A}_i|^{1/2}$ be the square root of $|\mathcal{A}_i|$. Set $\mathcal{U}_i = id - \mathcal{E}_i(0) - \mathcal{E}_i(0-)$. Then $\mathcal{U}_i$ commutes with $\mathcal{A}_i$, $|\mathcal{A}_i|$ and $|\mathcal{A}_i|^{1/2}$. Furthermore, $\mathcal{A}_i = \mathcal{U}_i|\mathcal{A}_i|$ is the polar decomposition of $\mathcal{A}_i$ (see [11, Theorem IV 3.3]). Let

$$H_i = \mathfrak{D}(|\mathcal{A}_i|^{1/2}), \qquad H_i^- = \mathcal{E}_i(0-)H_i, \qquad H_i^+ = [id - \mathcal{E}_i(0)]H_i.$$

For any $u_i \in H_i$, fixing $i = 1$ or $i = 2$, it is easy to see that $u_i = {u_i}^- + {u_i}^+$ with

$$u_i^- := \mathcal{E}_i(0-)u_i \in H_i^-, \qquad u_i^+ := [id - \mathcal{E}_i(0)]u_i \in H_i^+ \tag{2.1}$$

and

$$\mathcal{A}_i u_i^- = -|\mathcal{A}_i|u_i^-, \qquad \mathcal{A}_i u_i^+ = |\mathcal{A}_i|u_i^+, \qquad \forall\, u_i = {u_i}^- + {u_i}^+ \in H_i \cap \mathfrak{D}(\mathcal{A}_i). \tag{2.2}$$

For fixed i taking 1 or 2, we define an inner product

$$(u, v)_{H_i} = \left(|\mathcal{A}_i|^{1/2}u, |\mathcal{A}_i|^{1/2}v\right)_{L^2}, \quad u, v \in H_i \tag{2.3}$$

and the corresponding norm

$$\|u\|_{H_i} = \left\||\mathcal{A}_i|^{1/2}u\right\|_{L^2}, \quad u \in H_i,$$

where $(\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(\mathbb{R}^N)$, $\|\cdot\|_{L^s}$ stands for the usual $L^s(\mathbb{R}^N)$ norm, $1 \le s < \infty$. There are induced decompositions $H_i = H_i^- \oplus H_i^+$ which are orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)_{H_i}$. Then

$$\int_{\mathbb{R}^N} \left(|\nabla u_i|^2 + V_i(x)|u_i|^2\right) \mathrm{d}x = \|u_i^+\|_{H_i}^2 - \|u_i^-\|_{H_i}^2, \quad \forall\;\; u_i = u_i^- + u_i^+ \in H_i, \;\; i = 1, 2.$$

Under condition (V), $H_i^- \oplus H_i^+ = H_i = H^1(\mathbb{R}^N)$ with equivalent norms. Therefore, H_i embeds continuously in $L^s(\mathbb{R}^N)$ for all $2 \le s < 2^*$. Then, there exists a constant $\gamma_s > 0$ such that

$$\|z\|_s \le \gamma_s\|z\|, \quad \forall z \in E, s \in [2, 2^*], \tag{2.4}$$

where $\|\cdot\|_s$ stands for the usual $L^s(\mathbb{R}^N, \mathbb{R}^2)$ norm.

Let

$$E = H_1 \times H_2 \tag{2.5}$$

equipped with the inner product

$$\langle z, \xi\rangle = (u, \chi)_{H_1} + (v, \psi)_{H_2}, \quad z = (u, v),\ \xi = (\chi, \psi) \in E = H_1 \times H_2 \tag{2.6}$$

and the corresponding norm

$$\|z\| = \left[\|u\|_{H_1}^2 + \|v\|_{H_2}^2\right]^{1/2}, \quad z = (u, v) \in E. \tag{2.7}$$

For any $\varepsilon > 0$, (F1) and (F2) yield the existence of $C_\varepsilon > 0$ such that

$$|F_z(x, z)| \le \varepsilon|z| + C_\varepsilon|z|^{p-1}, \quad \forall\, (x, z) \in \mathbb{R}^N \times \mathbb{R}^2. \tag{2.8}$$

Under (V), a standard argument (see [8, 36]) shows that the solutions of problem (1.1) are critical points of the functional

$$\Phi(z)=\frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla u|^2+V_1(x)|u|^2+|\nabla v|^2+V_2(x)|v|^2\right]\mathrm{d}x-\int_{\mathbb{R}^N}F(x,z)\mathrm{d}x,\quad z=(u,v)\in E, \tag{2.9}$$

Φ is of class $\mathcal{C}^1(E,\mathbb{R})$, and

$$\begin{aligned}\langle\Phi'(z),\xi\rangle=&\int_{\mathbb{R}^N}(\nabla u\nabla\chi+V_1(x)u\chi)\,\mathrm{d}x+\int_{\mathbb{R}^N}(\nabla v\nabla\psi+V_2(x)v\psi)\,\mathrm{d}x\\&-\int_{\mathbb{R}^N}(F_u(x,z)\chi+F_v(x,z)\psi)\,\mathrm{d}x,\quad\forall\, z=(u,v),\ \xi=(\chi,\psi)\in E.\end{aligned}\tag{2.10}$$

Let

$$E^+=H_1^+\times H_2^+,\qquad E^-=H_1^-\times H_2^-, \tag{2.11}$$

then for any $z=(u,v)\in E$, (2.1) yields $z=z^++z^-$ with the corresponding summands

$$z^+=(u^+,v^+)\in E^+,\qquad z^-=(u^-,v^-)\in E^-. \tag{2.12}$$

Moreover, E^+ and E^- are orthogonal with respect to the inner products $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)_2$, where $(\cdot,\cdot)_2$ is chosen by $((u,v),(\chi,\psi))_2=(u,\chi)_{L^2}+(v,\psi)_{L^2}$ for any $(u,v),\ (\chi,\psi)\in L^2(\mathbb{R}^N,\mathbb{R}^2)$. Hence

$$E=E^+\oplus E^-.$$

It follows from (2.2), (2.3), (2.6) and (2.12) that

$$\begin{aligned}&\int_{\mathbb{R}^N}[\nabla u\nabla\chi+V_1(x)u\chi+\nabla v\nabla\psi+V_2(x)v\psi]\,\mathrm{d}x\\&\quad=(\mathcal{A}_1u,\chi)_{L^2}+(\mathcal{A}_2v,\psi)_{L^2}\\&\quad=(u_1^+,\chi_1^+)_{H_1}+(v_2^+,\psi_2^+)_{H_2}-(u_1^-,\chi_1^-)_{H_1}-(v_2^-,\psi_2^-)_{H_2}\\&\quad=\langle z^+,\xi^+\rangle-\langle z^-,\xi^-\rangle,\quad\forall\, z=(u,v),\ \xi=(\chi,\psi)\in E.\end{aligned}\tag{2.13}$$

and

$$\int_{\mathbb{R}^N}\left[|\nabla u|^2+V_1(x)|u|^2+|\nabla v|^2+V_2(x)|v|^2\right]\mathrm{d}x=\|z^+\|^2-\|z^-\|^2,\quad\forall\, z=(u,v)\in E. \tag{2.14}$$

Lemma 2.1. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then there exists* $\rho>0$ *such that*

$$\inf\{\Phi(z):z\in E^+,\|z\|=\rho\}>0. \tag{2.15}$$

We omit the proof here since it is standard.

Suppose that $G\in\mathbb{R}^N$ is a bounded domain. We can choose $\bar{e}:=(\bar{e}_u,\bar{e}_v)\in\mathcal{C}_0^\infty(\mathbb{R}^N,\mathbb{R}^+)\cap\mathcal{C}_0^\infty(G,\mathbb{R}^+)$ satisfying

$$\begin{aligned}\|\bar{e}^+\|^2-\|\bar{e}^-\|^2&=\int_{\mathbb{R}^N}\left[|\nabla\bar{e}_u|^2+V_1(x)|\bar{e}_u|^2+|\nabla\bar{e}_v|^2+V_2(x)|\bar{e}_v|^2\right]\mathrm{d}x\\&=\int_G\left[|\nabla\bar{e}_u|^2+V_1(x)|\bar{e}_u|^2+|\nabla\bar{e}_v|^2+V_2(x)|\bar{e}_v|^2\right]\mathrm{d}x\geq1,\end{aligned}$$

then $\bar{e}^+=(\bar{e}_u^+,\bar{e}_v^+)\neq(0,0)$.

Owing to prove $\mathcal{N}^-\neq\emptyset$, we also need the following lemma.

Lemma 2.2. *Assume that* (V), (F1), (F2) *and* (F5) *hold. Then* $\sup \Phi(E^- \oplus \mathbb{R}^+ \bar{e}^+) < \infty$ *and there is* $R_{\bar{e}} > 0$ *such that*

$$\Phi(z) \le 0, \quad \text{for } z \in E^- \oplus \mathbb{R}^+ \bar{e}^+ \ \text{ with } \ \|z\| \ge R_{\bar{e}}. \tag{2.16}$$

Proof. As the ideal of [34, Lemma 3.2 and Corollary 3.3], we can prove Lemma 2.2 by verifying that there is $r > \rho$ such that $\sup \Phi(\partial Q) \le 0$, where $Q = \{w + se^+ : w \in E^-, s \ge 0, \|w + se^+\| \le r\}$. □

Lemma 2.3. *Assume that* (V), (F1), (F2) *and* (F5) *hold. Then* $\mathcal{N}^- \ne \emptyset$.

Proof. From Lemma 2.1, $\Phi(t\bar{e}^+) > 0$ for small $t > 0$. Moreover, by Lemma 2.2, there exists $R_{\bar{e}} > 0$ such that $\Phi(z) \le 0$ for $z \in (E^- \oplus \mathbb{R}^+ \bar{e}^+) \setminus B_{R_{\bar{e}}}(0)$. Since that, $0 < \sup \Phi(E^- \oplus \mathbb{R}^+ \bar{e}^+) < \infty$. Hence, we can easily get that Φ is weakly upper semi-continuous on $E^- \oplus \mathbb{R}^+ \bar{e}^+$. Then, there exists $z_0 \in E^- \oplus \mathbb{R}^+ \bar{e}^+$ such that $\Phi(z_0) = \sup \Phi(E^- \oplus \mathbb{R}^+ \bar{e}^+)$. It is obvious that z_0 is a critical point of Φ, that is $\langle \Phi'(z_0), z_0 \rangle = \langle \Phi'(z_0), \zeta \rangle = 0$ for all $\zeta \in E^- \oplus \mathbb{R}^+ \bar{e}^+$. Therefore, $z_0 \in \mathcal{N}^- \cap (E^- \oplus \mathbb{R}^+ \bar{e}^+)$. □

3 The existence of ground state solutions

To prove Theorem 1.1 and Theorem 1.2, we define $I_\epsilon(z)$ for any $\epsilon \ge 0$ as follows:

$$I_\epsilon(z) = \Phi(z) - \epsilon \int_{\mathbb{R}^N} |z|^p \mathrm{d}x. \tag{3.1}$$

Let

$$\mathcal{N}_\epsilon^- = \{z \in E \setminus E^- : \langle I_\epsilon'(z), z \rangle = \langle I_\epsilon'(z), \zeta \rangle = 0, \quad \forall \zeta \in E^-\}. \tag{3.2}$$

Similar to Lemma 2.3, for $\epsilon \ge 0$, we have $\mathcal{N}_\epsilon^- \ne \emptyset$. Then we define $m_\epsilon := \inf_{\mathcal{N}_\epsilon^-} I_\epsilon$.

Lemma 3.1. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then*

$$I_\epsilon(z) \ge I_\epsilon(tz + \zeta) + \frac{1}{2}\|\zeta\|^2 + \frac{1-t^2}{2}\langle I_\epsilon'(z), z \rangle - t\langle I_\epsilon'(z), \zeta \rangle, \quad \forall t \ge 0,\ z \in E,\ \zeta \in E^-. \tag{3.3}$$

Proof. From (2.9), (2.10) and (3.1), we have

$$\begin{aligned}
&I_\epsilon(z) - I_\epsilon(tz + \zeta) \\
&\quad = \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \int_{\mathbb{R}^N} F(x,z)dx - \epsilon \int_{\mathbb{R}^N} |z|^p \mathrm{d}x \\
&\qquad - \frac{t^2}{2}\|z^+\|^2 + \frac{1}{2}\langle tz^- + \zeta, tz^- + \zeta \rangle + \int_{\mathbb{R}^N} F(x, tz + \zeta)dx - \epsilon \int_{\mathbb{R}^N} |tz + \zeta|^p \mathrm{d}x \\
&\quad = \frac{1}{2}\|\zeta\|^2 + \frac{1-t^2}{2}\langle I_\epsilon'(z), z \rangle - t\langle I_\epsilon'(z), \zeta \rangle \\
&\qquad + \frac{1-t^2}{2}\int_{\mathbb{R}^N} F_z(x,z)\cdot z dx - t\int_{\mathbb{R}^N} F_z(x,z)\cdot \zeta dx + \int_{\mathbb{R}^N} F(x, tz+\zeta)dx - \int_{\mathbb{R}^N} F(x,z)dx \\
&\qquad + \frac{1-t^2}{2}p\epsilon \int_{\mathbb{R}^N} |z|^p \mathrm{d}x - \epsilon \int_{\mathbb{R}^N} |z|^p \mathrm{d}x + \epsilon \int_{\mathbb{R}^N} |tz+\zeta|^p \mathrm{d}x - tp\epsilon \int_{\mathbb{R}^N} |z|^{p-2} z \cdot \zeta \mathrm{d}x.
\end{aligned} \tag{3.4}$$

From [35, Lemma 4.3], one has

$$\frac{1-t^2}{2}F_z(x,z)z - tF_z(x,z)\zeta + F(x, tz+\zeta) - F(x,z) \ge 0, \quad \forall z \in E,\ \zeta \in E^-,\ t \ge 0. \tag{3.5}$$

As in [28, Remark 6], we can get that

$$\frac{1-t^2}{2}p|z|^p - |z|^p + |tz+\zeta|^p - tp|z|^{p-2}z\cdot\zeta \ge 0, \quad \forall z \in E,\ \zeta \in E^-,\ t \ge 0. \tag{3.6}$$

Then, from (3.4), (3.5) and (3.6), we have

$$I_\epsilon(z) - I_\epsilon(tz+\zeta) \ge \frac{1}{2}\|\zeta\|^2 + \frac{1-t^2}{2}\langle I'_\epsilon(z), z\rangle - t\langle I'_\epsilon(z), \zeta\rangle.$$

The proof is completed. □

From the above lemma, we can get the following two corollaries.

Corollary 3.2. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then for* $z \in \mathcal{N}_\epsilon^-$,

$$I_\epsilon(z) \ge I_\epsilon(tz+\zeta), \quad \forall t \ge 0, \zeta \in E^-. \tag{3.7}$$

Corollary 3.3. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then*

$$I_\epsilon(z) \ge \frac{t^2}{2}\|z\|^2 - \int_{\mathbb{R}^N}\left[F(x,tz^+) + \epsilon|tz^+|^p\right]\mathrm{d}x + \frac{1-t^2}{2}\langle I'_\epsilon(z), z\rangle + t^2\langle I'_\epsilon(z), z^-\rangle, \quad \forall t \ge 0,\ z \in E. \tag{3.8}$$

Lemma 3.4. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then, for* $\epsilon \in [0,1]$,

(i) there exists $\hat{\kappa} > 0$ *which does not depend on* $\epsilon \in [0,1]$ *such that*

$$I_\epsilon(z) \ge m_\epsilon \ge \hat{\kappa}, \quad \forall z \in \mathcal{N}_\epsilon^-; \tag{3.9}$$

(ii) $\|z^+\| \ge \max\{\|z^-\|, \sqrt{2m_\epsilon}\}$ *for all* $z \in \mathcal{N}_\epsilon^-$.

Proof. (i) By (F1) and (F2), there exists a constant $C_4 > 0$ such that

$$F(x,z) + \epsilon|z|^p \le \frac{1}{4\gamma_2^2}|z|^2 + C_4|z|^p, \quad \forall x \in \mathbb{R}^N, z \in \mathbb{R}^2, \epsilon \in [0,1]. \tag{3.10}$$

In virtue of (2.4), (3.1), (3.7) and (3.10), one has

$$\begin{aligned} I_\epsilon(z) \ge I_\epsilon(tz^+) &= \frac{t^2}{2}\|z^+\|^2 - \int_{\mathbb{R}^N}\left[F(x,tz^+) + \epsilon|tz^+|^p\right]\mathrm{d}x \\ &\ge \frac{t^2}{4}\|z^+\|^2 - t^pC_4\|z^+\|_p^p \\ &\ge \frac{t^2}{4}\|z^+\|^2 - t^pC_4\gamma_p^p\|z^+\|^p, \quad \forall z \in \mathcal{N}_\epsilon^-,\ \epsilon \in [0,1],\ t \ge 0. \end{aligned} \tag{3.11}$$

Choose $t = t_z := \frac{1}{[2C_4\gamma_p^p p]^{\frac{1}{p-2}}\|z^+\|}$, then it follows from above inequality that

$$\begin{aligned} I_\epsilon(z) &\ge \frac{t_z^2}{4}\|z^+\|^2 - t_z^pC_4\gamma_p^p\|z^+\|^p \\ &= \frac{p-2}{4p\left[2C_4\gamma_p^p p\right]^{\frac{2}{p-2}}} =: \hat{\kappa} > 0, \quad \forall \epsilon \in [0,1], z \in \mathcal{N}_\epsilon^-. \end{aligned} \tag{3.12}$$

Hence, (3.9) holds.

(ii) (F4) shows that $F(x,z) \ge 0$. Then, it follows from (3.1), (3.2) and (3.9) that (ii) holds. □

Lemma 3.5. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then for any* $\epsilon \in (0,1]$*, there exists* $z_\epsilon \in \mathcal{N}_\epsilon^-$ *such that*

$$I_\epsilon(z_\epsilon) = m_\epsilon, \quad I'_\epsilon(z_\epsilon) = 0. \tag{3.13}$$

Proof. By virtue of [26, Lemma 4.2 and Lemma 4.3], we can get that there exists a bounded sequence $\{z_{\epsilon_n}\} \in E$ such that

$$I_\epsilon(z_{\epsilon_n}) \to c, \quad \|I'_\epsilon(z_{\epsilon_n})\|(1+\|z_{\epsilon_n}\|) \to 0, \quad n \to \infty, \tag{3.14}$$

where $c \in [\hat{\kappa}, m_\epsilon]$. Hence, there exists a constant $\tilde{C}_2 > 0$ such that $\|z_{\epsilon_n}\|_2 \le \tilde{C}_2$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |z_{\epsilon_n}|^2 \mathrm{d}x = 0,$$

applying Lion's concentration compactness principle [36, Lemma 1.21], $z_{\epsilon_n} \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. By (F1) and (F2), for $\epsilon = \frac{c}{4\tilde{C}_2^2} > 0$, there exists $\tilde{C}_\epsilon > 0$ such that

$$|F_z(x,z)| \le \epsilon|z| + \tilde{C}_\epsilon |z|^{p-1},$$
$$|F(x,z)| \le \epsilon|z|^2 + \tilde{C}_\epsilon |z|^p, \quad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R}^2.$$

Thus,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \left[\mathcal{F}(x, z_{\epsilon_n}) + \frac{p-2}{2}\epsilon_n |z_{\epsilon_n}|^p\right] \mathrm{d}x \le \frac{3}{2}\epsilon\tilde{C}_2^2 + \left(\frac{3}{2}\tilde{C}_\epsilon + \tilde{C}_3\right) \lim_{n\to\infty} \|z_{\epsilon_n}\|_p^p = \frac{3}{8}c. \tag{3.15}$$

From (3.1), (3.14) and (3.15), one has

$$\begin{aligned} c &= I_{\epsilon_n}(z_{\epsilon_n}) - \frac{1}{2}\langle I'_{\epsilon_n}(z_{\epsilon_n}), z_{\epsilon_n}\rangle + o(1) \\ &= \int_{\mathbb{R}^N} \left[\mathcal{F}(x, z_{\epsilon_n}) + \frac{p-2}{2}\epsilon_n |z_{\epsilon_n}|^p\right] \mathrm{d}x + o(1) \\ &\le \frac{3}{8}c + o(1). \end{aligned}$$

That is a contradiction, so we have $\delta > 0$.

Going if necessary to a subsequence, we may assume there exists $k_n \in \mathbb{Z}^N$ such that

$$\int_{B_{1+\sqrt{N}}(k_n)} |z_n|^2 \mathrm{d}x > \frac{\delta}{2}.$$

Define $w_n(x) := z_n(x + k_n)$ such that

$$\int_{B_{1+\sqrt{N}}(0)} |w_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{3.16}$$

In view of $V_i(x)$ and $F_z(x,z)$ are periodic on x, $i = 1,2$, we have $\|w_n\| = \|z_n\|$ and

$$I_{\epsilon_n}(w_n) \to c, \quad \|I'_{\epsilon_n}(w_n)\|(1+\|w_n\|) \to 0. \tag{3.17}$$

Going if necessary to a subsequence, we have $w_n \rightharpoonup \varpi$ in E, $w_n \to \varpi$ in $L^s_{loc}(\mathbb{R}^N), 2 < s < 2^*$ and $w_n \to \varpi$ a.e. on $\mathbb{R}^N$. Obviously, (3.16) implies that $\varpi \ne 0$. By a standard argument, we have $I'_{\epsilon_n}(\bar{w}) = 0$. Then $\bar{w} \in \mathcal{N}^-$ and $I_{\epsilon_n}(w_n) \geq m_\epsilon$. Moreover, from (3.17), (F4) and Fatou's Lemma, one has

$$\begin{aligned}
m_\epsilon \geq c &= \lim_{n\to\infty} \left[I_{\epsilon_n}(w_n) - \frac{1}{2}\langle I'_{\epsilon_n}(w_n), w_n \rangle \right] \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \left[\mathcal{F}(x, w_n) + \frac{p-2}{2}\epsilon_n |w_n|^p \right] \mathrm{d}x \\
&\geq \int_{\mathbb{R}^N} \lim_{n\to\infty} \left[\mathcal{F}(x, w_n) + \frac{p-2}{2}\epsilon_n |w_n|^p \right] \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left[\mathcal{F}(x, \bar{w}) + \frac{p-2}{2}\epsilon_n |\bar{w}|^p \right] \mathrm{d}x \\
&= I_{\epsilon_n}(\bar{w}) - \frac{1}{2}\langle I'_{\epsilon_n}(\bar{w}), \bar{w} \rangle = I_{\epsilon_n}(\bar{w}).
\end{aligned}$$

This shows that $I_{\epsilon_n}(\bar{w}) \leq m_\epsilon$ and then $I_{\epsilon_n}(\bar{w}) = m_\epsilon$. □

Lemma 3.6. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then for any $\epsilon \in (0,1]$ and $z \in E \setminus E^-$, there exist $t_\epsilon(z) > 0$ and $\zeta_\epsilon(z) \in E^-$ such that $t_\epsilon(z)z + \zeta_\epsilon(z) \in \mathcal{N}_\epsilon^-$.*

We can easily prove this lemma in a similar way as Lemma 2.3, so we omit it.

Proof of Theorem 1.1. Consider the case $N \geq 3$. By Lemma 3.5, there exists $z_\epsilon \in \mathcal{N}_\epsilon^-$ such that (3.13) holds, where $\epsilon \in (0,1]$.

By Lemma 2.3, $\mathcal{N}^- \neq \emptyset$. Then, for $z_0 \in \mathcal{N}^-$ and $\zeta \in E^-$, $\Phi(z_0) := \bar{c} \geq 0$ and $\langle \Phi'(z_0), z_0 \rangle = \langle \Phi'(z_0), \zeta \rangle = 0$ hold. In virtue of Lemma 3.6, there exist $t_\epsilon > 0$ and $\zeta_\epsilon \in E^-$ such that $t_\epsilon z_0 + \zeta_\epsilon \in \mathcal{N}_\epsilon^-$. By Corollary 3.2 and Lemma 3.4, one has

$$\begin{aligned}
\bar{c} = \Phi(z_0) &= I_0(z_0) \geq I_0(t_\epsilon z_0 + \zeta_\epsilon) \\
&\geq I_\epsilon(t_\epsilon z_0 + \zeta_\epsilon) \geq m_\epsilon \geq \hat{\kappa}, \quad \forall \epsilon \in (0,1).
\end{aligned} \tag{3.18}$$

Choose a sequence $\{\epsilon_n\} \subset (0,1]$ satisfy $\epsilon_n \to 0$ as $n \to \infty$, and

$$z_{\epsilon_n} \in \mathcal{N}_{\epsilon_n}^-, \quad I_{\epsilon_n}(z_{\epsilon_n}) = m_{\epsilon_n} \to \bar{m} \in [\hat{\kappa}, \bar{c}], \quad I'_{\epsilon_n}(z_{\epsilon_n}) = 0. \tag{3.19}$$

There are three steps to prove Theorem 1.1.

Step 1: We prove that $\{z_{\epsilon_n}\}$ is bounded in E.

Arguing by contradiction, suppose that $\|z_{\epsilon_n}\| \to \infty$. Set $w_n = \frac{z_{\epsilon_n}}{\|z_{\epsilon_n}\|}$, then $\|w_n\| = 1$. By the Sobolev embedding theorem, going if necessary to a subsequence, we have

$$\begin{cases}
w_n \rightharpoonup w, & \text{in } E; \\
w_n \to w, & \text{in } L^s_{loc}(\mathbb{R}^N),\ \forall s \in [2, 2^*); \\
w_n \to w, & \text{a.e. on } \mathbb{R}^N.
\end{cases}$$

From (3.19), we have

$$\bar{c} \geq I_{\epsilon_n}(z_{\epsilon_n}) - \frac{1}{2}\langle I'_{\epsilon_n}(z_{\epsilon_n}), z_{\epsilon_n} \rangle = \int_{\mathbb{R}^N} \left[\mathcal{F}(x, z_{\epsilon_n}) + \frac{p-2}{2}\epsilon_n |z_{\epsilon_n}|^p \right] \mathrm{d}x. \tag{3.20}$$

In view of Sobolev embedding theorem, there exists a constant $\tilde{C}_4 > 0$ such that $\|w_n\|_2 \leq \tilde{C}_4$. If

$$\delta := \limsup_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |w_n^+|^2 \mathrm{d}x = 0, \tag{3.21}$$

by Lion's concentration compactness principle, $w_n^+ \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Let $R > [2(1+\bar{c})]^{\frac{1}{2}}$. From (F1) and (F2), choose $\varepsilon = \frac{1}{4(R\tilde{C}_4)^2} > 0$, there exists $\tilde{C}_5 > 0$ such that

$$\begin{aligned}\limsup_{n\to\infty} \int_{\mathbb{R}^N} \left[F(x, Rw_n^+) + \epsilon_n |Rw_n^+|^p\right] \mathrm{d}x &\le \limsup_{n\to\infty} \left[\varepsilon R^2 \|w_n^+\|_2^2 + \tilde{C}_5 R^p \|w_n^+\|_p^p\right] \\ &\le \varepsilon (R\tilde{C}_4)^2 = \frac{1}{4}. \end{aligned} \tag{3.22}$$

Let $t_n = \frac{R}{\|z_{\epsilon_n}\|}$. From (3.19), (3.22) and Corollary 3.3, one has

$$\begin{aligned} \bar{c} \ge m_{\epsilon_n} &= I_{\epsilon_n}(z_{\epsilon_n}) \\ &\ge \frac{t_n^2}{2}\|z_{\epsilon_n}\|^2 - \int_{\mathbb{R}^N} \left[F(x, t_n z_{\epsilon_n}^+) + \epsilon_n |t_n z_{\epsilon_n}^+|^p\right] \mathrm{d}x \\ &= \frac{R^2}{2} - \int_{\mathbb{R}^N} \left[F(x, Rw_n^+) + \epsilon_n |Rw_n^+|^p\right] \mathrm{d}x \\ &\ge \frac{R^2}{2} - \frac{1}{4} + o(1) \\ &> \bar{c} + \frac{3}{4} + o(1), \end{aligned}$$

which is a contradiction, then $\delta > 0$.

Passing to a subsequence, we may assume there exists $k_n \in \mathbb{Z}^N$ such that

$$\int_{B_{1+\sqrt{n}}(k_n)} |w_n^+|^2 \mathrm{d}x > \frac{\delta}{2}.$$

Let $\tilde{w}_n = w_n(x + k_n)$. Since $V_1(x)$ and $V_2(x)$ are 1-periodic in each of $x_1, x_2, \dots, x_N$, then $\mathcal{A}_i = -\Delta + V_i$, E^+ and E^- are $\mathbb{Z}^N$-translation invariance. Thereby, $\|\tilde{w}_n\| = \|w_n\| = 1$, and

$$\int_{B_{1+\sqrt{n}}(0)} |\tilde{w}_n^+|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{3.23}$$

Going if necessary to a subsequence , we have

$$\begin{cases} \tilde{w}_n \rightharpoonup \tilde{w}, & \text{in } E; \\ \tilde{w}_n \to \tilde{w}, & \text{in } L^s_{loc}(\mathbb{R}^N),\ \forall s \in [2, 2^*); \\ \tilde{w}_n \to \tilde{w}, & \text{a.e. on } \mathbb{R}^N. \end{cases}$$

Then (3.23) shows that $\tilde{w} \ne 0$.

Define $\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n) = z_{\epsilon_n}(x + k_n)$, note that $z_{\epsilon_n} = (u_{\epsilon_n}, v_{\epsilon_n})$. Hence, $\frac{\tilde{z}_n}{\|z_{\epsilon_n}\|} = \tilde{w}_n \to \tilde{w}$ a.e. on $\mathbb{R}^N$ and $\tilde{w} \ne 0$, here $\tilde{w}_n = (\tilde{\eta}_n, \tilde{\theta}_n)$. For any $\varphi = (\mu, \nu) \in C_0^\infty(\mathbb{R}^N)$, let $\phi_n = (\mu_n, \nu_n) = \varphi(x - k_n)$. From (3.1) and (3.19), we have

$$\begin{aligned} 0 &= \langle I'_{\epsilon_n}(z_{\epsilon_n}), \|z_{\epsilon_n}\|\phi_n \rangle \\ &= \|z_{\epsilon_n}\| \int_{\mathbb{R}^N} (\nabla u_{\epsilon_n} \cdot \nabla \mu_n + V_1(x) u_{\epsilon_n} \cdot \mu_n + \nabla v_{\epsilon_n} \cdot \nabla \nu_n + V_2(x) v_{\epsilon_n} \cdot \nu_n)\, \mathrm{d}x \\ &\quad - \|z_{\epsilon_n}\| \int_{\mathbb{R}^N} \left[F_z(x, z_{\epsilon_n}) + p\epsilon_n |z_{\epsilon_n}|^{p-2} z_{\epsilon_n}\right] \varphi_n \mathrm{d}x \\ &= \|z_{\epsilon_n}\| \int_{\mathbb{R}^N} (\nabla \tilde{u}_n \cdot \nabla \mu + V_1(x) \tilde{u}_n \cdot \mu + \nabla \tilde{v}_n \cdot \nabla \nu + V_2(x) \tilde{v}_n \cdot \nu)\, \mathrm{d}x \\ &\quad - \|z_{\epsilon_n}\| \int_{\mathbb{R}^N} \left[F_z(x, \tilde{z}_n) + p\epsilon_n |\tilde{z}_n|^{p-2} \tilde{z}_n\right] \varphi \mathrm{d}x \end{aligned}$$

$$= \|z_{\epsilon_n}\|^2 \int_{\mathbb{R}^N} \left(\nabla\tilde{\eta}_n \cdot \nabla\mu + V_1(x)\tilde{\eta}_n \cdot \mu + \nabla\tilde{\theta}_n \cdot \nabla\nu + V_2(x)\tilde{\theta}_n \cdot \nu\right) \mathrm{d}x$$
$$- \|z_{\epsilon_n}\| \int_{\mathbb{R}^N} \left[F_z(x,\tilde{z}_n) + p\epsilon_n|\tilde{z}_n|^{p-2}\tilde{z}_n\right] \varphi \mathrm{d}x, \tag{3.24}$$

which implies

$$\int_{\mathbb{R}^N} \left(\nabla\tilde{\eta}_n \cdot \nabla\mu + V_1(x)\tilde{\eta}_n \cdot \mu + \nabla\tilde{\theta}_n \cdot \nabla\nu + V_2(x)\tilde{\theta}_n \cdot \nu\right) \mathrm{d}x$$
$$= \frac{1}{\|z_{\epsilon_n}\|} \int_{\mathbb{R}^N} \left[F_z(x,\tilde{z}_n) + p\epsilon_n|\tilde{z}_n|^{p-2}\tilde{z}_n\right] \varphi \mathrm{d}x. \tag{3.25}$$

By virtue of (F1), (F2), (F6), (3.20) and the Hölder inequality, one can get that

$$\begin{aligned}
&\frac{1}{\|z_{\epsilon_n}\|} \int_{\mathbb{R}^N} \left|\left[F_z(x,\tilde{z}_n) + p\epsilon_n|\tilde{z}_n|^{p-2}\tilde{z}_n\right] \varphi\right| \mathrm{d}x \\
&\le \frac{1}{\|z_{\epsilon_n}\|^{1-\sigma}} \int_{\tilde{z}_n\neq 0} \left(\frac{|F_z(x,\tilde{z}_n)|}{|\tilde{z}_n|^{\sigma}} + p\epsilon_n|\tilde{z}_n|^{p-1-\sigma}\right)|\tilde{w}_n|^{\sigma}|\varphi|\mathrm{d}x \\
&= \frac{1}{\|z_{\epsilon_n}\|^{1-\sigma}} \left[\int_{0<|\tilde{z}_n|<R_0} \left(\frac{|F_z(x,\tilde{z}_n)|}{|\tilde{z}_n|^{\sigma}} + p\epsilon_n|\tilde{z}_n|^{p-1-\sigma}\right)|\tilde{w}_n|^{\sigma}|\varphi|\mathrm{d}x\right. \\
&\quad \left.+ \frac{1}{\|z_{\epsilon_n}\|^{1-\sigma}} \int_{|\tilde{z}_n|\ge R_0} \left(\frac{|F_z(x,\tilde{z}_n)|}{|\tilde{z}_n|^{\sigma}} + p\epsilon_n|\tilde{z}_n|^{p-1-\sigma}\right)|\tilde{w}_n|^{\sigma}|\varphi|\mathrm{d}x\right] \\
&\le \frac{\|\tilde{w}_n\|_{2^*}^{\sigma}\|\varphi\|_{2^*}}{\|z_{\epsilon_n}\|^{1-\sigma}} \left[\int_{|\tilde{z}_n|\ge R_0} \left(\frac{|F_z(x,\tilde{z}_n)|}{|\tilde{z}_n|^{\sigma}} + p\epsilon_n|\tilde{z}_n|^{p-1-\sigma}\right)^{\frac{2^*}{2^*-1-\sigma}} \mathrm{d}x\right]^{\frac{2^*-1-\sigma}{2^*}} \\
&\quad + \frac{C_5\|\tilde{w}_n\|_2^{\sigma}\|\varphi\|_{\frac{2}{2-\sigma}}}{\|z_{\epsilon_n}\|^{1-\sigma}} \\
&\le \frac{C_6}{\|z_{\epsilon_n}\|^{1-\sigma}} \left\{\|\varphi\|_{\frac{2}{2-\sigma}} + \|\varphi\|_{2^*}\left[\int_{|\tilde{z}_n|\ge R_0} (\mathcal{F}(x,\tilde{z}_n) + \frac{p-2}{2}\epsilon_n|\tilde{z}_n|^p)\mathrm{d}x\right]^{\frac{2^*-1-\sigma}{2^*}}\right\} \\
&\le \frac{C_6}{\|z_{\epsilon_n}\|^{1-\sigma}} \left\{\|\varphi\|_{\frac{2}{2-\sigma}} + \|\varphi\|_{2^*}\left[\int_{\mathbb{R}^N} \left(\mathcal{F}(x,\tilde{z}_n) + \frac{p-2}{2}\epsilon_n|\tilde{z}_n|^p\right)\mathrm{d}x\right]^{\frac{2^*-1-\sigma}{2^*}}\right\} \\
&\le \frac{\tilde{C}_6}{\|z_{\epsilon_n}\|^{1-\sigma}} \left[\|\varphi\|_{\frac{2}{2-\sigma}} + \|\varphi\|_{2^*}\right] = o(1).
\end{aligned} \tag{3.26}$$

It follows from (3.25) and (3.26) that

$$\int_{\mathbb{R}^N} \left(\nabla\tilde{\eta}_n \cdot \nabla\mu + V_1(x)\tilde{\eta}_n \cdot \mu + \nabla\tilde{\theta}_n \cdot \nabla\nu + V_2(x)\tilde{\theta}_n \cdot \nu\right) \mathrm{d}x = o(1), \ \ \forall\ (\mu,\nu) \in C_0^{\infty}(\mathbb{R}^N). \tag{3.27}$$

In view of $\tilde{w}_n \rightharpoonup \tilde{w}$, one has

$$\int_{\mathbb{R}^N} \left(\nabla\tilde{\eta} \cdot \nabla\mu + V_1(x)\tilde{\eta} \cdot \mu + \nabla\tilde{\theta} \cdot \nabla\nu + V_2(x)\tilde{\theta} \cdot \nu\right) \mathrm{d}x = 0, \ \ \forall\ (\mu,\nu) \in C_0^{\infty}(\mathbb{R}^N). \tag{3.28}$$

This implies that $\mathcal{A}_i\tilde{w} = -\Delta\tilde{w} + V_i(x)\tilde{w} = 0$. Then $\tilde{w}$ is an eigenfunction of the operator $\mathcal{A}_i$, where $i = 1,2$. Note that $\mathcal{A}_i$ has only a continuous spectrum. That is a contradiction. Hence, $\{\|z_{\epsilon_n}\|\}$ is bounded.

Step 2: We prove that there exists $\bar{z} \in E$ such that $\Phi'(\bar{z}) = 0$ and $\Phi(\bar{z}) \ge m_0 := \inf_{\mathcal{N}_0^-} I_0 = \inf_{\mathcal{N}^-} \Phi$.

Applying Lion's concentration principle like in Step 1, we can deduce that there exist a constant $\delta_1 > 0$, a sequence $y_n \in \mathbb{Z}^N$ and a subsequence of $\{z_{\epsilon_n}\}$, which is still denoted by $\{z_{\epsilon_n}\}$, such that

$$\int_{B_1(y_n)} |z_{\epsilon_n}|^2 \mathrm{d}x > \delta_1. \tag{3.29}$$

Define $\hat{z}_n = z_{\epsilon_n}(x + y_n)$. By E^+ and E^- are $\mathbb{Z}^N$-translation invariance, we have $\|\hat{z}_n\| = \|z_{\epsilon_n}\|$ and

$$\hat{z}_n \in \mathcal{N}^-_{\epsilon_n}, \quad I_{\epsilon_n}(\hat{z}_n) = m_{\epsilon_n} \to \bar{m} \in [\hat{\kappa}, \bar{c}] \quad , I'_{\epsilon_n}(\hat{z}_n) = 0. \tag{3.30}$$

Hence, there exists $\bar{z} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that, going if necessary to a subsequence ,

$$\begin{cases} \hat{z}_n \rightharpoonup \bar{z}, & \text{in } H^1(\mathbb{R}^N); \\ \hat{z}_n \to \bar{z}, & \text{in } L^s_{loc}(\mathbb{R}^N),\ \forall s \in [1, 2^*); \\ \hat{z}_n \to \bar{z}, & \text{a.e. on } \mathbb{R}^N. \end{cases} \tag{3.31}$$

Noting that $\hat{z}_n = (\hat{u}_n, \hat{v}_n), \varphi = (\mu, \nu)$. By virtue of (2.10), (3.1) and (3.31), we have

$$\begin{aligned} \langle \Phi'(\bar{z}), \varphi \rangle &= \int_{\mathbb{R}^N} (\nabla \hat{u}_n \nabla \mu + V_1(x) \hat{u}_n \mu + \nabla \hat{v}_n \nabla \nu + V_2(x) \hat{v}_n \nu)\, \mathrm{d}x - \int_{\mathbb{R}^N} F_z(x, \bar{z}) \varphi \mathrm{d}x \\ &= \lim_{n\to\infty} \left\{ \int_{\mathbb{R}^N} (\nabla \hat{u}_n \nabla \mu + V_1(x) \hat{u}_n \mu + \nabla \hat{v}_n \nabla \nu + V_2(x) \hat{v}_n \nu)\, \mathrm{d}x \right. \\ &\quad \left. - \int_{\mathbb{R}^N} \left[F_z(x, \hat{z}_n) + \epsilon_n p |\hat{z}_n|^{p-2} \hat{z}_n \right] \varphi \mathrm{d}x \right\} \\ &= \lim_{n\to\infty} \langle I'_{\epsilon_n}(\hat{z}_n), \varphi \rangle = 0, \quad \forall \varphi \in C_0^\infty(\Omega). \end{aligned}$$

This implies that $\Phi'(\bar{z}) = 0$. Then, $\bar{z} \in \mathcal{N}^-$, $\Phi(\bar{z}) \geq m_0$.

Step 3: We prove that $\Phi(\bar{z}) = m_0$.

In view of (2.9), (2.10), (3.1), (3.30), (3.31) and Fatou's Lemma, we have

$$\begin{aligned} \bar{m} &= \lim_{n\to\infty} m_{\epsilon_n} \\ &= \lim_{n\to\infty} \left[I_{\epsilon_n}(\hat{z}_n) - \frac{1}{2} \langle I'_{\epsilon_n}(\hat{z}_n), \hat{z}_n \rangle \right] \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N} \left[\mathcal{F}(x, \hat{z}_n) + \frac{p-2}{2} \epsilon_n |\hat{z}_n|^p \right] \mathrm{d}x \\ &\geq \int_{\mathbb{R}^N} \mathcal{F}(x, \bar{z}) \mathrm{d}x = \Phi(\bar{z}) - \frac{1}{2} \langle \Phi'(\bar{z}), \bar{z} \rangle \geq m_0. \end{aligned} \tag{3.32}$$

Let $\varepsilon > 0$. Then there exists $w_\varepsilon \in \mathcal{N}^-$ such that $\Phi(w_\varepsilon) < m_0 + \varepsilon$. By Lemma 3.6, there exist $t_n > 0$ and $\zeta_n \in E^-$ such that $t_n w_\varepsilon + \zeta_n \in \mathcal{N}^-_{\epsilon_n}$. From (3.1) and Corollary 3.2, one has

$$m_0 + \varepsilon > \Phi(w_\varepsilon) = I_0(w_\varepsilon) \leq I_0(t_n w_\varepsilon + \zeta_n) \geq I_{\epsilon_n}(t_n w_\varepsilon + \zeta_n) \geq m_{\epsilon_n}. \tag{3.33}$$

Thus,

$$\bar{m} = \lim_{n\to\infty} m_{\epsilon_n} \leq m_0 + \varepsilon. \tag{3.34}$$

Since ε can be any positive number, we have $\bar{m} \leq m_0$. In view of (3.32), we can get that $\bar{m} = m_0 = \Phi(\bar{z})$.

Since the case $N = 1, 2$ can be dealt with similarly, we omit it. The proof is completed. □

Lemma 3.7. *Assume that (V), (F1)–(F3) and (F6) hold. Then*

(i) $\vartheta := \inf\{\|z\| : z \in K\} > 0$;

(ii) $\varrho := \inf\{\Phi(z) : z \in K\} > 0$.

Proof. We only consider the case where $N \geq 3$, since $N = 1, 2$ can be dealt with similarity.

(i) Similar to [26, Theorem 1.1], we have $K \neq \emptyset$. Let $\{z_n\} \subset K$ such that $\|z_n\| \to \vartheta$. From (2.10), we have

$$\|z_n\|^2 = \int_{\mathbb{R}^N} F_z(x, z_n)(z_n^+ - z_n^-) dx. \tag{3.35}$$

In view of $F(x,z) \geq 0$ and $\mathcal{F}(x,z) \geq 0$, then $F_z(x,z)z \geq 0$. From (F1), (F2), (2.4) and (3.35), one has

$$\begin{aligned}\|z_n\|^2 &= \int_{z_n \neq 0} \frac{F_z(x,z_n)}{z_n} \left(|z_n^+|^2 - |z_n^-|^2\right) dx \\ &\leq \frac{1}{2\gamma_2^2}\|z_n^+\|_2^2 + C_7\|z_n\|_p^{p-2}\|z_n^+\|_p^2 \\ &\leq \frac{1}{2}\|z_n\|_2^2 + C_8\|z_n\|^p,\end{aligned}$$

then,

$$\vartheta + o(1) = \|z_n\| \geq (2C_8)^{-\frac{1}{p-2}} > 0. \tag{3.36}$$

This implies that (i) holds.

(ii) Let $\{z_n\} \subset K$ such that $\Phi(z_n) \to \varrho$. Then $\langle \Phi'(z_n), \bar{z} \rangle = 0$ for any $\bar{z} \in E$. From (2.9) and (2.10), we have

$$\varrho + o(1) = \Phi(z_n) - \frac{1}{2}\langle \Phi'(z_n), z_n \rangle = \int_{\mathbb{R}^N} \mathcal{F}(x, z_n) dx. \tag{3.37}$$

Let $w_n = \frac{z_n}{\|z_n\|}$. Then $\|z_n\|^2 = 1$. Set

$$\Omega_n := \left\{ x \in \mathbb{R}^N : \frac{|F_z(x,z_n)|}{|z|} \leq \tau \right\}. \tag{3.38}$$

Since $\Lambda_0\|w_n^+\|_2^2 \leq \|w_n^+\|^2$, we have

$$\begin{aligned}&\int_{\Omega_n} \frac{F_z(x,z_n)}{z_n}|w_n|\left(|w_n^+| + |w_n^-|\right) dx \\ &\leq \tau\|w_n\|_2 \left[\int_{\mathbb{R}^N} (|w_n^+| + |w_n^-|)^2 dx\right]^{\frac{1}{2}} \\ &\leq \tau\|w_n\|_2 \left(\|w_n^+\|_2^2 + \|w_n^-\|_2^2\right)^{\frac{1}{2}} \leq 1 - \frac{\delta_0}{\Lambda_0}.\end{aligned} \tag{3.39}$$

From (F6), (3.36), (3.37) and the Hölder inequality, we have

$$\begin{aligned}&\frac{1}{\|z_n\|^{1-\delta}} \int_{\mathbb{R}^N \setminus \Omega_n} \frac{|F_z(x,z_n)|}{|z_n|^\sigma}|w_n|^\sigma|w_n^+ - w_n^-| dx \\ &\leq \frac{1}{\|z_n\|^{1-\delta}} \left[\int_{\mathbb{R}^N \setminus \Omega_n} \left(\frac{|F_z(x,z_n)|}{|z_n|^\sigma}\right)^{\frac{2^*}{2^*-1-\sigma}} dx\right]^{\frac{2^*-1-\sigma}{2^*}} \|w_n\|_{2^*}^\sigma \|w_n^+ - w_n^-\|_{2^*}\end{aligned}$$

$$\leq \frac{C_9}{\|w_n\|^{1-\sigma}} \left[\int_{\mathbb{R}^N \setminus \Omega_n} \mathcal{F}(x, z_n) \mathrm{d}x \right]^{\frac{2^*-1-\sigma}{2^*}} \leq C_{10}[\varrho + o(1)]^{\frac{2^*-1-\sigma}{2^*}}. \tag{3.40}$$

By virtue of (3.39), (3.40) and (2.10), one has

$$\begin{aligned} 1 &= \frac{\|z_n\|^2 - \langle \Phi'(z_n), z_n^+ - z_n^- \rangle}{\|z_n\|^2} \\ &= \frac{1}{\|z_n\|} \int_{\mathbb{R}^N} F_z(x, z_n)(z_n^+ - z_n^-) \mathrm{d}x \\ &= \int_{\Omega_n} \frac{F_z(x, z_n)}{z_n} \left[(w_n^+)^2 - (w_n^-)^2 \right] \mathrm{d}x + \frac{1}{\|z_n\|^{1-\sigma}} \int_{\mathbb{R}^N \setminus \Omega_n} \frac{F_z(x, z_n)}{|z_n|^\sigma} |w_n|^\sigma \left(w_n^+ - w_n^- \right) \mathrm{d}x \\ &\leq \int_{\Omega_n} \frac{F_z(x, z_n)}{z_n} (w_n^+)^2 \mathrm{d}x + \frac{1}{\|z_n\|^{1-\sigma}} \int_{\mathbb{R}^N \setminus \Omega_n} \frac{F_z(x, z_n)}{|z_n|^\sigma} |w_n|^\sigma (w_n^+ - w_n^-) \mathrm{d}x \\ &\leq 1 - \frac{\delta_0}{\Lambda_0} + C_{10}[\varrho + o(1)]^{\frac{2^*-1-\sigma}{2^*}}. \end{aligned}$$

Then we can get that $\varrho > 0$. □

Proof of Theorem 1.2. Let $z_n \in K$ such that $\Phi(z_n) \to \varrho$. As [26, Lemma 4.3], we can easily prove the boundedness of $\{z_n\}$ in E, so we omit it. Then, similar to the proof of Theorem 1.1, we can get that there exists $\bar{z} \in E \setminus \{0\}$ such that $\Phi'(\bar{z}) = 0$ and $\Phi(\bar{z}) = \varrho > 0$. □

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