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Nonlocal boundary value problems with BV-type data

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Only a fool can celebrate the years of upcoming death (George Bernard Shaw)

However, for you we make an exception: *Happy birthday, dear Jeff!*

Abstract. In this paper we present some existence and uniqueness results for solutions of second order boundary value problems, which are functions of bounded variation along with their derivatives. To this end, we apply fixed point theorems to an equivalent nonlinear perturbed Hammerstein integral equation. Here we consider non- standard boundary conditions like coupled boundary conditions, uncoupled boundary conditions, or integral-type boundary conditions. We also prove an abstract result concerning the spectral radii of some general classes of operators which applies to all boundary value problems mentioned above. The abstract results are throughout illustrated by a large number of examples.

Keywords: boundary value problem, bounded variation, spectral radius.

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1 Nonlocal boundary value problems

It is well known that nonlinear boundary value problems (BVPs) are closely related to Hammerstein integral equations, while nonlinear initial value problems (IVPs) are closely related to Volterra–Hammerstein integral equations. Since a linear Volterra operator has often spectral radius zero, solutions of IVPs are usually much easier to obtain than solutions of BVPs.

During the last decades, so-called nonlocal BVPs have found growing attention, mainly in view of their generality and applicability. In a very general formulation, a second-order

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nonlinear equation with nonlocal boundary conditions has the form [7]

$$\begin{aligned} &x''(t) + p(t)x'(t) + q(t)x(t) + r(t)g(t, x(t)) = 0 \qquad (0 \le t \le 1), \\ &ax(0) - bx'(0) = \alpha[x], \qquad cx(1) + dx'(1) = \beta[x]. \end{aligned} \tag{1.1}$$

Here $p, q, r : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are given functions, and $\alpha, \beta : C[0, 1] \to \mathbb{R}$ are linear functionals which are expressed by Riemann–Stieltjes integrals. The well-known multi-point BVPs are a special case of the problem (1.1).

Many important contributions to this problem have been given during the last 20 years by Webb [5–16], and Webb with Infante [2–4, 17–22]. While there is a vast literature on continuously differentiable solutions, considerably less is known on solutions with derivatives of bounded variation, although such solutions (e.g., monotone or convex solutions) have some interest in applications. An exception is the recent paper [1], where the authors prove, under suitable hypotheses, the existence of a continuous solution of bounded variation of the equation

$$x(t) = \alpha[x]v(t) + \beta[x]w(t) + \lambda \int_0^1 k(t, s)g(s, x(s))\, ds \qquad (0 \le t \le 1), \tag{1.2}$$

building on a variant of Krasnosel'skij's fixed point principle. We will study a similar equation and look for solutions with derivatives of bounded variation.

So in this paper we are going to consider the classical space BV equipped with the usual norm

$$\|x\|_{BV} = |x(0)| + \mathrm{Var}(x; [0, 1]), \tag{1.3}$$

where $\mathrm{Var}(x; [0, 1])$ denotes the total Jordan variation of x on the interval $[0, 1]$, as well as the higher order space

$$BV^m := \{x \in BV : x', x'', \ldots, x^{(m)} \in BV\},$$

equipped with the natural norm

$$\|x\|_{BV^m} = |x(0)| + \sum_{k=1}^{m} \|x^{(k)}\|_{BV}.$$

Observe that there is a peculiarity in the spaces BV^m for $m \ge 1$. Given $x \in BV^m$, the derivative $x^{(m)}$ belongs to BV and so can have only removable discontinuities or jumps; however, the well-known Darboux intermediate value theorem excludes such discontinuities. So the inclusion $BV^m \subseteq C^m$ holds, although the analogous "zero level" inclusion $BV \subseteq C$ is of course far from being true.

We will also need the space AC^m of all functions which have absolutely continuous derivatives up to order m, equipped with the norm inherited from BV^m. By the classical Vitali–Banach–Zaretskij theorem, the relation with the other spaces is then given by

$$AC^m \subset BV^m \subset C^m \ (m \ge 1), \qquad AC \subset BV \cap C \subset C, \tag{1.4}$$

where all inclusions are strict. In what follows, we will look for solutions $x \in AC^{m-1}$ of an m-th order nonlinear differential equation with nonlocal boundary conditions, with a particular emphasis on examples which illustrate how far our sufficient solvability conditions are from being necessary. If there are more than one sufficient condition we will also show their independence, in the sense that none of them implies the others.

2 Boundary value problems with BV data

To begin with, let us discuss the second order equation

$$x''(t) + \lambda g(t, x(t)) = 0 \qquad (0 \leq t \leq 1), \tag{2.1}$$

subject to the coupled boundary conditions

$$x(0) = \alpha[x], \qquad x(1) = \beta[x], \tag{2.2}$$

where $\alpha, \beta : BV \to \mathbb{R}$ are given linear functionals. This means that we take $p(t) = q(t) \equiv 0$, $r(t) \equiv 1$, $a = c = 1$, and $b = d = 0$ in (1.1). Occasionally, we will also consider more general data. Throughout this paper we suppose that the nonlinearity g in (2.1) satisfies the three hypotheses

(H1) $g(\cdot, u)$ is measurable for all $u \in \mathbb{R}$;

(H2) for each $R > 0$ there exists $a_R \in L_\infty[0,1]$ such that $|g(t,u)| \leq a_R(t)$ for $0 \leq t \leq 1$ and $|u| \leq R$;

(H3) $g(t, \cdot) \in C(\mathbb{R})$ for almost all $t \in [0,1]$.

In the sequel we refer to the problem (2.1)/(2.2) by the symbol (BVP). In order to solve this problem, we consider along with (BVP) the Hammerstein integral equation

$$x(t) = Ax(t) + \lambda \int_0^1 \kappa(t,s) g(s, x(s))\, ds \qquad (0 \leq t \leq 1), \tag{2.3}$$

where

$$\kappa(t,s) = \begin{cases} s(1-t) & \text{for } 0 \leq s \leq t \leq 1, \\ t(1-s) & \text{for } 0 \leq t < s \leq 1 \end{cases}$$

is the usual Green's function of the second order derivative, and A is a linear operator (to be specified below) from BV into itself. The bridge between (2.3) and our (BVP) is built by our first result

Proposition 2.1. *Let $A : BV \to BV$ be defined by*

$$Ax(t) := (1-t)\alpha[x] + t\beta[x] \qquad (0 \leq t \leq 1). \tag{2.4}$$

Then the following holds.

(a) Every function $x \in BV$ solving (2.3) belongs to AC^1 and solves (BVP) almost everywhere on $[0,1]$.

(b) If, in addition, g is continuous on $[0,1] \times \mathbb{R}$, then every solution x of (2.3) is of class C^2 and solves (BVP) everywhere on $[0,1]$.

(c) Conversely, if $x \in AC^1$ solves (BVP) almost everywhere on $[0,1]$, then x is a solution of the integral equation (2.3).

Proof. (a) Assume that (2.3) is satisfied for some $x \in BV$ and some $\lambda \in \mathbb{R}$. First observe that $h(s) := g(s, x(s))$ belongs to L_∞ because of our hypotheses (H1)/(H2)/(H3). Moreover, the function $\varphi : [0,1] \to \mathbb{R}$ defined by

$$\varphi(t) := \int_0^1 \kappa(t,s) g(s, x(s))\, ds = (1-t) \int_0^t s h(s)\, ds - t \int_1^t (1-s) h(s)\, ds$$

belongs to AC with

$$\varphi'(t) = -\int_0^t sh(s)\,ds - \int_1^t (1-s)h(s)\,ds$$

for almost all $t \in [0,1]$. But since the right hand side is again in AC we conclude that $\varphi \in AC^1$. Moreover,

$$\varphi''(t) = -th(t) - (1-t)h(t) = -h(t)$$

for almost all $t \in [0,1]$. In addition, by the definition (2.4) of A the function Ax is affine and hence of class C^2 with $(Ax)'' = 0$. From (2.3) it follows that

$$x(t) = Ax(t) + \lambda\varphi(t) \qquad (0 \le t \le 1);$$

in particular, this shows that (2.1) holds indeed almost everywhere in $[0,1]$. Moreover, since $\varphi(0) = \varphi(1) = 0$, $Ax(0) = \alpha[x]$, and $Ax(1) = \beta[x]$ the first part of the proof is complete.

(b) If, in addition, g is continuous, then so must be x'' which means that x is of class C^2 and solves (BVP) everywhere on $[0,1]$.

(c) Putting $h(t) = g(t, x(t))$ as before and integrating (2.1) twice over $[0,t]$ we obtain

$$x(t) = x(0) + x'(0)t - \lambda \int_0^t (t-s)h(s)\,ds.$$

Evaluating this at $t = 1$ yields

$$x'(0) = \beta[x] - \alpha[x] + \lambda \int_0^1 (1-s)h(s)\,ds$$

which gives the desired result. □

Observe that the problem discussed in Proposition 2.1 is of the form (1.2) with $v(t) = 1-t$ and $w(t) = t$. The inclusions (1.4) suggest that we cannot expect the solution of (2.3) to lie in BV^2, unless the function g is continuous.

Proposition 2.1 shows that the problem of solving (BVP) may be reduced to finding solutions $x \in BV$ of the Hammerstein equation (2.3). Of course, the structure of (2.3) suggests to use fixed point theorems, such as the Banach–Caccioppoli contraction mapping principle, the Schauder fixed point principle, or the Krasnosel'skij fixed point principle which is a combination of both. To this end, we have to make sure that the two functionals $\alpha, \beta \in BV^*$ behave in such a way that the norm $\|A^n\|_{BV\to BV}$ of the iterates A^n of the operator (2.4) shrinks below 1 for some $n \in \mathbb{N}$, and the integral operator in (2.3) is compact. Two conditions which fulfill the first requirement are given in the following

Theorem 2.2. *Assume that the two functionals $\alpha, \beta \in BV^*$ satisfy one of the conditions*

$$\|\alpha\|_{BV^*} + \|\alpha - \beta\|_{BV^*} < 1 \tag{2.5}$$

or

$$\alpha[e_0] = \beta[e_0] = 0, \qquad |\alpha[e_1] - \beta[e_1]| < 1, \tag{2.6}$$

where

$$e_k(t) := t^k \qquad (0 \le t \le 1,\ k = 0, 1, 2, \ldots). \tag{2.7}$$

Then for each $R > 0$ there is some $\rho > 0$ such that (BVP) has, for fixed $\lambda \in (-\rho, \rho)$, a solution $x \in AC^1$ satisfying $\|x\|_{BV} \le R$. If, in addition, g is continuous on $[0,1] \times \mathbb{R}$, then every such solution is of class C^2.

Proof. Define A as in Proposition 2.1, that is, $Ax = \alpha[x](e_0 - e_1) + \beta[x]e_1$. Since α and β are supposed to be bounded and linear, so is A. We show for either of the two options (2.5) and (2.6) that there is some $n \in \mathbb{N}$ such that $\|A^n\|_{BV\to BV} < 1$. Once this is done, standard solvability results for (2.3) give the claim. By Proposition 2.1, the solution x belongs to AC^1, has the correct boundary values according to (2.2), and satisfies (2.1) almost everywhere. If g is continuous, it follows easily from Proposition 2.1 (b) that x is then of class C^2.

So we claim that $\|A^n\|_{BV\to BV} < 1$ for some $n \in \mathbb{N}$ provided that α and β satisfy (2.5) or (2.6). Let us start with (2.5). For any $x \in BV$ we have

$$\begin{aligned}\|Ax\|_{BV} &= \|\alpha[x]e_0 + (\beta[x] - \alpha[x])e_1\|_{BV} \le \|e_0\|_{BV}\big|\alpha[x]\big| + \|e_1\|_{BV}\big|\beta[x] - \alpha[x]\big| \\ &\le \|\alpha\|_{BV^*}\|x\|_{BV} + \|\alpha - \beta\|_{BV^*}\|x\|_{BV},\end{aligned}$$

since $\|e_0\|_{BV} = \|e_1\|_{BV} = 1$. Consequently,

$$\|A\|_{BV\to BV} \le \|\alpha\|_{BV^*} + \|\alpha - \beta\|_{BV^*} < 1,$$

by (2.5), showing that A is a contraction. In this case, we may take $n = 1$.

We now assume that α and β satisfy option (2.6). Note that in this case, $Ae_0 = 0$. By induction, we first prove that the iterates of A are given by

$$A^{n+2}x = (\alpha[e_1](e_0 - e_1) + \beta[e_1]e_1)(\beta[e_1] - \alpha[e_1])^n(\beta[x] - \alpha[x]) \tag{2.8}$$

for $x \in BV$ and $n \in \mathbb{N}_0$, where we set $0^0 := 1$. First, using (2.6) we get

$$\begin{aligned}A(Ax) &= A(\alpha[x](e_0 - e_1) + \beta[x]e_1) = \alpha[x]A(e_0 - e_1) + \beta[x]Ae_1 \\ &= \alpha[x](\alpha[e_0](e_0 - e_1) + \beta[e_0]e_1) + (\beta[x] - \alpha[x])(\alpha[e_1](e_0 - e_1) + \beta[e_1]e_1) \\ &= (\alpha[e_1](e_0 - e_1) + \beta[e_1]e_1)(\beta[x] - \alpha[x]),\end{aligned}$$

and this is (2.8) for $n = 0$. Moreover,

$$\begin{aligned}\beta[Ax] - \alpha[Ax] &= (\beta - \alpha)[\alpha[x](e_0 - e_1) + \beta[x]e_1] \\ &= (\beta - \alpha)[e_0 - e_1]\alpha[x] + (\beta - \alpha)[e_1]\beta[x] = (\beta[e_1] - \alpha[e_1])(\beta[x] - \alpha[x]).\end{aligned}$$

From this we deduce that if (2.8) has been proved for some $n \in \mathbb{N}_0$, then

$$\begin{aligned}A^{n+3}x = A^{n+2}(Ax) &= (\alpha[e_1](e_0 - e_1) + \beta[e_1]e_1)(\beta[e_1] - \alpha[e_1])^n(\beta[Ax] - \alpha[Ax]) \\ &= (\alpha[e_1](e_0 - e_1) + \beta[e_1]e_1)(\beta[e_1] - \alpha[e_1])^{n+1}(\beta[x] - \alpha[x]).\end{aligned}$$

By induction, (2.8) is established. As a consequence we get for $n \ge 2$

$$\|A^n\|_{BV\to BV} \le \|\alpha[e_1](e_0 - e_1) + \beta[e_1]e_1\|_{BV}\big|\beta[e_1] - \alpha[e_1]\big|^{n-2}(\|\alpha\|_{BV^*} + \|\beta\|_{BV^*}).$$

Since $\big|\beta[e_1] - \alpha[e_1]\big| < 1$, by (2.6), and this is the only term depending on n, we find some $n \in \mathbb{N}$ such that $\|A^n\|_{BV\to BV} < 1$ as claimed. □

To illustrate the applicability of Theorem 2.2, we give now two examples. In the first example condition (2.5) works, but (2.6) does not, while in the second example condition (2.6) works, but (2.5) does not. Recall that we impose throughout the hypotheses (H1)/(H2)/(H3) on g.

Example 2.3. Consider the BVP

$$\begin{cases} x''(t) + \lambda g(t, x(t)) = 0 \quad (0 \le t \le 1), \\ x(0) = \frac{1}{7} x\left(\frac{1}{2}\right) + \frac{1}{6} x\left(\frac{2}{3}\right), \\ x(1) = \frac{1}{7} x\left(\frac{1}{4}\right) + \frac{1}{6} x\left(\frac{4}{5}\right). \end{cases} \tag{2.9}$$

The functionals

$$\alpha[x] := \frac{1}{7} x\left(\frac{1}{2}\right) + \frac{1}{6} x\left(\frac{2}{3}\right), \qquad \beta[x] := \frac{1}{7} x\left(\frac{1}{4}\right) + \frac{1}{6} x\left(\frac{4}{5}\right)$$

are obviously linear and bounded on BV and satisfy

$$\left|\alpha[x]\right| \le \frac{1}{7}\|x\|_\infty + \frac{1}{6}\|x\|_\infty \le \frac{13}{42}\|x\|_{BV}$$

where $\|\cdot\|_\infty$ denotes the supremum norm, and

$$\begin{aligned} \left|\alpha[x] - \beta[x]\right| &= \left|\frac{1}{7}\left[x\left(\frac{1}{2}\right) - x\left(\frac{1}{4}\right)\right] + \frac{1}{6}\left[x\left(\frac{2}{3}\right) - x\left(\frac{4}{5}\right)\right]\right| \\ &\le \frac{1}{7}\operatorname{Var}(x; [0,1]) + \frac{1}{6}\operatorname{Var}(x; [0,1]) \le \frac{13}{42}\|x\|_{BV}. \end{aligned}$$

Consequently,

$$\|\alpha\|_{BV^*} + \|\alpha - \beta\|_{BV^*} \le \frac{13}{21} < 1,$$

which means that α and β satisfy option (2.5) of Theorem 2.2. We conclude that (2.9) has for small $|\lambda|$ an AC^1-solution. On the other hand, α and β do not satisfy option (2.6), as $\alpha[e_0] = \beta[e_0] = 13/42 \ne 0$.

Example 2.4. Consider the BVP

$$\begin{cases} x''(t) + \lambda g(t, x(t)) = 0 \quad (0 \le t \le 1), \\ x(0) = 3x\left(\frac{1}{2}\right) - 3x\left(\frac{2}{3}\right), \\ x(1) = 2x\left(\frac{1}{4}\right) - 2x\left(\frac{4}{5}\right). \end{cases} \tag{2.10}$$

The functionals

$$\alpha[x] := 3x\left(\frac{1}{2}\right) - 3x\left(\frac{2}{3}\right), \qquad \beta[x] := 2x\left(\frac{1}{4}\right) - 2x\left(\frac{4}{5}\right)$$

are obviously linear and bounded on BV and satisfy

$$\alpha[e_0] = \beta[e_0] = 0, \qquad \left|\alpha[e_1] - \beta[e_1]\right| = \left|\frac{3}{2} - 2 - \frac{1}{2} + \frac{8}{5}\right| = \frac{3}{5} < 1,$$

which means that α and β satisfy option (2.6) of Theorem 2.2. We conclude that (2.10) has for small $|\lambda|$ an AC^1-solution. On the other hand, α and β do not satisfy option (2.5), because

$$\|\chi_{[0,1/2]}\|_{BV} = 1 + 1 = 2, \qquad \alpha[\chi_{[0,1/2]}] = 3,$$

and so $\|\alpha\|_{BV^*} \ge 3/2 > 1$.

3 A refinement of Theorem 2.2

The preceding two examples show that the crucial conditions (2.5) and (2.6) in Theorem 2.2 are independent. As one could expect, there exist BVPs where neither (2.5) nor (2.6) can be used. Here is a simple example.

Example 3.1. Consider the BVP

$$\begin{cases} x''(t) + \lambda g(t, x(t)) = 0 \quad (0 \leq t \leq 1), \\ x(0) = x\left(\frac{1}{3}\right) + x\left(\frac{2}{3}\right), \\ x(1) = -\frac{1}{2}x\left(\frac{1}{3}\right) - \frac{1}{2}x\left(\frac{2}{3}\right). \end{cases} \tag{3.1}$$

The functionals

$$\alpha[x] := x\left(\tfrac{1}{3}\right) + x\left(\tfrac{2}{3}\right), \qquad \beta[x] := -\tfrac{1}{2}x\left(\tfrac{1}{3}\right) - \tfrac{1}{2}x\left(\tfrac{2}{3}\right)$$

are obviously linear and bounded on BV. However, $\alpha[e_0] = 2 \neq 0$, so option (2.6) cannot be used. The same relation shows that $\|\alpha\|_{BV^*} \geq 2$, and so option (2.5) cannot be used either.

In view of Example 3.1 the question arises how to generalize the ideas of Theorem 2.2 in order to cover a larger range of applications. Due to the special structure of the linear operator (2.4) it is possible to give an exact formula for its spectral radius. For this purpose we prove now an abstract result about the spectral radius of an even slightly more general class of operators which might be of interest on its own.

Proposition 3.2. *Let $(X, \|\cdot\|)$ be a Banach space, $v, w \in X$ fixed, and $\alpha, \beta \in X^*$. Define $A : X \to X$ by*

$$Ax := \alpha[x]v + \beta[x]w \qquad (x \in X). \tag{3.2}$$

Then the matrix

$$\mathcal{A} := \begin{pmatrix} \alpha[v] & \beta[v] \\ \alpha[w] & \beta[w] \end{pmatrix} \in \mathbb{R}^{2\times 2} \tag{3.3}$$

and the operator A have the same spectral radius.

Proof. We first show that $\mathfrak{R}(A) \leq \mathfrak{R}(\mathcal{A})$, where $\mathfrak{R}$ denotes the spectral radius, by means of the classical Gel'fand formula. The iterates of A can be written in the form

$$A^n x = \alpha_n[x]v + \beta_n[x]w \quad (x \in X),$$

where $\alpha_n, \beta_n \in X^*$ satisfy for all $x \in X$ the linear recursions

$$\alpha_1[x] := \alpha[x], \qquad \alpha_{n+1}[x] = \alpha_n[v]\alpha[x] + \alpha_n[w]\beta[x] \tag{3.4}$$

and

$$\beta_1[x] := \beta[x], \qquad \beta_{n+1}[x] = \beta_n[v]\alpha[x] + \beta_n[w]\beta[x]. \tag{3.5}$$

Indeed, once the formula for A^n has been established, we get

$$\begin{aligned} A^{n+1}x = A^n(Ax) &= \alpha_n[Ax]v + \beta_n[Ax]w \\ &= (\alpha_n[v]\alpha[x] + \alpha_n[w]\beta[x])v + (\beta_n[v]\alpha[x] + \beta_n[w]\beta[x])w \\ &= \alpha_{n+1}[x]v + \beta_{n+1}[x]w. \end{aligned}$$

Plugging v and w for x into the recursion formulas (3.4) and (3.5) we see that the four numbers $\alpha_n[v]$, $\alpha_n[w]$, $\beta_n[v]$ and $\beta_n[w]$ in turn satisfy the matrix recursions $B_{n+1} = \mathcal{A}B_n$, where

$$B_k := \begin{pmatrix} \alpha_k[v] & \beta_k[v] \\ \alpha_k[w] & \beta_k[w] \end{pmatrix}.$$

Thus, $B_1 = \mathcal{A}$ and, more generally, $B_k = \mathcal{A}^k$. Setting

$$M := \max\,\{\|v\|\,\|\alpha\|_{X^*}, \|v\|\,\|\beta\|_{X^*}, \|w\|\,\|\alpha\|_{X^*}, \|w\|\,\|\beta\|_{X^*}\},$$

our recursion for A^{n+1} implies

$$\begin{aligned} \|A^{n+1}\|_{X\to X} &\le \|v\|\left(|\alpha_n[v]|\,\|\alpha\|_{X^*} + |\alpha_n[w]|\,\|\beta\|_{X^*}\right) + \|w\|\left(|\beta_n[v]|\,\|\alpha\|_{X^*} + |\beta_n[w]|\,\|\beta\|_{X^*}\right) \\ &\le M\left(|\alpha_n[v]| + |\alpha_n[w]| + |\beta_n[v]| + |\beta_n[w]|\right) \le 2M\|B_n\|_\infty = 2M\|\mathcal{A}^n\|_\infty, \end{aligned}$$

where $\|\cdot\|_\infty$ denotes the row sum norm of a matrix. Taking the n-th root in this estimate, Gel'fand's formula yields

$$\mathfrak{R}(A) = \lim_{n\to\infty} \|A^{n+1}\|_{X\to X}^{1/n} \le \lim_{n\to\infty} (2M\|\mathcal{A}^n\|_\infty)^{1/n} = \mathfrak{R}(\mathcal{A}).$$

We now prove the reverse estimate and distinguish the two cases when the set $\{v, w\}$ is linearly dependent or linearly independent in X.

<u>1st case</u>: Assume $w = \lambda v$ for some $\lambda \in \mathbb{R}$. In this case the matrix (3.3) reads

$$\mathcal{A} = \begin{pmatrix} \alpha[v] & \beta[v] \\ \lambda\alpha[v] & \lambda\beta[v] \end{pmatrix},$$

so $\mathfrak{R}(\mathcal{A}) = |\alpha[v] + \lambda\beta[v]|$. Moreover, the functional $\gamma := \alpha + \lambda\beta \in X^*$ satisfies

$$Ax = \gamma[x]v, \quad A^2x = \gamma[v]\gamma[x]v,$$
$$A^3x = \gamma[v]^2\gamma[x]v, \ \ldots, \ A^nx = \gamma[v]^{n-1}\gamma[x]v$$

for all $n \in \mathbb{N}$ and $x \in X$. In case $v = o$ we also have $w = o$, hence $\mathfrak{R}(A) = \mathfrak{R}(\mathcal{A}) = 0$. We therefore assume $v \ne o$. If $\gamma[x] = 0$ for all $x \in X$ we have $\alpha = -\lambda\beta$ which implies, on the one hand, $Ax = 0$, hence $\mathfrak{R}(A) = 0$, and

$$\mathcal{A} = \begin{pmatrix} -\lambda\beta[v] & \beta[v] \\ -\lambda^2\beta[v] & \lambda\beta[v] \end{pmatrix} = \beta[v]\begin{pmatrix} -\lambda & 1 \\ -\lambda^2 & \lambda \end{pmatrix}$$

hence $\mathfrak{R}(\mathcal{A}) = 0$, on the other. So suppose that there is some $y \in X$ with $\|y\| = 1$ and $\gamma[y] \ne 0$. Then from our recursion formula for the iterates of A we conclude that

$$\|A^n\|_{X\to X} \ge \|A^ny\| = |\gamma[v]|^{n-1}|\gamma[y]|\|v\|.$$

Consequently,

$$\mathfrak{R}(A) = \lim_{n\to\infty} \|A^n\|_{X\to X}^{1/n} \ge |\gamma[v]| = |\alpha[v] + \lambda\beta[v]| = \mathfrak{R}(\mathcal{A})$$

as claimed.

<u>2nd case</u>: Assume $w \neq \mu v$ for all $\mu \in \mathbb{R}$. We use the fact that the spectral radius of an operator $A : X \to X$ on a real space X coincides with the spectral radius of its complexification $A_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$. Recall that $X_{\mathbb{C}} := \{x + iy : x, y \in X\}$ is equipped with the norm

$$\|x + iy\|_{X_{\mathbb{C}}} := \max_{0 \le t \le 2\pi} \|(\cos t)x + (\sin t)y\|,$$

and $A_{\mathbb{C}}$ is defined by $A_{\mathbb{C}}(x + iy) := Ax + iAy$. Similarly, the functionals α and β are complexified by putting

$$\alpha_{\mathbb{C}}[x + iy] := \alpha[x] + i\alpha[y], \qquad \beta_{\mathbb{C}}[x + iy] := \beta[x] + i\beta[y].$$

Note that $\|A_{\mathbb{C}}\|_{X_{\mathbb{C}} \to X_{\mathbb{C}}} = \|A\|_{X \to X}$, $\|\alpha_{\mathbb{C}}\|_{X^*_{\mathbb{C}}} = \|\alpha\|_{X^*}$, and $\|\beta_{\mathbb{C}}\|_{X^*_{\mathbb{C}}} = \|\beta\|_{X^*}$. The relation (3.2) translates then into complexifications in the form

$$A_{\mathbb{C}} z = \alpha_{\mathbb{C}}[z]v + \beta_{\mathbb{C}}[z]w \qquad (z \in X_{\mathbb{C}}).$$

Let now $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}^T$ with eigenvector $u = (u_1, u_2) \in \mathbb{C}^2$. This means that $u^T \mathcal{A} = \lambda u^T$, i.e. in components,

$$\alpha[v]u_1 + \alpha[w]u_2 = \lambda u_1, \qquad \beta[v]u_1 + \beta[w]u_2 = \lambda u_2.$$

Since $\{v, w\}$ is linearly independent in X, by hypothesis, we find $x, y \in X$ such that

$$Ax = \mathrm{Re}(u_1)v + \mathrm{Re}(u_2)w, \qquad Ay = \mathrm{Im}(u_1)v + \mathrm{Im}(u_2)w.$$

The element $z := x + iy \in X_{\mathbb{C}}$ satisfies then

$$A_{\mathbb{C}} z = A_{\mathbb{C}}(x + iy) = Ax + iAy = vu_1 + wu_2.$$

But from $u = (u_1, u_2) \neq (0, 0)$ we conclude that $A_{\mathbb{C}} z \neq o$, hence

$$\begin{aligned} A_{\mathbb{C}}(A_{\mathbb{C}} z) &= \alpha_{\mathbb{C}}[A_{\mathbb{C}} z]v + \beta_{\mathbb{C}}[A_{\mathbb{C}} z]w = (u_1\alpha[v] + u_2\alpha[w])v + (u_1\beta[v] + u_2\beta[w])w \\ &= \lambda(u_1 v + u_2 w) = \lambda A_{\mathbb{C}} z. \end{aligned}$$

Since $Az_{\mathbb{C}} \neq o$, we conclude that $A_{\mathbb{C}} z \in X_{\mathbb{C}}$ is an eigenvector of $A_{\mathbb{C}}$ corresponding to the eigenvalue λ. This implies that $\mathfrak{R}(A) \ge \mathfrak{R}(\mathcal{A}^T) = \mathfrak{R}(\mathcal{A})$ which completes the proof. □

Let us illustrate Proposition 3.2 by two simple examples, the first being one-dimensional, the second infinite dimensional, which we collect in the following

Example 3.3. The simplest case is of course $X = \mathbb{R}$. Then we have $Ax = (v\alpha + w\beta)x$, where x, v, w, α, and β are all real numbers, and A represents a straight line with slope $v\alpha + w\beta$. Since $A^n x = (v\alpha + w\beta)^n x$, the linear map A has the spectral radius $|v\alpha + w\beta|$. On the other hand, the matrix (3.3) is here

$$\mathcal{A} = \begin{pmatrix} \alpha[v] & \beta[v] \\ \alpha[w] & \beta[w] \end{pmatrix} = \begin{pmatrix} v\alpha & v\beta \\ w\alpha & w\beta \end{pmatrix}$$

which has the two eigenvalues 0 and $v\alpha + w\beta$, and therefore the same spectral radius as A.

A slightly less trivial example reads as follows. In the space $X = C[0, 1]$, let A be given by (3.2), where $v(t) \equiv 1$, $\alpha[x] := x(0)$, $w(t) := t$, and $\beta[x] := x(1)$. A trivial calculation shows then that

$$A^n x(t) = x(0) + ((n-1)x(0) + x(1))t, \qquad \|A^n\|_{X \to X} = n + 1.$$

Consequently, the linear operator A has spectral radius 1. On the other hand, the matrix (3.3) is here

$$\mathcal{A} = \begin{pmatrix} \alpha[v] & \beta[v] \\ \alpha[w] & \beta[w] \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which has the double eigenvalue 1, and therefore the same spectral radius as A.

The following refinement of Theorem 2.2 is now an immediate consequence of Proposition 3.2.

Theorem 3.4. *Let $\alpha, \beta \in BV^*$ be bounded linear functionals satisfying*

$$\mathfrak{R}(\mathcal{A}) < 1, \tag{3.6}$$

where $\mathcal{A}$ denotes the matrix (3.3), $\mathfrak{R}(\mathcal{A})$ its spectral radius, and

$$v(t) := e_1(1-t) = 1-t, \qquad w(t) := e_1(t) = t.$$

Then for each $R > 0$ there is some $\rho > 0$ such that (BVP) has, for fixed $\lambda \in (-\rho, \rho)$, a solution $x \in AC^1$ satisfying $\|x\|_{BV} \leq R$. If, in addition, g is continuous on $[0,1] \times \mathbb{R}$, then every such solution is of class C^2.

Proof. The argument is similar as in the proof of Theorem 2.2. Accordingly, we only have to show that the operator A in (3.2) satisfies $\|A^n\|_{BV \to BV} < 1$ for some $n \in \mathbb{N}$. But this is clear, since (3.6) in combination with Proposition 3.2 yields $\mathfrak{R}(A) < 1$. □

We point out that Theorem 2.2 is completely covered by Theorem 3.4. Indeed, in the proof of Theorem 2.2 we have shown that each of the hypotheses (2.5) or (2.6) implies that $\mathfrak{R}(A) < 1$, and so also $\mathfrak{R}(\mathcal{A}) < 1$, by Proposition 3.2, with $\mathcal{A}$ given by (3.3). Moreover, Theorem 3.4 has several advantages. First, it does not use the operator norm $\|\cdot\|_{BV \to BV}$, but the spectral radius, which is invariant when passing to an equivalent norm. For example, if we replace the norm (1.3) by the (larger, but equivalent) norm

$$|||x|||_{BV} = \|x\|_\infty + \mathrm{Var}(x;[0,1]) = \sup_{0 \leq t \leq 1} |x(t)| + \mathrm{Var}(x;[0,1]),$$

we must impose in Theorem 2.2, instead of (2.5), the stronger condition

$$|||\alpha|||_{BV^*} + 2|||\alpha - \beta|||_{BV^*} < 1,$$

because in this norm we have $|||e_k|||_{BV} = 2$. Second, condition (3.6) is easier to verify than the conditions imposed in Theorem 2.2. Third, Theorem 3.4 covers more cases than Theorem 2.2, as we show now by means of an example.

Example 3.5. Consider again the BVP (3.1) from Example 3.1. As we have seen there, neither (2.5) nor (2.6) applies to this BVP. On the other hand, taking into account the form of the functionals α and β and the definition of v and w used in Theorem 3.4 we get here

$$\begin{cases} \alpha[v] = v\left(\frac{1}{3}\right) + v\left(\frac{2}{3}\right) = 1, \\ \beta[v] = -\frac{1}{2}v\left(\frac{1}{3}\right) - \frac{1}{2}v\left(\frac{2}{3}\right) = -\frac{1}{2} \\ \alpha[w] = w\left(\frac{1}{3}\right) + w\left(\frac{2}{3}\right) = 1, \\ \beta[w] = -\frac{1}{2}w\left(\frac{1}{3}\right) - \frac{1}{2}w\left(\frac{2}{3}\right) = -\frac{1}{2}. \end{cases}$$

So in this case the matrix $\mathcal{A}$ has the eigenvalues 0 and $1/2$, which shows that Theorem 3.4 applies, while Theorem 2.2 does not.

We may summarize our discussion as follows. In all examples discussed so far we imposed, similarly as in (1.1), boundary conditions of the form

$$x(0) = ax(\sigma_1) - bx(\sigma_2), \quad x(1) = cx(\tau_1) + dx(\tau_2), \tag{3.7}$$

where $\sigma_1, \sigma_2, \tau_1, \tau_2 \in (0,1)$ are fixed. Theorem 3.4 applies to equation (2.1) with these boundary conditions if and only if

$$\mathfrak{R}(\mathcal{M}) < 1, \tag{3.8}$$

where $\mathcal{M} = \mathcal{M}(a, b, c, d, \sigma_1, \sigma_2, \tau_1, \tau_2)$ is the matrix

$$\mathcal{M} = \begin{pmatrix} a(1-\sigma_1) - b(1-\sigma_2) & c(1-\tau_1) + d(1-\tau_2) \\ a\sigma_1 - b\sigma_2 & c\tau_1 + d\tau_2 \end{pmatrix}. \tag{3.9}$$

For instance, in Example 3.1 we have $a = 1$, $b = -1$, $c = d = -1/2$, $\sigma_1 = \tau_1 = 1/3$, and $\sigma_2 = \tau_2 = 2/3$, which gives

$$\mathcal{M} = \begin{pmatrix} 1 & -1/2 \\ 1 & -1/2 \end{pmatrix}$$

and implies the solvability of (3.1), as we have seen in Example 3.5. On the other hand, since condition $\mathfrak{R}(\mathcal{M}) < 1$ is both necessary and sufficient, we may easily construct a BVP which is not covered even by Theorem 3.4.

Example 3.6. Consider the BVP

$$\begin{cases} x''(t) + \lambda g(t, x(t)) = 0 \quad (0 \leq t \leq 1), \\ x(0) = ax\left(\frac{1}{2}\right), \quad x(1) = cx\left(\frac{1}{2}\right). \end{cases} \tag{3.10}$$

The functionals

$$\alpha[x] := ax\left(\tfrac{1}{2}\right), \qquad \beta[x] := cx\left(\tfrac{1}{2}\right)$$

are obviously linear and bounded on BV. Since $b = d = 0$, $\sigma_1 = \tau_1 = 1/2$, the matrix (3.9) is here

$$\mathcal{M} = \frac{1}{2}\begin{pmatrix} a & c \\ a & c \end{pmatrix}.$$

Since this matrix has spectral radius $|a + c|/2$, Theorem 3.4 applies to the BVP (3.10) if and only if $-2 < a + c < 2$.

We point out that condition (3.8) is necessary for the applicability of Theorem 3.4, but *not* for the existence of a solution $x \in AC^1$ of (BVP). This is illustrated by the following

Example 3.7. Consider the BVP

$$\begin{cases} x''(t) - 2(1 + 2t^2)x(t) = 0 \quad (0 \leq t \leq 1), \\ x(0) = e^{-1/4}x\left(\frac{1}{2}\right), \quad x(1) = e^{3/4}x\left(\frac{1}{2}\right). \end{cases} \tag{3.11}$$

Obviously, the nonlinearity $g(t, u) = (2 + 4t^2)u$ satisfies (H1)/(H2)/(H3). In the notation of (3.7) we have here $a = e^{-1/4}$, $c = e^{3/4}$, $b = d = 0$, and $\sigma_1 = \tau_1 = 1/2$. Consequently, the matrix (3.9) reads

$$\mathcal{M} = \frac{1}{2}\begin{pmatrix} e^{-1/4} & e^{3/4} \\ e^{-1/4} & e^{3/4} \end{pmatrix}$$

which has spectral radius

$$\mathfrak{R}(\mathcal{M}) = \frac{1 + e}{2e^{1/4}} > 1.$$

So Theorem 3.4, let alone Theorem 2.2, does not apply. Nevertheless, it is easy to check that $x(t) := e^{t^2}$ is an (even analytic) solution of the boundary value problem (3.11).

4 Integral type boundary conditions

Theorem 3.4 applies not only to "pointwise" boundary conditions like (3.7), but also to "global" boundary conditions of the form

$$x(0) = \int_0^1 k_0(s)x(s)\,ds, \qquad x(1) = \int_0^1 k_1(s)x(s)\,ds, \tag{4.1}$$

where $k_0, k_1 \in L_1$ are given. The functionals α and β are defined here by the integrals in (4.1), and so Theorem 3.4 applies if and only if $\Re(\mathcal{M}) < 1$, where $\mathcal{M} = \mathcal{M}(k_0, k_1)$ is the matrix

$$\mathcal{M} = \begin{pmatrix} \displaystyle\int_0^1 k_0(s)(1-s)\,ds & \displaystyle\int_0^1 k_1(s)(1-s)\,ds \\ \displaystyle\int_0^1 k_0(s)s\,ds & \displaystyle\int_0^1 k_1(s)s\,ds \end{pmatrix}. \tag{4.2}$$

We illustrate this by another simple example which contains a free parameter $c \in \mathbb{R}$.

Example 4.1. Consider the BVP

$$\begin{cases} x''(t) + \lambda g(t, x(t)) = 0 \quad (0 \le t \le 1), \\ x(0) = \displaystyle\int_0^1 x(s)\,ds, \quad x(1) = c\displaystyle\int_0^1 x(s)\,ds, \end{cases} \tag{4.3}$$

where g satisfies (H1)/(H2)/(H3). Here we have $k_0(s) \equiv 1$ and $k_1(s) \equiv c$, so the matrix $\mathcal{M} = \mathcal{M}(1, c)$ becomes

$$\mathcal{M} = \begin{pmatrix} \displaystyle\int_0^1 k_0(s)(1-s)\,ds & \displaystyle\int_0^1 k_1(s)(1-s)\,ds \\ \displaystyle\int_0^1 k_0(s)s\,ds & \displaystyle\int_0^1 k_1(s)s\,ds \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & c \\ 1 & c \end{pmatrix}.$$

Since this matrix has spectral radius $(c+1)/2$, we may guarantee the solvability of problem (4.3) for small $|\lambda|$ if $-3 < c < 1$.

5 A higher order problem

The theory developed in the preceding sections may be applied to other similar boundary value problems than those we have considered in the examples so far. For instance, we can modify our constructions to cover a third order problem like

$$\begin{cases} x'''(t) + \lambda g(t, x(t)) = 0 \quad (0 \le t \le 1), \\ x'(0) = \alpha[x], \quad x'(1) = \beta[x] \end{cases} \tag{5.1}$$

with $\alpha, \beta \in BV^*$ as before. We do not state a formal theorem, since we do not want the reader to get drowned in too many technicalities, but just sketch an outline of the idea, because the arguments are similar as those used before.

We are looking for solutions $x \in AC^2$ that satisfy the differential equation in (5.1) almost everywhere in $[0, 1]$ and have the correct boundary values $x'(0) = \alpha[x]$ and $x'(1) = \beta[x]$. In order to find such a solution we solve the integral equation

$$x(t) = Ax(t) + \lambda \int_0^t \int_0^1 \kappa(\tau, s) g(s, x(s))\,ds\,d\tau \qquad (0 \le t \le 1) \tag{5.2}$$

in the space BV, where κ is the same Green's function as before, and the linear operator $A : BV \to BV$ is again given by (3.2), where now

$$v(t) := -\frac{1}{2}e_2(1-t) = -\frac{1}{2}(1-t)^2, \qquad w(t) := \frac{1}{2}e_2(t) = \frac{1}{2}t^2.$$

For $x \in AC^2$ the outer integral in (5.2) defines a differentiable function. Similarly as in Proposition 2.1 one may show that any function $x \in BV$ satisfying (5.2) is a solution in AC^2 to the boundary value problem (5.1), and vice versa. Note that for the first derivative of a solution x of (5.2) we have

$$x'(t) = (1-t)\alpha[x] + t\beta[x] + \lambda \int_0^1 \kappa(t,s)g(s,x(s))\,ds \qquad (0 \le t \le 1), \tag{5.3}$$

and so indeed $x'(0) = \alpha[x]$ and $x'(1) = \beta[x]$.

Now, in order to solve (5.2) we can use Fubini's Theorem to reduce the double integral to a single one and transform the integral equation into

$$x(t) = Ax(t) + \lambda \int_0^1 \hat{\kappa}(t,s)g(s,x(s))\,ds \qquad (0 \le t \le 1), \tag{5.4}$$

where

$$\hat{\kappa}(t,s) := \int_0^t \kappa(\tau,s)\,d\tau = \begin{cases} \frac{1}{2}s(2t - t^2 - s) & \text{for } 0 \le s \le t \le 1, \\ \frac{1}{2}t^2(1-s) & \text{for } 0 \le t \le s \le 1. \end{cases}$$

Consequently, under the hypotheses of Theorem 3.4 (with v and w as above), we may solve (5.2) and therefore also (5.1) exactly as we solved (BVP). Instead of going into details, let us close this section with an example.

Example 5.1. Consider the third order BVP

$$\begin{cases} x'''(t) - 4t(2t^2+3)x(t) = 0 \quad (0 \le t \le 1), \\ x'(0) = 0, \quad x'(1) = 2e^{3/4}x\left(\frac{1}{2}\right). \end{cases} \tag{5.5}$$

Here the integral equation (5.4) is

$$x(t) = e^{3/4}x\left(\tfrac{1}{2}\right)t^2 + 2\int_0^t s^2(2t - t^2 - s)(2s^2+3)x(s)\,ds + 2t^2\int_t^1 (1-s)s(2s^2+3)x(s)\,ds.$$

A somewhat cumbersome, but straightforward calculation shows that $x(t) = e^{t^2}$ is a solution. However, if we are only interested in the existence of a solution without constructing it explicitly, we may use Proposition 2.1 and calculate the spectral radius of the matrix

$$\mathcal{A} = \begin{pmatrix} \alpha[v] & \beta[v] \\ \alpha[w] & \beta[w] \end{pmatrix} = \frac{e^{3/4}}{4}\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix},$$

which turns out to be $e^{3/4}/4 < 1$. So in contrast to Example 3.7 we may now apply Theorem 3.4.

6 Initial value problems with BV data

To conclude, let us briefly discuss the second order equation (2.1), but now subject to the uncoupled initial conditions

$$x(0) = \alpha[x], \qquad x'(0) = \beta[x], \tag{6.1}$$

where $\alpha, \beta : BV \to \mathbb{R}$ are given linear functionals. Here we do not repeat all the results which are parallel to those for boundary value problems, but rather point out the differences. In the sequel we refer to the problem (2.1)/(6.1) by the symbol (IVP).

In order to solve this problem, we consider along with (IVP) the Hammerstein–Volterra integral equation

$$x(t) = Ax(t) + \lambda \int_0^t \nu(t,s)g(s,x(s))\,ds \qquad (0 \le t \le 1), \tag{6.2}$$

where the Volterra kernel is given by

$$\nu(t,s) = \begin{cases} s-t & \text{for } 0 \le s \le t \le 1, \\ 0 & \text{for } 0 \le t < s \le 1, \end{cases}$$

and $A : BV \to BV$ is a linear operator. The following is then a perfect analogue to Proposition 2.1.

Proposition 6.1. *Let $A : BV \to BV$ be defined by*

$$Ax(t) := \alpha[x] + t\beta[x] \qquad (0 \le t \le 1). \tag{6.3}$$

Then the following holds:

(a) Every function $x \in BV$ solving (6.2) belongs to AC^1 and solves (IVP) almost everywhere on $[0,1]$.

(b) If, in addition, g is continuous on $[0,1] \times \mathbb{R}$, then every solution x of (6.2) is of class C^2 and solves (IVP) everywhere on $[0,1]$.

(c) Conversely, if $x \in AC^1$ solves (IVP) almost everywhere on $[0,1]$, then x is a solution of the integral equation (6.2).

The proof is very similar to that of Proposition 2.1, with the difference that we now define the function $\varphi : [0,1] \to \mathbb{R}$ by

$$\varphi(t) := \int_0^1 \nu(t,s)g(s,x(s))\,ds = \int_0^t (s-t)h(s)\,ds$$

and use the fact that $\varphi \in AC^1$ with $\varphi(0) = \varphi'(0) = 0$.

The sufficient condition (2.5) imposed in Theorem 2.2 becomes now even easier: since Ax is, for fixed $x \in BV$, a straight line joining the points $(0, \alpha[x])$ and $(1, \alpha[x] + \beta[x])$, we can calculate its BV norm explicitly and obtain

$$\|Ax\|_{BV} = |Ax(0)| + \mathrm{Var}(Ax;[0,1]) = |\alpha[x]| + |\beta[x]| \le (\|\alpha\|_{BV^*} + \|\beta\|_{BV^*})\|x\|_{BV}.$$

Thus, the estimate

$$\|\alpha\|_{BV^*} + \|\beta\|_{BV^*} < 1 \tag{6.4}$$

which is parallel to (2.5) now guarantees that $\|A\|_{BV\to BV} < 1$ and makes it possible to apply Krasnosl'skij's fixed point principle to (6.2) for sufficiently small $|\lambda|$.

Of course, as in Section 2 we could easily find specific IVPs to illustrate the applicability of (6.4). Instead, it is more interesting to compare (2.5) and (6.4). It is tempting to think that (2.5) implies (6.4), or vice versa. But no such implication is true, as the following two examples show.

Example 6.2. Define two functionals $\alpha, \beta \in BV^*$ by

$$\alpha[x] := \beta[x] := \tfrac{1}{2}x\left(\tfrac{1}{2}\right).$$

Then $\|\alpha\|_{BV^*} = \|\beta\|_{BV^*} = 1/2$ and $\|\alpha - \beta\|_{BV^*} = 0$. Thus, condition (2.5) is fulfilled, while condition (6.4) is violated.

Example 6.3. On the other hand, if we define $\alpha, \beta \in BV^*$ by

$$\alpha[x] := \tfrac{1}{3}x\left(\tfrac{1}{2}\right), \qquad \beta[x] := -\tfrac{1}{3}x\left(\tfrac{1}{2}\right),$$

it is easy to see that condition (2.5) is violated, while condition (6.4) is fulfilled.

We now jump to Theorem 3.4 and see how it looks like in the setting of (IVP). Since the structure of the linear operator A in (6.3) is covered by Proposition 3.2, we have a general method to calculate the spectral radius of A. Accordingly, the following analogue to Theorem 3.4 holds true.

Theorem 6.4. *Let $\alpha, \beta \in BV^*$ be bounded linear functionals satisfying (3.6), where $\mathcal{A}$ denotes the matrix (3.3) for $v := e_0$ and $w := e_1$. Then for each $R > 0$ there is some $\rho > 0$ such that (IVP) has, for fixed $\lambda \in (-\rho, \rho)$, a solution $x \in AC^1$ satisfying $\|x\|_{BV} \le R$. If, in addition, g is continuous on $[0,1] \times \mathbb{R}$, then every such solution is of class C^2.*

Since the argument is similar, we skip the proof of Theorem 6.4. Instead, let us go back to Example 6.2, where condition (6.4) fails. Even worse, it is clear that $Ax = \alpha[x]e_0 + \beta[x]e_1 = \alpha[x](e_0 + e_1)$ cannot be a contraction in BV, because $\|Ae_0\|_{BV} = 1$. However, we have

$$\mathfrak{R}\begin{pmatrix} \alpha[e_0] & \beta[e_0] \\ \alpha[e_1] & \beta[e_1] \end{pmatrix} = \mathfrak{R}\begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 1/4 \end{pmatrix} = \frac{3}{4} < 1,$$

and so Theorem 6.4 tells us that the IVP in Example 6.2 has as a solution $x \in AC^1$ for small $|\lambda|$.

Finally, let us look at an initial value condition which corresponds to the very general boundary condition (3.7). Its analogue has the form

$$x(0) = ax(\sigma_1) - bx(\sigma_2), \qquad x'(0) = cx(\tau_1) + dx(\tau_2), \tag{6.5}$$

where $\sigma_1, \sigma_2, \tau_1, \tau_2 \in (0,1)$ are fixed. Theorem 6.4 applies to equation (2.1) with these initial conditions if and only if

$$\mathfrak{R}(\mathcal{N}) < 1, \tag{6.6}$$

where $\mathcal{N} = \mathcal{N}(a, b, c, d, \sigma_1, \sigma_2, \tau_1, \tau_2)$ is the matrix

$$\mathcal{N} = \begin{pmatrix} a - b & c + d \\ a\sigma_1 - b\sigma_2 & c\tau_1 + d\tau_2 \end{pmatrix}. \tag{6.7}$$

In the next example we show that neither of the conditions $\mathfrak{R}(\mathcal{M}) < 1$ or $\mathfrak{R}(\mathcal{N}) < 1$ implies the other, where $\mathcal{M}$ is given by (3.9).

Example 6.5. Let $\sigma_1 := 1/3$, $\sigma_2 := 2/3$, and $c = d := 0$ which means that we consider both the BVP

$$\begin{cases} x''(t) + \lambda g(t, x(t)) = 0 & (0 \le t \le 1), \\ x(0) = ax\left(\frac{1}{3}\right) - bx\left(\frac{2}{3}\right), & x(1) = 0, \end{cases} \tag{6.8}$$

and simultaneously the IVP

$$\begin{cases} x''(t) + \lambda g(t, x(t)) = 0 & (0 \le t \le 1), \\ x(0) = ax\left(\frac{1}{3}\right) - bx\left(\frac{2}{3}\right), & x'(0) = 0. \end{cases} \tag{6.9}$$

Then

$$\mathcal{M} = \frac{1}{3}\begin{pmatrix} 2a - b & 0 \\ a - 2b & 0 \end{pmatrix}, \qquad \mathcal{N} = \frac{1}{3}\begin{pmatrix} 3a - 3b & 0 \\ a - 2b & 0 \end{pmatrix}.$$

The matrix $\mathcal{M}$ has spectral radius $|2a - b|/3$, the matrix $\mathcal{N}$ has spectral radius $|a - b|$. Consequently, for $a := -1/2$ and $b := -5/2$ we have $\mathfrak{R}(\mathcal{M}) = 1/2$, but $\mathfrak{R}(\mathcal{N}) = 2$ (which ensures the solvability of (6.8), but not of (6.9)). For $a := 11/2$ and $b := 5$, however, it is exactly the other way round.

At this point the same warning as in Section 3 is in order. Condition (6.6) is necessary and sufficient for the applicability of Theorem 6.4, but only sufficient for the solvability of (IVP). This is illustrated by our final

Example 6.6. Consider the IVP

$$\begin{cases} x''(t) + 4\left(4t^2 x(t) + \sqrt{1 - x(t)^2}\right) & (0 \le t \le 1), \\ x(0) = \sqrt{2}x\left(\sqrt{\pi/8}\right), & x'(0) = 0. \end{cases} \tag{6.10}$$

Clearly, the nonlinearity $g(t, u) = 4(4t^2 u + \sqrt{1 - u^2})$ satisfies (H1)/(H2)/(H3). In the notation of (6.5) we have here $a = \sqrt{2}$, $b = c = d = 0$, and $\sigma_1 = \sqrt{\pi/8}$. Consequently, the matrix (6.7) reads

$$\mathcal{N} = \begin{pmatrix} \sqrt{2} & 0 \\ \sqrt{\pi}/2 & 0 \end{pmatrix} \tag{6.11}$$

which has spectral radius

$$\mathfrak{R}(\mathcal{N}) = \sqrt{2} > 1,$$

so Theorem 6.4 does not apply. Nevertheless, it is easy to check that $x(t) := \cos(2t^2)$ is an (even analytic) solution of the initial value problem (6.10).

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