Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 68, 1–13; https://doi.org/10.14232/ejqtde.2021.1.68 www.math.u-szeged.hu/ejqtde/

A nonzero solution for bounded selfadjoint operator equations and homoclinic orbits of Hamiltonian systems

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Received 20 November 2020, appeared 10 September 2021
Communicated by Petru Jebelean

Abstract. We obtain an existence theorem of nonzero solution for a class of bounded selfadjoint operator equations. The main result contains as a special case the existence result of a nontrivial homoclinic orbit of a class of Hamiltonian systems by Coti Zelati, Ekeland and Séré. We also investigate the existence of nontrivial homoclinic orbit of indefinite second order systems as another application of the theorem.

Keywords: bounded selfadjoint operator equations, nonzero solution, homoclinic orbit, Hamiltonian systems, indefinite second order systems.

2020 Mathematics Subject Classification: 34A34, 34C37, 37N05, 58E05.

1 Introduction

In recent years several authors studied the existence of homoclinic orbits for first or second order Hamiltonian systems via variational methods and critical point theory, see for instance [2, 4–6, 9, 12–16]. In particular, with the aid of a bounded self-adjoint linear operator and the dual action principle, Coti Zelati, Ekeland and Séré [4] obtained some existence theorems of nonzero homoclinic orbit for first order Hamiltonian systems

$$\begin{cases} x' = JAx + JH'(t,x), \\ x(\pm\infty) = 0, \end{cases}$$

via the Ambrosetti–Rabinowitz mountain-pass theorem and concentration compactness principle. Inspired by the ideas of [4], we consider the more generalized operator equation

$$Lu - G'(t,u) = 0, \tag{1.1}$$

where $L : L^\beta(\mathbb{R}, \mathbb{R}^N) \to W^{1,\beta}(\mathbb{R}, \mathbb{R}^N) \bigcap L^\gamma(\mathbb{R}, \mathbb{R}^N)$ is a bounded linear operator for all $\gamma \geq \beta$ and for some $\beta \in (1,2)$ and $\int_{\mathbb{R}}((Lu)(t), v(t))dt = \int_{\mathbb{R}}((Lv)(t), u(t))dt$ for all $u, v \in L^\beta(\mathbb{R}, \mathbb{R}^N)$,

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$G : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $G'(t,u)$ denotes the gradient of G with respect to u. $u = u(t) \in L^\beta(\mathbb{R}, \mathbb{R}^N)$ is called a solution of (1.1) if $(Lu)(t) - G'(t, u(t)) = 0$ a.e. $t \in \mathbb{R}$.

We need the following assumptions:

(L_1) For any bounded $\{u_n\} \subset L^\beta(\mathbb{R}, \mathbb{R}^N)$ and $R > 0$, there exists a subsequence $\{u_{n_j}\}$ such that $Lu_{n_j} \to w$ in $C([-R, R], \mathbb{R}^N)$.

(L_2) There exists $v_0 \in L^\beta(\mathbb{R}, \mathbb{R}^N)$ such that $\int_{-\infty}^{+\infty}(Lv_0, v_0)dt > 0$.

(L_3) $(Lu(\cdot + T))(t) = (Lu)(t + T)$ for all $t \in \mathbb{R}$, where $T > 0$ is a constant.

(L_4) $|(Lu)(t)| \le c_0 \int_{-\infty}^{+\infty} e^{-l|t-\tau|}|u(\tau)|d\tau$ for all $u \in L^\beta(\mathbb{R}, \mathbb{R}^N)$, where $c_0, l > 0$ are two constants.

(G_1) $G(t, \cdot)$ and $G'(t, \cdot)$ are continuous for a.e. $t \in \mathbb{R}$, $G(\cdot, u)$ and $G'(\cdot, u)$ are measurable for all $u \in \mathbb{R}^N$, $G(t, \cdot)$ is convex for all $t \in \mathbb{R}$ and $G^{*\prime}(t, \cdot)$ exists for a.e. $t \in \mathbb{R}$.

(G_2) $G(t + T, u) = G(t, u)$ for all $t \in \mathbb{R}$.

(G_3) $c_1|u|^\beta \le G(t, u) \le c_2|u|^\beta$, where $c_2 \ge c_1 > 0$ are two constants.

(G_4) $0 \le \frac{1}{\beta}(G'(t, u), u) \le G(t, u)$.

(G_5) $|G'(t, u)| \le c_3|u|^{\beta-1}$, where $c_3 > 0$ is a constant.

Now we state our main result as follows.

Theorem 1.1. *Assume L and G satisfy* (L_1)–(L_4) *and* (G_1)–(G_5). *Then* (1.1) *has a nonzero solution.*

Remark 1.1. Although the equation (1.1) also appeared in the proof of Theorem 4.2 in [4], the bounded linear operator L there equal (2.2) which comes from first order Hamiltonian systems. In this paper, L discussed in (1.1) contains not only (2.2) but also (2.4) coming from indefinite second order Hamiltonian systems. In addition, introducing the condition (L_1) makes the proof of conclusion clearer and simpler.

The rest of this paper is organized as follows. In Section 2, we firstly establish a preliminary lemma, and then, we give two application examples for homoclinic orbit of Hamiltonian systems. In Section 3, we give the proof of our main result.

2 Preliminaries and examples

To complete the proof of Theorem 1.1, we need a lemma.

Lemma 2.1. *Let* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

(1) *If* $u \in L^\beta(\mathbb{R}, \mathbb{R}^N)$ *and* $b > a > 0$, *then*

$$\left(\int_{|t|\ge b}\left(\int_{-a}^{a} e^{-l|t-\tau|}|u(\tau)|d\tau\right)^\alpha dt\right)^{\frac{1}{\alpha}} \le 2(\alpha l)^{-\frac{2}{\alpha}} e^{-l(b-a)}\left(\int_{-a}^{a}|u(\tau)|^\beta d\tau\right)^{\frac{1}{\beta}}.$$

(2) *If* $w, u \in L^{\beta}(\mathbb{R}, \mathbb{R}^N)$ *and* $b \geq a > r \geq 0$, *then*

$$\int_{|t|\geq b} |u(t)| \int_{a\geq|\tau|\geq r} e^{-l|t-\tau|}|w(\tau)|d\tau dt \\ \leq 2(\alpha l)^{-\frac{2}{\alpha}} e^{-l(b-a)} \left(\int_{|t|\geq b} |u(t)|^{\beta} dt\right)^{\frac{1}{\beta}} \left(\int_{a\geq|\tau|\geq r} |w(\tau)|^{\beta} d\tau\right)^{\frac{1}{\beta}}.$$

(3) *If* $w, u \in L^{\beta}(\mathbb{R}, \mathbb{R}^N)$ *and* $b > a > r > 0$, *then*

$$\int_{a\leq|t|\leq b} |u(t)| \int_{|\tau|\leq a} e^{-l|t-\tau|}|w(\tau)|d\tau dt \\ \leq 2(\alpha l)^{-\frac{2}{\alpha}} \|u\|_{L^{\beta}} \left[e^{-l(a-r)}\|w\|_{L^{\beta}} + \left(\int_{r\leq|\tau|\leq a} |w(\tau)|^{\beta} d\tau\right)^{\frac{1}{\beta}}\right].$$

Proof. For $u \in L^{\beta}(\mathbb{R}, \mathbb{R}^N)$ and $b > a > 0$, by some simple calculations, we have

$$\begin{aligned} &\left(\int_{|t|\geq b} \left(\int_{-a}^{a} e^{-l|t-\tau|}|u(\tau)|d\tau\right)^{\alpha} dt\right)^{\frac{1}{\alpha}} \\ &\leq \left(\left(\int_{b}^{+\infty} + \int_{-\infty}^{-b}\right) \int_{-a}^{a} e^{-\alpha l|t-\tau|} d\tau dt\right)^{\frac{1}{\alpha}} \left(\int_{-a}^{a} |u(\tau)|^{\beta} d\tau\right)^{\frac{1}{\beta}} \\ &= 2^{\frac{1}{\alpha}} (\alpha l)^{-\frac{2}{\alpha}} \left(1 - e^{-2\alpha a l}\right)^{\frac{1}{\alpha}} e^{-l(b-a)} \left(\int_{-a}^{a} |u(\tau)|^{\beta} d\tau\right)^{\frac{1}{\beta}} \\ &\leq 2(\alpha l)^{-\frac{2}{\alpha}} e^{-l(b-a)} \left(\int_{-a}^{a} |u(\tau)|^{\beta} d\tau\right)^{\frac{1}{\beta}}, \end{aligned}$$

which implies that (1) holds. The same arguments also prove that (2) holds.

By (2), we have

$$\begin{aligned} &\int_{a\leq|t|\leq b} |u(t)| \int_{|\tau|\leq a} e^{-l|t-\tau|}|w(\tau)|d\tau dt \\ &= \int_{a\leq|t|\leq b} |u(t)| \left(\int_{|\tau|\leq r} + \int_{r\leq|\tau|\leq a}\right) e^{-l|t-\tau|}|w(\tau)|d\tau dt \\ &\leq 2(\alpha l)^{-\frac{2}{\alpha}} \|u\|_{L^{\beta}} \left[e^{-l(a-r)}\|w\|_{L^{\beta}} + \left(\int_{r\leq|\tau|\leq a} |w(\tau)|^{\beta} d\tau\right)^{\frac{1}{\beta}}\right]. \end{aligned}$$

This shows that (3) holds. □

Next, we return to applications to homoclinic orbit of Hamiltonian systems. For systematic researches of homoclinic orbit of Hamiltonian systems, we refer to the excellent papers [2,4–6,9,12–16] and references therein.

As the first example we consider

$$\begin{cases} x' = JAx + JH'(t,x), \\ x(\pm\infty) = 0, \end{cases} \tag{2.1}$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ is the standard symplectic matrix in $\mathbb{R}^{2N}$, A is a $2N \times 2N$ symmetric matrix and all the eigenvalues of JA have non-zero real part, $H(t,x)$ satisfies

(H_1) $H \in C(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R}), H' \in C(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R}^{2N})$ and $H(t,\cdot)$ is strictly convex;

(H_2) $H(t+T,x) = H(t,x)$ for some $T > 0$;

(H_3) $k_1|x|^\alpha \le H(t,x) \le k_2|x|^\alpha$ for some $\alpha > 2$ and $0 < k_1 \le k_2$;

(H_4) $H(t,x) \le \frac{1}{\alpha}(H'(t,x),x)$.

As in [4], define $G(t,u) = \sup_{x\in\mathbb{R}^{2N}}\{(u,x) - H(t,x)\}$ and G satisfies (G_1)–(G_5).

Define $L : L^\beta(\mathbb{R},\mathbb{R}^{2N}) \to W^{1,\beta}(\mathbb{R},\mathbb{R}^N)\bigcap L^\alpha(\mathbb{R},\mathbb{R}^{2N})$ by $z = Lu$ satisfies

$$-Jz' - Az = u, z(\pm\infty) = 0$$

Then

$$z(t) = \int_{-\infty}^{t} e^{E(t-\tau)}P_sJu(\tau)d\tau - \int_{t}^{+\infty} e^{E(t-\tau)}P_uJu(\tau)d\tau, \tag{2.2}$$

where $E = JA, \mathbb{R}^{2N} = E_u \oplus E_s$ and P_s and P_u are the projections onto E_s and E_u respectively satisfying $|e^{tE}P_s\xi| \le ke^{-bt}|\xi|$ for $t \ge 0$ and $|e^{tE}P_u\xi| \le ke^{bt}|\xi|$ for $t \le 0, \xi \in \mathbb{R}^{2N}$ and some $b, k > 0$. So

$$\begin{aligned}|(Lu)(t)| &\le \int_{-\infty}^{t} ke^{-b(t-\tau)}|u(\tau)|d\tau + \int_{t}^{+\infty} ke^{b(t-\tau)}|u(\tau)|d\tau \\ &= k\int_{-\infty}^{+\infty} e^{-b|t-\tau|}|u(\tau)|d\tau,\end{aligned}$$

which implies that (L_4) holds. From Lemma 2.1 of [4], we know that $L : L^\beta(\mathbb{R},\mathbb{R}^{2N}) \to W^{1,\beta}(\mathbb{R},\mathbb{R}^{2N})\bigcap L^\gamma(\mathbb{R},\mathbb{R}^{2N})$ is a bounded linear operator for $\gamma \ge \beta, \beta \in (1,2)$ and

$$\int_{\mathbb{R}}((Lu)(t),v(t))dt = \int_{\mathbb{R}}((Lv)(t),u(t))dt$$

for all $u, v \in L^\beta(\mathbb{R},\mathbb{R}^{2N})$.

By $z'(t) = Ju(t) + Ez(t)$ for all $t \in \mathbb{R}$, we have

$$\begin{aligned}|z(t_1) - z(t_2)| &= \left|\int_{t_1}^{t_2}(Ju(t) + Ez(t))dt\right| \\ &\le |t_2 - t_1|^{\frac{1}{\alpha}}\|u\|_{L^\beta} + M_0|t_2 - t_1|\|z\|_\infty\end{aligned}$$

where $M_0 > 0$, which implies that (L_1) holds. Note that the proof of (b) of Lemma 4.1 in [4], we see that there exists $v_0 \in L^\beta(\mathbb{R},\mathbb{R}^{2N})$ such that (L_2) holds. The validity of (L_3) is obvious.

Moreover, $G^*(t,x) = H(t,x)$ and a solution $u \in L^\beta(\mathbb{R},\mathbb{R}^{2N}) \setminus \{0\}$ of $Lu - G'(t,u) = 0$ corresponds to a nonzero solution $x = Lu$ of

$$\begin{cases}-Jx' - Ax = H'(t,x), \\ x(\pm\infty) = 0.\end{cases}$$

Therefore, we have the following corollary.

Corollary 2.2 ([4, Theorem 4.2]). *Assume H satisfies* (H_1)–(H_4). *Then* (2.1) *has a nonzero solution, i.e., the Hamiltonian system*

$$-Jx' - Ax = H'(t,x)$$

has at least one nontrivial homoclinic orbit.

Remark 2.1. The above corollary was essentially [4, Theorem 4.2] by Coti Zelati, Ekeland and Séré using the Ekeland variational principle and concentration compactness principle, and the equation (1.1) also appeared in the proof the theorem already.

As a second example we consider

$$\begin{cases} Dx'' - Bx = V'(t,x), \\ x(\pm\infty) = 0, \end{cases} \tag{2.3}$$

where D, B are $N \times N$ symmetric matrix, $(\pm\sigma(D)) \bigcap (0, +\infty) \neq \emptyset$, D is invertible, $D^{-1}B = Q^2$ with Q being a $N \times N$ matrix and all the eigenvalues of Q have positive real part, $V : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $V'(t,x)$ denotes the gradient of V with respect to x. The system was called indefinite second order system in [3].

Let

$$\begin{cases} Dx'' - Bx = u, \\ x(\pm\infty) = 0. \end{cases}$$

Then

$$\begin{cases} x'' - D^{-1}Bx = x'' - Q^2 x = D^{-1}u, \\ x(\pm\infty) = 0 \end{cases}$$

and

$$\begin{cases} \left[e^{tQ}(x' - Qx)\right]' = e^{tQ}D^{-1}u, \\ x(\pm\infty) = 0. \end{cases}$$

Assume $x'(-\infty) = 0$ (and this will be verified later). Then

$$x' - Qx = e^{-tQ}\int_{-\infty}^{t} e^{\tau Q}D^{-1}u(\tau)d\tau$$

and

$$(e^{-tQ}x)' = e^{-2tQ}\int_{-\infty}^{t} e^{\tau Q}D^{-1}u(\tau)d\tau.$$

So, we have

$$\begin{aligned} x &= -e^{tQ}\int_{t}^{+\infty} e^{-2sQ}\left(\int_{-\infty}^{s} e^{\tau Q}D^{-1}u(\tau)d\tau\right)ds \\ &= -\frac{Q^{-1}}{2}e^{tQ}\int_{-\infty}^{t} e^{-2tQ}e^{\tau Q}D^{-1}u(\tau)d\tau - \frac{Q^{-1}}{2}e^{tQ}\int_{t}^{+\infty} e^{-\tau Q}D^{-1}u(\tau)d\tau \\ &= -\frac{Q^{-1}}{2}\int_{-\infty}^{+\infty} e^{-|t-\tau|Q}D^{-1}u(\tau)d\tau. \end{aligned}$$

For $u \in L^{\beta}(\mathbb{R}, \mathbb{R}^N)$, set

$$x = Lu = -\frac{Q^{-1}}{2}\int_{-\infty}^{+\infty} e^{-|t-\tau|Q}D^{-1}u(\tau)d\tau. \tag{2.4}$$

We claim that

$$x = Lu \in W^{1,\beta}(\mathbb{R}, \mathbb{R}^N) \bigcap L^{\gamma}(\mathbb{R}, \mathbb{R}^N)$$

for $\gamma \geq \beta, \beta \in (1,2)$ and $u \in L^{\beta}(\mathbb{R}, \mathbb{R}^N)$. In fact, from all the eigenvalues of Q have positive real part, we know that there exist $\lambda_0 > 0$ and $c_4 > 0$ such that $|e^{-|t|Q}\xi| \leq c_4 e^{-\lambda_0|t|}|\xi|$ for $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$. By $\int_{-\infty}^{+\infty} e^{-\eta|t|}dt = \frac{2}{\eta}$, we have

$$e^{-\lambda_0|t|} \in L^{\eta}(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad \|e^{-\lambda_0|t|}\|_{L^\eta}^{\eta} = \frac{2}{\lambda_0 \eta} \quad \forall \eta \geq 1.$$

Using the convolution inequality, we have

$$\begin{aligned}\left(\int_{-\infty}^{+\infty} |Lu|^r dt\right)^{\frac{1}{r}} &\leq \frac{c_4\|Q^{-1}\| \cdot \|D^{-1}\|}{2} \left(\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} e^{-\lambda_0|t-\tau|}|u(\tau)|d\tau\right)^r dt\right)^{\frac{1}{r}} \\ &\leq \frac{c_4\|Q^{-1}\| \cdot \|D^{-1}\|}{2} \|e^{-\lambda_0|t|}\|_{L^p} \cdot \|u\|_{L^\beta} \end{aligned} \tag{2.5}$$

for $\frac{1}{r} = \frac{1}{p} + \frac{1}{\beta} - 1$ and $r, p \geq 1$, which shows that $Lu \in L^r(\mathbb{R}, \mathbb{R}^N) \ \ \forall r \in [\beta, +\infty]$. Similarly, from the equation

$$x' = Qx + \int_{-\infty}^{t} e^{-(t-\tau)Q} D^{-1} u(\tau) d\tau, \tag{2.6}$$

it is easy to see that $Lu \in W^{1,\beta}(\mathbb{R}, \mathbb{R}^N)$. Moreover, by (2.5), we can also see that $L\colon L^{\beta}(\mathbb{R}, \mathbb{R}^N) \to W^{1,\beta}(\mathbb{R}, \mathbb{R}^N) \bigcap L^{\gamma}(\mathbb{R}, \mathbb{R}^N)$ is a bounded linear operator for $\gamma \geq \beta$. This implies $x(\pm\infty) = 0$ and $x'(-\infty) = 0$ via the above equation.

Let $x = Lu$ and $y = Lv$. Then

$$\begin{aligned}\int_{\mathbb{R}} ((Lu)(t), v(t))dt &= \int_{-\infty}^{+\infty} (x, Dy'' - By))dt \\ &= \int_{-\infty}^{+\infty} (Dx'' - Bx), y)dt \\ &= \int_{\mathbb{R}} (u(t), (Lv)(t))dt\end{aligned}$$

for all $u, v \in L^{\beta}(\mathbb{R}, \mathbb{R}^N)$, which implies that $L : L^{\beta} \to L^{\alpha}$ is self-adjoint.

By (2.6) for all $t_1, t_2 \in \mathbb{R}$, we have

$$\begin{aligned}|x(t_1) - x(t_2)| &= \left|\int_{t_1}^{t_2} \left(Qx + \int_{-\infty}^{t} e^{-(t-\tau)Q} D^{-1} u(\tau) d\tau\right) dt\right| \\ &\leq \|Q\| \cdot \|x\|_{\infty} \cdot |t_2 - t_1| + c_4(\lambda_0\alpha)^{\frac{-1}{\alpha}} \|D^{-1}\| \cdot \|u\|_{L^\beta} \cdot |t_2 - t_1|,\end{aligned}$$

which implies that (L_1) holds.

Since $(\pm\sigma(D)) \bigcap (0, +\infty) \neq \emptyset$, we know that there exist $\lambda_1 < 0$ and $\xi_0 \in \mathbb{R}^N \backslash \{0\}$ such that $|\xi_0| = 1$ and $D\xi_0 = \lambda_1 \xi_0$. Let

$$x_0(t) = \begin{cases} \xi_0 \sin kt, & t \in [0, 2m\pi], \\ \xi_0[\frac{k}{\pi^2}(t - 2m\pi - \pi)^3 + \frac{k}{\pi}(t - 2m\pi - \pi)^2], & t \in [2m\pi, 2m\pi + \pi], \\ 0, & t \geq 2m\pi + \pi, \\ -x_0(-t), & t < 0, \end{cases}$$

where $k, m \in \mathbf{N} \backslash \{0\}$. Then

$$\begin{cases} Dx_0'' - Bx_0 = v_0, \\ x_0(\pm\infty) = 0 \end{cases}$$

and

$$\begin{aligned}\int_{-\infty}^{+\infty}(Lv_0,v_0)dt &= 2\int_0^{+\infty}(Dx_0''(t)-Bx_0(t),x_0(t))dt\\ &= 2\left(\int_0^{2m\pi}+\int_{2m\pi}^{2m\pi+\pi}\right)(Dx_0''(t)-Bx_0(t),x_0(t)dt\\ &\geq 2\left(\int_0^{2m\pi}+\int_{2m\pi}^{2m\pi+\pi}\right)\left[-\lambda_1|x_0'(t)|^2-\|B\|\cdot|x_0(t)|^2\right]dt\\ &= -2\lambda_1\left(k^2m\pi+\frac{2}{15}k^2\pi\right)-2\|B\|\cdot\left(m\pi+\frac{k^2\pi^3}{105}\right)\\ &> -2\lambda_1 mk^2-2\pi\|B\|\cdot(m+k^2)\\ &> 0\end{aligned}$$

provided $m=k^2$ and $k^2>\frac{2\pi\|B\|}{-\lambda_1}$. This shows that there exists $v_0\in L^\beta(\mathbb{R},\mathbb{R}^N)$ such that (L_2) holds. The validity of (L_3) and (L_4) are obvious.

Further, assume V satisfies (H_1)–(H_4) with $H(t,x)$ replaced with $V(t,x)$ and $2N$ replaced with N. Define $V^*(t,u)=\sup_{x\in\mathbb{R}^N}\{(u,x)-V(t,x)\}$. Then $V^*(t,u)$ satisfies (G_1)–(G_5) with $G(t,u)$ replaced with $V^*(t,u)$. By the Legendre reciprocity formula

$$V^{*\prime}(t,u)=x\Leftrightarrow u=V'(t,x),$$

we see that (2.3) is equivalent to

$$Lu-V^{*\prime}(t,u)=0,\qquad u\in L^\beta(\mathbb{R},\mathbb{R}^N). \tag{2.7}$$

Therefore, we have the following result from Theorem 1.1.

Corollary 2.3. *Assume V satisfies (H_1)–(H_4) with $H(t,x)$ replaced with $V(t,x)$ and $2N$ replaced with N. Then (2.3) has a nonzero solution.*

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. The method comes from [4] with some modifications.

Proof of Theorem 1.1. We define the functional I on $L^\beta(\mathbb{R},\mathbb{R}^N)$ by

$$I(u)=\int_{\mathbb{R}}G(t,u)dt-\frac{1}{2}\int_{\mathbb{R}}(Lu,u)dt \tag{3.1}$$

for all $u\in L^\beta(\mathbb{R},\mathbb{R}^N)$. From (G_3), we have

$$0\leq\int_{\mathbb{R}}G(t,u)dt\leq c_2\int_{\mathbb{R}}|u|^\beta dt<+\infty.$$

Noticing that $Lu\in L^\alpha(\mathbb{R},\mathbb{R}^N)$ and L is a bounded linear operator, then $\int_{\mathbb{R}}(Lu,u)dt$ is well defined. Since $G(t,\cdot)$ and $G'(t,\cdot)$ are continuous for a.e. $t\in\mathbb{R}$, from (G_5), we know that the functional I is a C^1 functional. Moreover, a solution of (1.1) correspond to a critical point of the functional I.

Next, we take five steps to prove the existence of the critical point of the functional I.

Step 1. There exists a sequence $\{u_n\}\subset L^\beta(\mathbb{R},\mathbb{R}^N)$ such that $I(u_n)\to c>0$ and $I'(u_n)\to 0$.

By (L_2) and (G_3), for $v_0 \in L^\beta(\mathbb{R}, \mathbb{R}^N), \beta \in (1,2)$ and $s > 0$ we have

$$\begin{aligned} I(sv_0) &= \int_{\mathbb{R}} G(t, sv_0)dt - \frac{s^2}{2}\int_{\mathbb{R}} (Lv_0, v_0)dt \\ &\le c_2 s^\beta \int_{\mathbb{R}} |v_0|^\beta dt - \frac{s^2}{2}\int_{\mathbb{R}} (Lv_0, v_0)dt \\ &\to -\infty \quad \text{as } s \to +\infty, \end{aligned}$$

which shows there is $s_0 > 0$ such that $I(s_0 v_0) < 0$. Set $u_0 = s_0 v_0$ and define

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma([0,1])} I(u),$$

where $\Gamma = \{\gamma \in C([0,1], L^\beta(\mathbb{R}, \mathbb{R}^N)) | \gamma(0) = 0, \gamma(1) = u_0\}$.

By (G_3), we have

$$\begin{aligned} I(u) &\ge c_1 \int_{\mathbb{R}} |u|^\beta dt - \frac{1}{2}\int_{\mathbb{R}} (Lu, u)dt \\ &\ge c_1 \|u\|^\beta_{L^\beta} - \frac{M}{2}\|u\|^2_{L^\beta}, \end{aligned}$$

where $M > 0$ and $\|Lu\|_{L^\alpha} \le M\|u\|_{L^\beta}$. Since $\beta \in (1,2)$, there exists $r \in (0, \|u_0\|_{L^\beta})$ such that $c_1 r^\beta - \frac{M}{2}r^2 > 0$. So $\sup_{u\in\gamma([0,1])} I(u) \ge c_1 r^\beta - \frac{M}{2} r^2 > 0$ and $c > 0$. By [7, Theorem V.1.6], the result follows.

Step 2. We prove that the sequence $\{u_n\} \subset L^\beta(\mathbb{R}, \mathbb{R}^N)$ is bounded and there exist $\delta_2 > \delta_1 > 0$ such that $\|u_n\|_{L^\beta} \in [\delta_1, \delta_2]$.

Clearly,

$$\langle I'(u_n), u_n \rangle = \int_{\mathbb{R}} (G'(t, u_n), u_n)dt - \int_{\mathbb{R}} (Lu_n, u_n)dt.$$

Using (G_3) and (G_4), we have

$$\begin{aligned} I(u_n) + \frac{1}{2}\|I'(u_n)\|_{L^\alpha} \cdot \|u_n\|_{L^\beta} &\ge I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \\ &= \int_{\mathbb{R}} G(t, u_n)dt - \frac{1}{2}\int_{\mathbb{R}} (G'(t, u_n), u_n)dt \\ &\ge (1 - \frac{\beta}{2}) \int_{\mathbb{R}} G(t, u_n)dt \\ &\ge (1 - \frac{\beta}{2}) c_1 \|u_n\|^\beta_{L^\beta}. \end{aligned}$$

Since $c_1 > 0, 1 < \beta < 2$, $I(u_n) \to c > 0$ and $\|I'(u_n)\|_{L^\alpha} \to 0$, we deduce that $\{u_n\}$ is bounded in $L^\beta(\mathbb{R}, \mathbb{R}^N)$.

Again, from (3.1) and (G_3), we have

$$\begin{aligned} I(u_n) &\le c_2 \int_{\mathbb{R}} |u_n|^\beta dt - \frac{1}{2}\int_{\mathbb{R}} (Lu_n, u_n)dt \\ &\le c_2 \|u_n\|^\beta_{L^\beta} + \frac{M}{2}\|u_n\|^2_{L^\beta}. \end{aligned}$$

If there is a subsequence$\{u_{n_k}\}$ such that $\|u_{n_k}\|_{L^\beta} \to 0$, then

$$I(u_{n_k}) \le c_2 \|u_{n_k}\|^\beta_{L^\beta} + \frac{M}{2}\|u_{n_k}\|^2_{L^\beta} \to 0 \Rightarrow c \le 0,$$

which contradicts $c > 0$.

Set $\rho_n(t) = \frac{|u_n(t)|^\beta}{\|u_n\|_{L^\beta}^\beta}$. Then $\int_{-\infty}^{+\infty} \rho_n(t)dt = 1$. By [4, page 145, Lemma] (also see [10,11]), we have three possibilities:

(i) vanishing

$$\sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} \rho_n(t)dt \to 0 \;\text{ as }\; n \to \infty \;\forall R > 0;$$

(ii) concentration

$$\exists y_n \in \mathbb{R} : \forall \varepsilon > 0 \;\exists R > 0 : \int_{y_n-R}^{y_n+R} \rho_n(t)dt \geq 1 - \varepsilon \;\forall n;$$

(iii) dichotomy

$\exists y_n \in \mathbb{R}, \exists \lambda \in (0,1), \exists R_n^1, R_n^2 \in \mathbb{R}$ such that

(a) $R_n^1, R_n^2 \to +\infty$, $\frac{R_n^1}{R_n^2} \to 0$;

(b) $\int_{y_n-R_n^1}^{y_n+R_n^1} \rho_n(t)dt \to \lambda$ as $n \to \infty$;

(c) $\forall \varepsilon > 0 \;\; \exists R > 0$ such that $\int_{y_n-R}^{y_n+R} \rho_n(t)dt \geq \lambda - \varepsilon \;\forall n$;

(d) $\int_{y_n-R_n^2}^{y_n+R_n^2} \rho_n(t)dt \to \lambda$ as $n \to \infty$.

Step 3. Vanishing cannot occur.

Otherwise, there exists a nonnegative sequence $\varepsilon_n \to 0$ such that

$$\int_{s-1}^{s+1} |u_n(t)|^\beta dt \leq \varepsilon_n \|u_n\|_{L^\beta}^\beta \qquad \forall s \in \mathbb{R}.$$

By (L_4), we have

$$\begin{aligned}
|(Lu_n)(t)| &\leq c_0 \int_{-\infty}^{+\infty} e^{-l|t-\tau|}|u_n(\tau)|d\tau \\
&= c_0 \int_{t}^{+\infty} e^{-l|t-\tau|}|u_n(\tau)|d\tau + c_0 \int_{-\infty}^{t} e^{-l|t-\tau|}|u_n(\tau)|d\tau \\
&\leq c_0 e^{lt} \sum_{k=0}^{+\infty} \left(\int_{t+k}^{t+k+1} e^{-\alpha l\tau} d\tau \right)^{\frac{1}{\alpha}} \left(\int_{t+k}^{t+k+1} |u_n(\tau)|^\beta d\tau \right)^{\frac{1}{\beta}} \\
&\quad + c_0 e^{-lt} \sum_{k=0}^{+\infty} \left(\int_{t-k-1}^{t-k} e^{\alpha l\tau} d\tau \right)^{\frac{1}{\alpha}} \left(\int_{t-k-1}^{t-k} |u_n(\tau)|^\beta d\tau \right)^{\frac{1}{\beta}} \\
&\leq 2c_0 \varepsilon_n^{\frac{1}{\beta}} \|u_n\|_{L^\beta} \left(\frac{1-e^{-\alpha l}}{\alpha l} \right)^{\frac{1}{\alpha}} \cdot \frac{1}{1-e^{-l}} \to 0
\end{aligned}$$

as $n \to \infty$ uniformly for $t \in \mathbb{R}$, which implies that $\|Lu_n\|_\infty \to 0$.

From $L : L^\beta(\mathbb{R}, \mathbb{R}^N) \to W^{1,\beta}(\mathbb{R}, \mathbb{R}^N) \bigcap L^\gamma(\mathbb{R}, \mathbb{R}^N)$ is a bounded linear operator for $\gamma \geq \beta$, we obtain $\|Lu_n\|_{L^\beta} \leq c_5 \|u_n\|_{L^\beta}$, where $c_5 > 0$. Since

$$\|Lu_n\|_{L^\alpha}^\alpha = \int_{\mathbb{R}} |Lu_n|^\alpha dt \leq \|Lu_n\|_\infty^{\alpha-\beta} \int_{\mathbb{R}} |Lu_n|^\beta dt \leq c_5 \|u_n\|_{L^\beta} \|Lu_n\|_\infty^{\alpha-\beta},$$

we have $\|Lu_n\|_{L^\alpha} \to 0$. By ($G_3$) and the convexity of $G(t,\cdot)$, $G(t,0) \equiv 0$ and $G(t,u_n) \le (G'(t,u_n),u_n)$. So

$$\int_{\mathbb{R}} |u_n|^\beta dt \le \frac{1}{c_1}\int_{\mathbb{R}} (G'(t,u_n),u_n)dt \le \frac{1}{c_1}\|G'(t,u_n)\|_{L^\alpha}\cdot\|u_n\|_{L^\beta} \to 0,$$

since $G'(t,u_n) = Lu_n + I'(u_n) \to 0$ in $L^\alpha(\mathbb{R},\mathbb{R}^N)$. This is a contradiction to $\|u_n\|_{L^\beta} \ge \delta_1 > 0$.

Step 4. Concentration implies the existence of a nontrivial solution of (1.1).

If concentration occurs, we set

$$w_n(t) = u_n(t+y_n), \qquad v_n(t) = \frac{w_n(t)}{\|w_n\|_{L^\beta}}.$$

Then $\int_{\mathbb{R}} |v_n(t)|^\beta dt = 1$ and for every $\varepsilon_1 > 0$ there exists $R > 0$ such that

$$1-\varepsilon_1 \le \int_{-R}^{R} |v_n(t)|^\beta dt \le 1. \tag{3.2}$$

We claim there is $\bar{z}$ and a subsequence denoted also by itself such that

$$Lv_n \to \bar{z} \quad \text{in } L^\alpha(\mathbb{R},\mathbb{R}^N). \tag{3.3}$$

In fact it suffices to show that for every $\varepsilon > 0$ there exist $z_\varepsilon \in L^\alpha(\mathbb{R},\mathbb{R}^N)$ and subsequence v_{n_j} such that

$$\|Lv_{n_j} - z_\varepsilon\|_{L^\alpha} \le \varepsilon.$$

Let $v_n^{(1)}(t) = v_n(t)\chi_{[-R,R]}(t)$ and $v_n^{(2)}(t) = v_n(t) - v_n^{(1)}(t)$. By ($L_1$), for every $t_0 > 0$ there exist $\{v_{n_j}^{(1)}\}$ and $u_\varepsilon^{(1)} \in C([-t_0,t_0],\mathbb{R}^N)$ such that $Lv_{n_j}^{(1)} \to u_\varepsilon^{(1)}$ in $C([-t_0,t_0],\mathbb{R}^N)$. Define $u_\varepsilon(t) = u_\varepsilon^{(1)}(t)$ for $t \in [-t_0,t_0]$ and $u_\varepsilon(t) = 0$ otherwise. Then

$$\|Lv_{n_j} - u_\varepsilon\|_{L^\alpha} \le \|Lv_{n_j}^{(2)}\|_{L^\alpha} + \|Lv_{n_j}^{(1)} - u_\varepsilon\|_{L^\alpha} \le M\varepsilon_1^{\frac{1}{\beta}} + \|Lv_{n_j}^{(1)} - u_\varepsilon\|_{L^\alpha},$$

and

$$\begin{aligned}
&\left(\int_{-\infty}^{+\infty} |Lv_{n_j}^{(1)} - u_\varepsilon|^\alpha dt\right)^{\frac{1}{\alpha}} \\
&\le \left(\int_{|t|\ge t_0} |Lv_{n_j}^{(1)}|^\alpha dt + 2t_0\|Lv_{n_j}^{(1)} - u_\varepsilon\|_{C[-t_0,t_0]}^\alpha\right)^{\frac{1}{\alpha}} \\
&\le c_0\left(\int_{|t|\ge t_0}\left(\int_{-R}^{R} e^{-l|t-\tau|}|v_{n_j}^{(1)}(\tau)|d\tau\right)^\alpha dt\right)^{\frac{1}{\alpha}} + (2t_0)^{\frac{1}{\alpha}}\|Lv_{n_j}^{(1)} - u_\varepsilon\|_{C[-t_0,t_0]} \\
&\le 2c_0(\alpha l)^{-\frac{2}{\alpha}}e^{-l(t_0-R)}\left(\int_{-R}^{R}|v_{n_j}^{(1)}(\tau)|^\beta d\tau\right)^{\frac{1}{\beta}} + (2t_0)^{\frac{1}{\alpha}}\|Lv_{n_j}^{(1)} - u_\varepsilon\|_{C[-t_0,t_0]} \\
&\le 2c_0(\alpha l)^{-\frac{2}{\alpha}}e^{-l(t_0-R)} + (2t_0)^{\frac{1}{\alpha}}\|Lv_{n_j}^{(1)} - u_\varepsilon\|_{C[-t_0,t_0]}
\end{aligned}$$

via (1) of Lemma 2.1 and $\int_{\mathbb{R}} |v_n(t)|^\beta dt = 1$, where $t_0 > R$.

For any $\varepsilon > 0$, there is $\varepsilon_1 > 0$ such that $M\varepsilon_1^{\frac{1}{\beta}} \le \frac{\varepsilon}{3}$, and there exists $R = R(\varepsilon_1) > 0$ such that (3.2) is satisfied. For the above $R > 0$, there exists $t_0 > R$ such that

$$2c_0(\alpha l)^{-\frac{2}{\alpha}}e^{-l(t_0-R)} \le \frac{\varepsilon}{3}.$$

Then we can choose subsequence v_{n_j} such that

$$(2t_0)^{\frac{1}{\alpha}}\|Lv_{n_j}^{(1)} - u_\varepsilon\|_{C[-t_0,t_0]} \le \frac{\varepsilon}{3}$$

via (L_1). It follows that

$$\|Lv_{n_j} - u_\varepsilon\|_{L^\alpha} \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

From (3.3) and the boundedness of $\|w_n\|_{L^\beta}$, there exists $z \in L^\alpha(\mathbb{R}, \mathbb{R}^N)$ such that $Lw_n \to z$ in $L^\alpha(\mathbb{R}, \mathbb{R}^N)$. We assume $\frac{y_n}{T} \in \mathbf{Z}$. It follows that $I(w_n) = I(u_n)$ and that $I'(w_n)(t) = I'(u_n)(t + y_n)$, and $I(w_n) \to c$, $I'(w_n) \to 0$ in $L^\alpha(\mathbb{R}, \mathbb{R}^N)$. Then

$$z_n(t) = G'(t, w_n) = I'(w_n)(t) + (Lw_n)(t) \to z \quad \text{in } L^\alpha(\mathbb{R}, \mathbb{R}^N).$$

We have $w_n = G^{*\prime}(t, z_n) \to G^{*\prime}(t, z) = w$ on $L^\beta(\mathbb{R}, \mathbb{R}^N)$. Taking limit on both sides of

$$G'(t, w_n) - Lw_n = I'(w_n),$$

we have $G'(t, w) - Lw = 0$, i.e., $u = w$ is a nontrivial solution of (1.1).

Step 5. Dichotomy also leads to a nontrivial solution of (1.1).

If dichotomy occurs, we set

$$\begin{aligned}
w_n(t) &= u_n(t + y_n),\\
w_n^{(1)}(t) &= w_n(t)\chi_{[-R_n^1, R_n^1]}(t),\\
w_n^{(2)}(t) &= w_n(t)(1 - \chi_{[-R_n^2, R_n^2]}(t)),\\
w_n^{(3)}(t) &= w_n(t) - w_n^{(1)}(t) - w_n^{(2)}(t),\\
v_n^{(1)}(t) &= \frac{w_n^{(1)}(t)}{\|w_n^{(1)}\|_{L^\beta}}.
\end{aligned}$$

By (b) of the dichotomy, we have

$$\int_{-\infty}^{+\infty} \frac{|w_n^{(1)}(t)|^\beta}{\|w_n\|_{L^\beta}^\beta} dt = \int_{-R_n^1}^{R_n^1} \frac{|w_n(t)|^\beta}{\|w_n\|_{L^\beta}^\beta} dt \to \lambda > 0.$$

From $\delta_2 \ge \|w_n\|_{L^\beta} = \|u_n\|_{L^\beta} \ge \delta_1$, we can see that there exists $\delta_3 > 0$ such that $\|w_n^{(1)}\|_{L^\beta} > \delta_3$. By Step 4 and (L_1), $w_n^{(1)}(t) \to z$ in $L^\beta(\mathbb{R}, \mathbb{R}^N)$ and $\|z\|_{L^\beta} \ge \delta_3$. We will show that $I'(w_n^{(1)}) \to 0$, and hence $I'(z) = 0$, that is, $u = z$ is a nontrivial solution of (1.1). In fact, for any $u \in L^\beta(\mathbb{R}, \mathbb{R}^N)$, as the splitting of w_n, $u = u^{(1)} + u^{(2)} + u^{(3)}$, and

$$\begin{aligned}
\langle I'(w_n^{(1)}), u\rangle &= \int_{-\infty}^{+\infty} (G'(t, w_n^{(1)}), u^{(1)}) dt - \int_{-\infty}^{+\infty} (Lw_n^{(1)}, u) dt\\
&= \langle I'(w_n), u^{(1)}\rangle - \int_{-\infty}^{+\infty} (Lw_n^{(1)}, u^{(2)} + u^{(3)}) dt + \int_{-\infty}^{+\infty} (L(w_n^{(2)} + w_n^{(3)}), u^{(1)}) dt.
\end{aligned}$$

In the following we assume $\|u\|_{L^\beta} \le 1$ and the limits will be taken as $n \to +\infty$. From (b) and (d) of the dichotomy, we have

$$\|w_n^{(3)}\|_{L^\beta}^\beta = \int_{-\infty}^{+\infty} |w_n^{(3)}|^\beta dt = \int_{|t| \le R_n^2} |w_n|^\beta dt - \int_{|t| \le R_n^1} |w_n^{(1)}|^\beta dt \to 0,$$

which shows that

$$\left|\int_{-\infty}^{+\infty}(Lw_n^{(3)},u^{(1)})dt\right| \le M\|w_n^{(3)}\|_{L^\beta}\|u^{(1)}\|_{L^\beta} \le M\|w_n^{(3)}\|_{L^\beta} \to 0. \tag{3.4}$$

Using (L$_4$), (2) of Lemma 2.1 and (a) of the dichotomy, we have

$$\begin{aligned}\left|\int_{-\infty}^{+\infty}(Lw_n^{(2)},u^{(1)})dt\right| &\le c_0\int_{-R_n^1}^{R_n^1}|u(t)|\int_{|\tau|\ge R_n^2}e^{-l|t-\tau|}|w_n(\tau)|d\tau dt\\ &\le 2c_0(\alpha l)^{-\frac{2}{\alpha}}e^{-l(R_n^2-R_n^1)}\|u\|_{L^\beta}\|w_n\|_{L^\beta}\\ &\le 2c_0(\alpha l)^{-\frac{2}{\alpha}}e^{-l(R_n^2-R_n^1)}\delta_2 \to 0\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\left|\int_{-\infty}^{+\infty}(Lw_n^{(1)},u^{(2)})dt\right| &= \left|\int_{-\infty}^{+\infty}(Lu^{(2)},w_n^{(1)})dt\right|\\ &\le 2c_0(\alpha l)^{-\frac{2}{\alpha}}e^{-l(R_n^2-R_n^1)}\delta_2 \to 0.\end{aligned} \tag{3.6}$$

By (c) of the dichotomy, we have that for any $\varepsilon_1 > 0$ there is $R > 0$ such that $\int_{-R}^{R}\frac{|w_n(t)|^\beta}{\|w_n\|_{L^\beta}^\beta}dt \ge \lambda - \varepsilon_1$. Using (b) of the dichotomy, we obtain $\int_{R\le|\tau|\le R_n^1}|w_n(\tau)|^\beta d\tau \le \varepsilon_1\|w_n\|_{L^\beta}^\beta$. By (L$_4$), (3) of Lemma 2.1 and (a) of the dichotomy, we have

$$\begin{aligned}\left|\int_{-\infty}^{+\infty}(Lw_n^{(1)},u^{(3)})dt\right| &\le c_0\int_{R_n^1\le|t|\le R_n^2}|u(t)|\int_{|\tau|\le R_n^1}e^{-l|t-\tau|}|w_n(\tau)|d\tau dt\\ &\le 2c_0(\alpha l)^{-\frac{2}{\alpha}}\|u\|_{L^\beta}\left[e^{-l(R_n^1-R)}\|w_n\|_{L^\beta}+\left(\int_{R\le|\tau|\le R_n^1}|w_n(\tau)|^\beta d\tau\right)^{\frac{1}{\beta}}\right]\\ &\le 2c_0(\alpha l)^{-\frac{2}{\alpha}}\|u\|_{L^\beta}\|w_n\|_{L^\beta}\left(e^{-l(R_n^1-R)}+\varepsilon_1^{\frac{1}{\beta}}\right)\\ &\le 2c_0(\alpha l)^{-\frac{2}{\alpha}}\delta_2\left(e^{-l(R_n^1-R)}+\varepsilon_1^{\frac{1}{\beta}}\right)\\ &\to 2c_0(\alpha l)^{-\frac{2}{\alpha}}\delta_2\varepsilon_1^{\frac{1}{\beta}}.\end{aligned} \tag{3.7}$$

Noticing $I'(w_n)\to 0$, from (3.4)–(3.7), for any $\epsilon > 0$ choosing $\varepsilon_1 > 0$ satisfying $2c_0(\alpha l)^{-\frac{2}{\alpha}}\delta_2\varepsilon_1^{\frac{1}{\beta}} \le \epsilon$, we find that $\limsup_{n\to+\infty}\|I'(w_n^{(1)})\|_{L^\beta} \le \epsilon$ and hence $I'(w_n^{(1)}) \to 0$. The proof is complete. □

Acknowledgements

This research was supported by the National Natural Science Foundation of China (Grant No. 12071219).

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