Electronic Journal of Qualitative Theory of Differential Equations
2020, No. **80**, 1–10; https://doi.org/10.14232/ejqtde.2020.1.80 www.math.u-szeged.hu/ejqtde/

Existence of periodic solutions of pendulum-like ordinary and functional differential equations

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

László Hatvani ✉

Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, Szeged, H–6720, Hungary

Received 10 August 2020, appeared 21 December 2020
Communicated by Tibor Krisztin

Abstract. The equation

$$x''(t) = a(t, x(t)) + b(t, x) + d(t, x)e(x'(t))$$

is considered, where $a : \mathbb{R}^2 \to \mathbb{R}$, $b, d : \mathbb{R} \times C(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$, $e : \mathbb{R} \to \mathbb{R}$ are continuous, and a, b, d are T-periodic with respect to t. Using the Leray–Schauder degree theory we prove that a sign condition, in which a dominates b, is sufficient for the existence of a T-periodic solution. The main theorem is applied to the equation of the forced damped pendulum.

Keywords: Leray–Schauder degree, forced damped pendulum.

2020 Mathematics Subject Classification: 34C25, 34K13.

1 Introduction

Second order differential equations of the type

$$x'' = h(t, x, x')$$

are basic models in mechanics: h is the resultant force acting on the system. When h is T-periodic with respect to t then it is an important problem to find conditions for the existence of T-periodic answer, T-periodic motions of the system. A simple model is the periodically forced damped mathematical pendulum

$$x'' + g(t, x, x') + a \sin x = e(t), \tag{1.1}$$

where e is T-periodic, g is T-periodic with respect to t and satisfies the following Nagumo-type condition: there exists a constant C such that every possible solution x of (1.1) satisfying $\sup_{[0,T]} |x| < 3\pi/2$ has the property $|x'(t)| < C$ ($t \in \mathbb{R}$). H. W. Knobloch [8] proved that if

✉ Email: hatvani@math.u-szeged.hu

$\sup_{[0,T]} |e| < a$, then equation (1.1) has T-periodic solutions. J. Mawhin and M. Willem [10,12] extended this result to more general equations.

In the practice many important technical models connected with the pendulum are described by more general differential equations than (1.1). As particular cases we will consider in detail the mathematical pendulum with periodically vibrating suspension point and a functional differential equation model. The equations cannot be handled by Knobloch's or by Mawhin's and Willem's extensions. We extend the Leray–Schauder method for more general pendulum-like equations, i.e., differential equations containing a main part satisfying the same sign condition as the sine function in the pendulum equation but admitting also periodic perturbations.

In this paper we introduce a wide class of pendulum-like differential equations admitting a variety of perturbations including ordinary and functional terms even with unbounded delays. The proof of the existence of periodic solutions is based upon the Leray–Schauder continuation method [5,6,9,10].

2 The main theorem and its proof

For a fixed $T > 0$ we will use the standard notations:

$$C := \{\varphi : \mathbb{R} \to \mathbb{R} \mid \varphi \text{ is continuous}\};$$
$$C^1 := \{\psi : \mathbb{R} \to \mathbb{R} \mid \psi \text{ is continuously differentiable}\};$$
$$C_T := \{\varphi \in C : \varphi \text{ is } T\text{-periodic}\}, \qquad C_T^1 := \{\psi \in C^1 : \psi \text{ is } T\text{-periodic}\}.$$

If $\varphi \in C$ is bounded, $\psi \in C^1$, and ψ, ψ' are bounded on $\mathbb{R}$, then define

$$\|\varphi\|_0 := \sup_{t\in\mathbb{R}} |\varphi(t)|, \qquad \|\psi\|_1 := \max\left\{\sup_{t\in\mathbb{R}} |\psi(t)|;\ \sup_{t\in\mathbb{R}} |\psi'(t)|\right\}.$$

Consider the equation

$$x''(t) = a(t, x(t)) + b(t, x) + d(t, x)e(x'(t)), \tag{2.1}$$

where functions $a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $b, d : \mathbb{R} \times C \to \mathbb{R}$; $e : \mathbb{R} \to \mathbb{R}$ are continuous, and $e(0) = 0$. Moreover, we suppose that for every fixed $\overline{u} \in \mathbb{R}$, $\overline{\varphi} \in C$ functions $t \mapsto a(t, \overline{u}), b(t, \overline{\varphi}), d(t, \overline{\varphi})$ are T-periodic.

Functions a, b, d, e generate the following operators:

$$\begin{aligned}
A &: C \to C, & \varphi &\mapsto A\varphi, & (A\varphi)(t) &:= a(t, \varphi(t));\\
B &: C \to C, & \varphi &\mapsto B\varphi, & (B\varphi)(t) &:= b(t, \varphi);\\
D &: C \to C, & \varphi &\mapsto D\varphi, & (D\varphi)(t) &:= d(t, \varphi);\\
D_e &: C^1 \to C, & \psi &\mapsto D_e\psi, & (D_e\psi)(t) &:= d(t, \psi)e(\psi'(t)).
\end{aligned}$$

For $R > 0$, $S > 0$ given we define the subset

$$C_T(-R, S) := \{\varphi \in C_T : -R \le \varphi(t) \le S \ (t \in \mathbb{R})\}.$$

By the use of the notations $f : \mathbb{R} \times C^1 \to \mathbb{R}$, $F : C^1 \to C$,

$$f(t, \psi) := a(t, \psi(t)) + b(t, \psi) + d(t, \psi)e(\psi'(t)),$$
$$F\psi := f(\cdot, \psi) = a(\cdot, \psi(\cdot)) + b(\cdot, \psi) + d(\cdot, \psi)e(\psi'(\cdot)) = A\psi + B\psi + D_e\psi$$

equation (2.1) can be rewritten in the shortened form

$$x''(t) = f(t,x) = Fx(t). \tag{2.2}$$

Theorem 2.1. *Suppose that there exist positive constants R, S and a continuous nondecreasing function $\phi : (0,\infty) \to (0,\infty)$ such that*

(i)
$$a(t,S) > \sup\{|b(t,\varphi)| : \varphi \in C_T(-R,S)\} =: \beta_{-R,S}(t),$$
$$a(t,-R) < -\beta_{-R,S}(t) \qquad (t \in \mathbb{R});$$

(ii) *operators B and D map bounded sets of C_T into bounded sets of C_T;*

(iii)
$$\int_1^\infty \frac{u}{\phi(u)}\,\mathrm{d}u = \infty, \qquad |e(u)| \le \phi(|u|) \qquad (u \in \mathbb{R})$$

hold.

Then there exists a T-periodic solution $x \in C_T(-R,S)$ of (2.1).

Proof. We use the Leray–Schauder degree for completely continuous perturbation of the identity operator [5, 6, 9, 10, 13]. We suppose that the reader is familiar with the definition of the Brouwer degree and the Leray–Schauder degree and their most basic properties (see, e.g., [4]).

Now we sketch the main steps of the proof. We find an open bounded set $\Omega \subset C_T^1$ and a family of mappings $M_\lambda : \overline{\Omega} \to C_T^1$ $(\lambda \in [0,1])$ having the following properties:

(a) if x is a fixed point of M_1 in Ω, then x is the desired periodic solution of (2.1), i.e., $x \in C_T(-R,S)$, and x is a solution of (2.1);

(b) the function
$$M^* : \overline{\Omega} \times [0,1] \to C_T^1, \quad M^*(\psi,\lambda) = M_\lambda \psi$$
is completely continuous;

(c) if $\varphi \in \partial\Omega$ and $\lambda \in [0,1]$, then $\varphi \neq M_\lambda \varphi$;

(d) if $I : C \to C$ is the identity operator and $d[I - M_\lambda, \Omega, 0]$ denotes the Leray–Schauder degree of M_λ with respect to Ω, then $d[I - M_0, \Omega, 0] \neq 0$.

Then an application of basic theorems of the theory of the Leray–Schauder degree yields the assertion of the theorem.

For the definition of $\Omega \subset C_T^1$ we need a Nagumo-type result [13] for the family of equations

$$x''(t) = \lambda f(t,x) \qquad (\lambda \in [0,1]) \tag{2.3}$$

associated with (2.2).

Lemma 2.2. *Suppose that conditions (i)–(iii) in Theorem 2.1 are satisfied. Then there is a $K > 1$ such that for any $\lambda \in [0,1]$ and for an arbitrary solution $x \in C_T(-R,S)$ of (2.3) the inequality*

$$|x'(t)| \le K - 1 \qquad (t \in \mathbb{R})$$

holds.

Proof. Consider an arbitrary solution $x \in C_T(-R,S)$ of (2.1). By conditions (ii) and (iii) there exist constants K_1 and K_2 independent of $\lambda \in [0,1]$ and the solution x such that

$$|x''(t)| \le \max\{|a(s,u)| : 0 \le s \le T, -R \le u \le S\} + K_1 + K_2\phi(|x'(t)|) \qquad (0 \le t \le T).$$

Let us define

$$\tilde{\phi}(v) := K_1 + K_2\phi(v) \qquad (v > 0).$$

Then

$$\frac{v}{\tilde{\phi}(v)} \ge \frac{1}{2K_2}\frac{v}{\phi(v)},$$

provided that $\phi(v) \ge K_1/K_2$. The Nagumo–Hartman Lemma [7, Lemma XII. 5.1] and condition (iii) of the theorem imply the existence of the desired K. □

Now we can define the basic set Ω and the homotopy mapping M_λ for the Leray–Schauder degree. Let K be the constant associated with R, S by Lemma 2.2 and consider the set

$$\Omega := \Omega_{R,S,K} := \left\{\psi \in C^1_T : -R < \psi(t) < S, |\psi'(t)| < K \quad (t \in [0,T])\right\}. \tag{2.4}$$

This set is open and bounded in C^1_T.

To define the family of mappings $M_\lambda : \overline{\Omega} \to C^1_T$ ($\lambda \in [0,1]$) we need further notation. The mean value operator $P : C_T \to C_T$ is defined by

$$(P\varphi)(t) := \frac{1}{T}\int_0^T \varphi(t)\,\mathrm{d}t \qquad (\varphi \in C_T).$$

Introduce the subspace $C_{T,I-P} := \{\varphi \in C_T : P\varphi = 0\}$ and the operator of the primitivation $H : C_{T,I-P} \to C_{T,I-P} \cap C^1_T$ by

$$(H\varphi)(t) := \int_0^t \varphi(s)\,\mathrm{d}s - \frac{1}{T}\int_0^T \left(\int_0^t \varphi(s)\,\mathrm{d}s\right)\mathrm{d}t.$$

It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(H\left(I-P\right)\varphi\right)(t) = \varphi(t) - P\varphi \qquad (\varphi \in C_T). \tag{2.5}$$

Now for $\lambda \in [0,1]$ we define the mapping:

$$M_\lambda : \overline{\Omega} \to C^1_T, \qquad M_\lambda\psi := M^*(\psi,\lambda), \tag{2.6}$$

where

$$M^* : C^1_T \times [0,1] \to C^1_T, \qquad M^*(\psi,\lambda) := P\psi - PF\psi + \lambda H^2(I-P)F\psi. \tag{2.7}$$

Property (a) is a consequence of the following lemma.

Lemma 2.3. *For $\lambda \in (0,1]$ a function $\psi \in C^1_T$ is a fixed point of M_λ, i.e., $\psi = M_\lambda\psi$ if and only if ψ is a T-periodic solution of* (2.3).

Function $\psi \in C^1_T$ is a fixed point of M_0 if and only if

$$\psi = P\psi \quad \textit{and} \quad PFP\psi = 0. \tag{2.8}$$

Proof. Suppose that $\lambda \in (0,1]$ is fixed, and $\psi \in C^1_T$ is a fixed point of M_λ:

$$\psi = P\psi - PF\psi + \lambda H^2(I-P)F\psi. \tag{2.9}$$

Applying functional P to both sides we get $PF\psi = 0$. By (2.9) ψ is two times differentiable and we obtain $\psi''(t) = \lambda f(t,\psi)$ $(t \in \mathbb{R})$, which means that ψ is a solution of (2.3).

On the other hand, if ψ is a T-periodic solution of (2.3) then

$$P\psi'' = \frac{1}{T}\int_0^T \psi''(t)\,\mathrm{d}t = \frac{1}{T}\big(\psi'(T) - \psi'(0)\big) = 0,$$

consequently $PF\psi = 0$, and we can write

$$\psi''(t) = \lambda\{f(t,\psi) - PF\psi\}.$$

Integrating this equality we obtain

$$\psi'(t) = \psi'(0) + \lambda\int_0^t (f(s,\psi) - PF\psi)\,\mathrm{d}s,$$

which, together with the definition of H, gives

$$\psi' = \psi'(0) + \frac{\lambda}{T}\int_0^T \left(\int_0^t (f(s,\psi) - PF\psi)\,\mathrm{d}s\right)\mathrm{d}t + \lambda H(I-P)F\psi.$$

Apply functional P to both sides of this equality. Since $P\psi' = 0$ we have

$$\psi'(0) + \frac{\lambda}{T}\int_0^T \left(\int_0^t (f(s,\psi) - PF\psi)\,\mathrm{d}s\right)\mathrm{d}t = 0,$$

therefore $\psi' = \lambda H(I-P)F\psi$. Integration yields

$$\psi = \text{const.} + \lambda H^2(I-P)F\psi.$$

From the definition of H there follows const. $= P\psi$, which, together with $PF\psi = 0$, shows that ψ is a fixed point of M_λ, i.e., (2.9) holds.

Now we turn to the proof of the second statement of the lemma concerning the case $\lambda = 0$. Suppose that $\psi \in C^1_T$ is a fixed point of $M_0 = P - PF$, i.e.,

$$\psi = P\psi - PF\psi. \tag{2.10}$$

Obviously, $\psi = P\psi$ and, consequently, (2.8) holds.

On the other hand, if (2.8) holds, then

$$\psi = P\psi = P\psi + PFP\psi = P\psi + PF\psi = M_0\psi.$$

In other words, ψ is a fixed point of M_0. □

Step (b) is contained in the following lemma.

Lemma 2.4. *Under the conditions of Theorem 2.1 function M^* is completely continuous on the set $\overline{\Omega} \times [0,1]$, provided that the norm $|||\cdot|||$ in $\overline{\Omega} \times [0,1]$ is defined by*

$$|||(\psi,\lambda)||| := \|\psi\|_1 + |\lambda| \qquad ((\psi,\lambda) \in \overline{\Omega} \times [0,1]).$$

Proof. The continuity of M^* follows from the conditions on a, b, d, e. In fact, to this property it is enough to prove the continuity of $F : C_T^1 \to C_T$. Obviously, $A, B, D : C_T^1 \to C_T$ are continuous. For $D_e : C_T^1 \to C_T$, let us fix a $\overline{\psi} \in C_T^1$ and consider the sets

$$Q := \left\{\psi \in C_T^1 : \|\psi - \overline{\psi}\|_1 \le 1\right\} \subset C_T^1,$$
$$Q_1 := \left\{v \in \mathbb{R} : \min_{[0,T]} \overline{\psi}'(t) - 1 \le v \le \max_{[0,T]} \overline{\psi}'(t) + 1\right\} \subset \mathbb{R}.$$

There are constants K_0, K_1 such that

$$|d(t,\psi)| \le K_0 \quad \text{if } \|\psi - \overline{\psi}\|_1 \le 1,\ 0 \le t \le T,$$
$$|e(\overline{\psi}'(t))| \le K_1 \quad \text{if } 0 \le t \le T.$$

Let $\varepsilon > 0$ be arbitrary. Function e is uniformly continuous on Q_1, and D is continuous at $\overline{\psi}$. Therefore there is a δ $(0 < \delta < 1)$ such that$\|\psi - \overline{\psi}\|_1 < \delta$ and $v_1, v_2 \in Q_1$, $|v_1 - v_2| < \delta$ imply

$$\|D\psi - D\overline{\psi}\|_0 < \frac{\varepsilon}{2K_1}, \qquad |e(v_1) - e(v_2)| < \frac{\varepsilon}{2K_0}.$$

If $\|\psi - \overline{\psi}\|_1 < \delta$, then

$$\begin{aligned}
&|d(t,\psi)e(\psi'(t)) - d(t,\overline{\psi})e(\overline{\psi}'(t))| \\
&\quad \le |d(t,\psi)||e(\psi'(t)) - e(\overline{\psi}'(t))| + |d(t,\psi) - d(t,\overline{\psi})||e(\overline{\psi}'(t))| \\
&\quad \le K_0\frac{\varepsilon}{2K_0} + K_1\frac{\varepsilon}{2K_1} = \varepsilon,
\end{aligned}$$

i.e., D_e is continuous.

Finally, we prove that M^* maps $\overline{\Omega} \times [0,1]$ into a precompact set in C^1. It is easy to see that $\|H\varphi\|_1 \le (2T+1)\|\varphi\|_0$ $(\varphi \in C_{T,I-P})$. Continuity of a, e and condition (ii) in Theorem 2.1 imply the existence of K_2, K_3 such that

$$|||(\psi,\lambda)||| \le K_2, \qquad \|F\psi\|_0 \le K_3 \qquad ((\psi,\lambda) \in \overline{\Omega} \times [0,1]).$$

Therefore

$$\begin{aligned}
\|M^*(\psi,\lambda)\|_0 &\le \|\psi\|_0 + \|F\psi\|_0 + 2(2T+1)^2\|F\psi\|_0 \\
&\le K_2 + (1 + 2(2T+1)^2)K_3 \qquad ((\psi,\lambda) \in \overline{\Omega} \times [0,1]).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|M^*(\psi,\lambda)'\|_0 &\le \|\lambda H(I-P)F\psi\|_0 \\
&\le 2(2T+1)\|F\psi\|_0 \le 2(2T+1)K_3, \\
\|M^*(\psi,\lambda)''\|_0 &\le \|\lambda(F\psi - PF\psi)\|_0 \\
&\le 2\|F\psi\|_0 \le K_3 \qquad ((\psi,\lambda) \in \overline{\Omega} \times [0,1]),
\end{aligned}$$

consequently the elements of $M^*(\overline{\Omega} \times [0,1]) \subset C_T^1$ are uniformly bounded and equicontinuous. By the Arzelà–Ascoli Theorem [7, Selection Theorem I.2.3] $M^*(\overline{\Omega} \times [0,1])$ is precompact. □

In general, step (c) is the biggest challenge in proofs of Leray–Schauder type; it depends most strongly on the specialities of the differential equation.

Lemma 2.5. *Under the conditions of Theorem 2.1, if $\psi \in \overline{\Omega}$ is a fixed point of M_λ for some $\lambda \in [0,1]$, then $\psi \notin \partial\Omega$.*

Proof. Suppose that the statement is not true, i.e., $\psi \in \partial\Omega$. If $\lambda \in (0,1]$, then by Lemma 2.3 ψ is a solution of (2.3). According to Lemma 2.2 there exists at least one $\tau \in [0,T)$ such that the function $t \mapsto r(t) := \psi^2(t)$ $(t \in \mathbb{R})$ has a total maximum at $t = \tau$, therefore $r'(\tau) = \psi'(\tau) = 0$, $r''(\tau) \le 0$, and either $\psi(\tau) = S$ or $\psi(\tau) = -R$. Condition (i) implies that either

$$\begin{aligned} r''(\tau) &= 2\psi(\tau)\psi''(\tau) = 2\lambda\psi(\tau)\{a(\tau,\psi(\tau)) + b(\tau,\psi)\} \\ &\ge 2\lambda|\psi(\tau)|\{a(\tau,\psi(\tau))\operatorname{sign}(\psi(\tau)) - \beta_{-R,S}(\tau)\} \\ &= 2\lambda S\{a(\tau,S) - \beta_{-R,S}(\tau)\} > 0, \end{aligned} \tag{2.11}$$

or

$$r''(\tau) \ge 2\lambda R\{-a(\tau,-R) - \beta_{-R,S}(\tau)\} > 0. \tag{2.12}$$

Both of them contradict $r''(\tau) \le 0$.

If $\lambda = 0$, then from (2.8) we know that $\psi(t) \equiv \psi_0 = \text{const.}$ and

$$m(\psi_0) := \frac{1}{T}\int_0^T (a(t,\psi_0) + b(t,\psi_0))\,\mathrm{d}t = 0. \tag{2.13}$$

On the other hand, we also know that either $\psi_0 = S$ or $\psi_0 = -R$. In the first case from condition (i) we get

$$|a(t,\psi_0) + b(t,\psi_0)| > a(t,S) - \beta_{-R,S}(t) > 0 \qquad (t \in \mathbb{R}), \tag{2.14}$$

which contradicts (2.13). The second case is similar. □

Lemma 2.6. *Under conditions of Theorem 2.1,*

$$d[I - M_0, \Omega, 0] = d[m, (-R,S), 0], \tag{2.15}$$

and the Brower degree on the right-hand side is equal to 1.

Proof. (2.15) is a consequence of (2.8). By virtue of condition (i) we have

$$\begin{aligned} m(-R) &= \frac{1}{T}\int_0^T (a(t,-R) + b(t,-R))\,\mathrm{d}t \\ &< \frac{1}{T}\int_0^T (a(t,-R) + \beta_{-R,S}(t))\,\mathrm{d}t < 0, \end{aligned}$$

$$\begin{aligned} m(S) &= \frac{1}{T}\int_0^T (a(t,S) + b(t,S))\,\mathrm{d}t \\ &< \frac{1}{T}\int_0^T (a(t,S) - \beta_{-R,S}(t))\,\mathrm{d}t > 0. \end{aligned}$$

But $d[m,(-R,S),0]$ depends only on $m(-R)$ and $m(S)$, and for the linear function connecting $m(-R)$ and $m(S)$ the degree is equal to 1, so $d[m,(-R,S),0] = 1$. □

Lemmas 2.3–2.4–2.5 make it possible to apply the theorem of invariance of the Leray–Schauder degree with respect to homotopy to the mapping M^* defined by (2.7), consequently

$$d[I - M_1, \Omega, 0] = d[m, (-R,S), 0] = 1.$$

On the basis of the Kronecker Existence Theorem [13] and Lemma 2.3 this means that (2.1) has a T-periodic solution $x \in C_T(-R,S)$. □

3 Applications

3.1 The forced mathematical pendulum with vibrating suspension point

The mathematical pendulum is one of the most important model equations in the nonlinear mechanics (see, e.g., [2]). When it is under the action of an outer periodic force then its motions are described by the equation

$$\varphi'' + \frac{g}{l}\sin\varphi = q(t) \tag{3.1}$$

where φ denotes the angle between the direction vertically downward and the rod of the pendulum measured anticlockwise, l is the length of the rod, g denotes the constant of gravity, and $q : \mathbb{R} \to \mathbb{R}$ is a T-periodic continuous function. A great number of papers have been devoted to the problem of finding T-periodic solutions of the equation (see an excellent history and literature in [11]). H. W. Knobloch [8], using the degree theory and taking also some damping, proved that the equation

$$\varphi'' + |\varphi'|\varphi' + \frac{g}{l}\sin\varphi = q(t) \tag{3.2}$$

has at least one T-periodic solution, provided that

$$\|q\|_\infty := \max_{[0,T]} |q(t)| < \frac{g}{l}. \tag{3.3}$$

Using the same technique, J. Mawhin and M. Willem [12] could guarantee multiple periodic solutions.

In the technical practice it often happens that the suspension point of the rod is vibrating in the plane of the motions of the pendulum. Consider now the case of the vibration

$$x_0(t) = Ue_1\cos\omega t, \qquad y_0(t) = Ue_2\sin\omega t \qquad (t \in \mathbb{R}),$$

where the x-axis is directed vertically downward, $U > 0$ is the amplitude, $\omega := m\pi/T$ is the frequency of the vibration; $m \in \mathbb{N}$ and the unit vector $(e_1, e_2) \in \mathbb{R}^2$ are fixed. It can be seen that Lagrange's equation of motion of the second kind has the form

$$\begin{aligned} \varphi'' - \frac{U}{l}\omega\sin\omega t(e_1\cos\varphi + e_2\sin\varphi)\varphi' \\ + \left(\frac{g}{l} + \frac{U}{l}\omega^2 e_1\cos\omega t\right)\sin\varphi - \frac{U}{l}\omega^2 e_2\cos\omega t\cos\varphi \\ = b_1(t,\varphi) - d(t,\varphi)e(\varphi'). \end{aligned} \tag{3.4}$$

Here the force function $b_1 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous, the function $b_1(\cdot, u)$ is T-periodic, $d : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $e : \mathbb{R} \to \mathbb{R}$ are continuous, $d(\cdot,\varphi)$ is T-periodic, and $e(0) = 0$. Introduce the notation

$$V := \max\left\{|b_1(t,u)| : 0 \le t \le T, \frac{\pi}{2} \le u \le \frac{3\pi}{2}\right\}.$$

Corollary 3.1. *Suppose that there exists a continuous function $\phi : (0,\infty) \to (0,\infty)$ ($\phi(r) \ge r$) such that the condition (iii) in Theorem 2.1 is satisfied. If*

$$U\omega^2 + Vl < g, \tag{3.5}$$

then equation (3.4) has a T-periodic solution φ such that $\pi/2 \le \varphi(t) \le 3\pi/2$ ($t \in \mathbb{R}$).

Proof. In the new variable $\theta := \varphi - \pi$ equation (3.4) has the form

$$\begin{aligned}\theta'' = & -\frac{U}{l}\omega \sin \omega t(e_1 \cos\theta + e_2 \sin\theta)\theta' + \left(\frac{g}{l} + \frac{U}{l}\omega^2 e_1 \cos\omega t\right)\sin\theta \\ & -\frac{U}{l}\omega^2 e_2 \cos\omega t \cos\theta + b_1(t, \theta+\pi) - d(t,\theta+\pi)e(\theta').\end{aligned} \tag{3.6}$$

There are constants c_1, c_2 such that

$$\left|\frac{U}{l}\omega \sin\omega t(e_1\cos\theta + e_2\sin\theta)\theta'\right| + |d(t,\theta+\pi)e(\theta')| \le c_1|\theta'| + c_2\phi(|\theta'|) \le (c_1+c_2)\phi(|\theta'|),$$

so condition (iii) in Theorem 2.1 is satisfied. We can choose $a(t,u) := (g/l)\sin u$, $R := \pi/2$, $S := 3\pi/2$. Then $\beta_{-R,S}(t) \equiv V$ and we apply Theorem 2.1 to equation (3.6) to get the corollary. □

Condition (3.5) can be considered as a generalization of (3.3) to (3.6). In Knobloch's special case (3.5) gives (3.3).

3.2 A second order integro-differential equation with unbounded delay

Consider the equation

$$x''(t) = a(t, x(t)) + \int_{-\infty}^{\infty} k(t,s)x(s)\,\mathrm{d}s + d_1(t, x_t)e(x'(t)) + p(t), \quad (t \in \mathbb{R}) \tag{3.7}$$

where $k : \mathbb{R}^2 \to \mathbb{R}$ is continuous, $k(t+T, s+T) \equiv k(t,s)$ $(t, s \in \mathbb{R})$, $d_1 : \mathbb{R} \times C((-\infty, 0]; \mathbb{R}) \to \mathbb{R}$ is continuous, $d_1(t+T, \chi) \equiv d_1(t,\chi)$ $(\chi \in C((-\infty,0];\mathbb{R}))$, $p : \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic. We used the standard notation $x_t(\tau) := x(t+\tau)$ $(t \in \mathbb{R}, \tau \le 0)$.

Equation (3.7) can be considered as a perturbation of the pendulum equation (3.1). As we will see in the following corollary, function sin will be replaced by a function a satisfying a sign condition like the sine function and dominating the other terms in the equation. By example of (3.7) we would like to illuminate that our main result Theorem 2.1 is robust in the sense that it makes possible a variety of applications where different types of equations appear such as functional differential equations even with unbounded delays. Actually, such equations can occur among others in mechanics (see, e.g., [1, 4.3. Examples]) and population dynamics [3].

The following corollary is a direct consequence of Theorem 2.1.

Corollary 3.2. *Suppose that there exists a continuous function $\phi : (0,\infty) \to (0,\infty)$ such that the condition (iii) in Theorem 2.1 is satisfied. If there are positive constants R, S such that*

$$\begin{aligned} a(t,S) &> \max\{R,S\}\int_{-\infty}^{\infty}|k(t,s)|\,\mathrm{d}s + \|p\|_0 =: \beta_{-R,S}(t), \\ a(t,-R) &< -\beta_{-R,S}(t) \qquad (t \in \mathbb{R}), \end{aligned} \tag{3.8}$$

and d_1 transforms every bounded set contained in $\mathbb{R} \times C((0,\infty];\mathbb{R})$ into a bounded set of $\mathbb{R}$, then there exists a T-periodic solution $x \in C_T(-R,S)$ of (3.7).

Acknowledgements

Supported by the National Reserch, Development and Innovation Fund, NKFIH-K-129322.

*

The author is very grateful to the referee for the valuable comments and remarks.

References

[1] T. A. Burton, *Volterra integral and differential equations*, Mathematics in Science and Engineering, Vol. 167, Academic Press, New York, 1983. MR2155102

[2] C. Chicone, *Ordinary differential equations with applications*, Texts in Applied Mathematics, Vol. 34, Springer-Verlag, New York, 1999. https://doi.org/10.1007/0-387-35794-7; MR1707333

[3] J. M. Cushing, *Integrodifferential equations and delay models in population dynamics*, Lecture Notes in Biomathematics, Vol. 20, Springer-Verlag, Berlin, 1977. https://doi.org/10.1007/978-3-642-93073-7; MR0496838

[4] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985. https://doi.org/10.1007/978-3-662-00547-7; MR0787404

[5] J. Cronin, *Fixed points and topological degree in nonlinear analysis*, Mathematical Surveys, Vol. 11, American Mathematical Society, Providence, R.I., 1964. MR0164101

[6] R. Gaines, J. Mawhin, *Coincidence degree and nonlinear differential equations*, Lecture Notes in Mathematics, Vol. 568, Springer-Verlag, Berlin-New York, 1977. MR0637067

[7] P. Hartman, *Ordinary differential equations*, Birkhäuser, Boston–Basel–Stuttgart, 1982. MR0658490

[8] H. W. Knobloch, Eine neue Methode zur Approximation periodischer Lösungen nichtlinearer Differenzialgleichungen zweiter Ordnung, *Math. Z.* **82**(1963), 177–197. https://doi.org/10.1007/BF01111422; MR0158124

[9] J. Mawhin, Periodic solutions of nonlinear functional differential equations, *J. Differential Equations* **10**(1971), 240–261. https://doi.org/10.1016/0022-0396(71)90049-0; MR0294823

[10] J. Mawhin, Topological degree methods in nonlinear boundary value problems, in: *Expository lectures from the CBMS Regional Conference held at Harvey Mudd College, Claremont, Calif., June 9–15, 1977*, CBMS Regional Conference Series in Mathematics, Vol. 40, American Mathematical Society, Providence, R.I., 1979. MR0525202

[11] J. Mawhin, Periodic oscillations of forced pendulum-like equations, in: *Ordinary and Partial Differential Equations (Dundee, 1982)*, Lecture Notes in Math., Vol. 964, Springer, Berlin-New York, 1982, 458–476. https://doi.org/10.1007/BFb0065017; MR0693131

[12] J. Mawhin, M. Willem, Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations, *J. Differential Equations* **52**(1984), 264–287. https://doi.org/10.1016/0022-0396(84)90180-3; MR0741271

[13] N. Rouche, J. Mawhin, *Ordinary differential equations. Stability and periodic solutions*, Surveys and Reference Works in Mathematics, Vol. 5, Pitman (Advanced Publishing Program), Boston, Mass.–London, 1980. MR0615082