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Ground-state solutions to a class of modified Kirchhoff-type transmissiom problems with critical perturbation

Ying Zhang, **Zhanping Liang**, **Xiaoli Zhu** and **Fuyi Li**✉

School of Mathematical Sciences, Shanxi University, Taiyuan 030006, Shanxi, P.R.China

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Abstract. This paper discusses a class of modified Kirchhoff-type transmission problems with critical perturbation. We establish an existence result of the ground-state solutions by using perturbation methods. Meanwhile, the limit properties of solution sequence are investigated.

Keywords: modified Kirchhoff-type transmission problem, critical perturbation, ground-state solution.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\Gamma := \partial\Omega$, $\Omega_1 \subset \mathbb{R}^3$ be a subdomain of Ω with a smooth boundary $\Sigma := \partial\Omega_1$ and $\overline{\Omega}_1 \subset \Omega$. Assume that $\Omega_2 = \Omega \setminus \overline{\Omega}_1$ is connected. Obviously, $\Gamma \cap \Sigma = \emptyset$ and $\partial\Omega_2 = \Gamma \cup \Sigma$. In the present paper we study the existence of solutions for the following Kirchhoff-type transmission problem

$$\begin{cases} \alpha\left(\int_{\Omega_1} g^2(u)|\nabla u|^2\right)\left[-\operatorname{div}\left(g^2(u)\nabla u\right)+g(u)g'(u)|\nabla u|^2\right]=f(u)+\lambda\phi(u), & \text{in } \Omega_1,\\ \beta\left(\int_{\Omega_2} g^2(v)|\nabla v|^2\right)\left[-\operatorname{div}\left(g^2(v)\nabla v\right)+g(v)g'(v)|\nabla v|^2\right]=h(v)+\lambda\psi(v), & \text{in } \Omega_2,\\ v=0, & \text{on } \Gamma,\\ u=v, & \text{on } \Sigma,\\ \alpha\left(\int_{\Omega_1} g^2(u)|\nabla u|^2\right)\dfrac{\partial u}{\partial \nu}=\beta\left(\int_{\Omega_2} g^2(v)|\nabla v|^2\right)\dfrac{\partial v}{\partial \nu}, & \text{on } \Sigma, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}_+ := [0,\infty)$ and ν is the unit outward normal vector to $\partial\Omega_1$. This system is a modified version of Kirchhoff-type transmission problem because the appearance of nonlocal terms $\int_{\Omega_1} g^2(u)|\nabla u|^2$ and $\int_{\Omega_2} g^2(v)|\nabla v|^2$.

✉Corresponding author. E-mail: fyli@sxu.edu.cn.

There are two motivations for studying equation (1.1). The first one is the generalized quasilinear Schrödinger equations. The second one is the classical Kirchhoff-type transmission problem.

In 2015, Deng, Peng, and Yan in [9] researched the generalized quasilinear Schrödinger equations

$$-\operatorname{div}\left(g^2(u)\nabla u\right)+g(u)g'(u)|\nabla u|^2+V(x)u=f(x,u), \qquad x\in\mathbb{R}^N, \tag{1.2}$$

where $N \geqslant 3$, the potential function $V \in C(\mathbb{R}^N)$ and $f \in C(\mathbb{R}^N \times \mathbb{R})$. If we take $g^2(t) = 1+\left[(l(t^2))'\right]^2/2$ for $t \in \mathbb{R}$ and l being a suitable function defined on $\mathbb{R}_+$, then the equation (1.2) turns into

$$-\Delta u+V(x)u-\Delta[l(u^2)]l'(u^2)u=f(x,u), \qquad x\in\mathbb{R}^N. \tag{1.3}$$

Solutions of (1.3) is related to the existence of solitary wave solutions for the following quasilinear Schrödinger equation

$$i\partial_t z=-\Delta z+V(x)z-f(x,z)-\Delta[l(|z|^2)]l'(|z|^2)z, \qquad x\in\mathbb{R}^N. \tag{1.4}$$

This quasilinear version of Schrödinger equations is derived from several models of various physical phenomena. The equation (1.4) is called the superfluid film equation in plasma physics when $l(t) = t$ for $t \in \mathbb{R}_+$, see [13] or [14, 15]. If $l(t) = (1+t)^{1/2}$ for $t \in \mathbb{R}_+$, the equation (1.4) was used for the self-channeling of a high-power ultrashort laser in matter, see [4,5,7,24]. In mathematics, many results about the equation (1.3) with $l(t) = t^\alpha$ for some $\alpha \geqslant 1$ have been obtained, see [1, 2, 6, 8, 10, 18–20, 22, 23, 29–31] and the references therein. Equation (1.3) with a general l was studied in the recent papers [9, 25]. We can see that the equation (1.2) is more general and more practical than the equation (1.3).

If we choose $g(t) = 1$ for $t \in \mathbb{R}$ and $\lambda = 0$, then the equation (1.1) becomes the classical Kirchhoff-type transmission problem

$$\begin{cases} -\alpha\left(\int_{\Omega_1}|\nabla u|^2\right)\Delta u=f(u), & \text{in } \Omega_1,\\ -\beta\left(\int_{\Omega_2}|\nabla v|^2\right)\Delta v=h(v), & \text{in } \Omega_2,\\ v=0, & \text{on } \Gamma,\\ u=v, & \text{on } \Sigma,\\ \alpha\left(\int_{\Omega_1}|\nabla u|^2\right)\dfrac{\partial u}{\partial \nu}=\beta\left(\int_{\Omega_2}|\nabla v|^2\right)\dfrac{\partial v}{\partial \nu}, & \text{on } \Sigma. \end{cases} \tag{1.5}$$

It is well known that this problem is related to the stationary analogue of the problem

$$\begin{cases} u_{tt}-\alpha\left(\int_{\Omega_1}|\nabla u|^2\right)\Delta u=f(u), & x\in\Omega_1, t>0,\\ v_{tt}-\beta\left(\int_{\Omega_2}|\nabla v|^2\right)\Delta v=g(v), & x\in\Omega_2, t>0,\\ v=0, & \text{on } \Gamma,\\ u=v, & \text{on } \Sigma,\\ \alpha\left(\int_{\Omega_1}|\nabla u|^2\right)\dfrac{\partial u}{\partial \nu}=\beta\left(\int_{\Omega_2}|\nabla v|^2\right)\dfrac{\partial v}{\partial \nu}, & \text{on } \Sigma,\\ u(0)=u_0, u_t(0)=u_1, & x\in\Omega_1,\\ v(0)=v_0, v_t(0)=v_1, & x\in\Omega_2, \end{cases} \tag{1.6}$$

which models the transverse vibrations of a membrane composed of two different materials in Ω_1 and Ω_2. According to [21], we call the problem (1.6) a transmission problem because the boundary conditions $u = v$ and $\alpha\big(\int_{\Omega_1}|\nabla u|^2\big)\frac{\partial u}{\partial \nu} = \beta\big(\int_{\Omega_2}|\nabla v|^2\big)\frac{\partial v}{\partial \nu}$ on Σ. This transmission problem (1.6) arises in physics and biology phenomena, such as in the study of electromagnetic processes in ferromagnetic media with different dielectric constants [3], and in thinking about the population distribution of subjects living in an environment composed of different ecological media. In 2003, Ma and Muñoz Rivera [21] discussed the existence and nonexistence of positive solution to the Kirchhoff-type transmission problem (1.5) by using minimization arguments with f and g having subcritical growth. In [16], Li, Zhang, Zhu, and Liang investigated the existence of the ground-state solutions to the following Kirchhoff-type transmission problem with critical perturbation

$$\begin{cases} -\alpha\left(\int_{\Omega_1}|\nabla u|^2\right)\Delta u = f(u) + \lambda u^5, & \text{in } \Omega_1,\\ -\beta\left(\int_{\Omega_2}|\nabla v|^2\right)\Delta v = g(v) + \lambda v^5, & \text{in } \Omega_2,\\ v = 0, & \text{on } \Gamma,\\ u = v, & \text{on } \Sigma,\\ \alpha\left(\int_{\Omega_1}|\nabla u|^2\right)\dfrac{\partial u}{\partial \nu} = \beta\left(\int_{\Omega_2}|\nabla v|^2\right)\dfrac{\partial v}{\partial \nu}, & \text{on } \Sigma. \end{cases} \tag{1.7}$$

Here, we will establish the existence of ground-state solutions to Kirchhoff-type transmission problem with more general g and more general perturbation terms ϕ and ψ. To obtain the existence of ground-state solutions to the more general Kirchhoff-type transmission problem (1.1), we assume that four pairs of functions (α, g, f), (β, g, h), (α, g, ϕ), and (β, g, ψ) belong to the set $\mathcal{A}$, where a pair of functions (α, g, f) is said to belongs to $\mathcal{A}$, if (α, g, f) satisfies the following assumptions

(A_0) $\alpha \in C^1(\mathbb{R}_+)$ is an increasing function and $\alpha(0) > 0$;

(A_1) there exists $\gamma \in (0, 2)$ such that $[\alpha(s) - \alpha(0)]/s^\gamma$ is decreasing on $(0, \infty)$;

(G) $g \in C^1(\mathbb{R}, \mathbb{R}_+)$ is even with $g'(s) \geqslant 0$ for $s \in \mathbb{R}_+$ and $g(0) = 1$;

(F_0) $f \in C^1(\mathbb{R}, \mathbb{R})$ and $\lim_{s\to 0} f(s)/s = 0$;

(F_1) there exists $l_f \in \mathbb{R}$ such that

$$\lim_{|s|\to\infty} \frac{f(s)}{g(s)G^5(s)} = l_f,$$

where $G(s) = \int_0^s g(t)\mathrm{d}t$ for $s \in \mathbb{R}$. And if $l_f = 0$, we call that f has a quasicritical growth; if $l_f \neq 0$, we call that f has a critical growth;

(F_2) $f(s)/(g(s)|G(s)|^{2\gamma}G(s))$ is nondecreasing on $(0, \infty)$ and nonincreasing on $(-\infty, 0)$, and $\lim_{|s|\to\infty} F(s)/|G(s)|^{2\gamma+2} = \infty$, where $F(s) = \int_0^s f(t)\mathrm{d}t$ for $s \in \mathbb{R}$ and γ is as in (A_1).

Remark 1.1. Assuming that g satisfies (G) and $\gamma \in (0, 2)$, let $f(s) = g(s)|G(s)|^{2\gamma}G(s)\ln|G(s)|$ and $\phi(s) = g(s)(G(s))^5$ for $s \in \mathbb{R}$. Then f and ϕ satisfy (F_0), (F_1), and (F_2).

Example 1.2. Let $\alpha(s) = 1 + s^2$ for $s \in \mathbb{R}_+$, and for $\gamma \in (0, 2)$, define $g(s) = s^2 + 1$, $f(s) = (s^2+1)\left|s^3/3+s\right|^{2\gamma}(s^3/3+s)\ln\left|s^3/3+s\right|$, $\phi(s) = (s^2+1)\left(s^3/3+s\right)^5$ for $s \in \mathbb{R}$. Then (α, g, f) and (α, g, ϕ) belong to $\mathcal{A}$.

Example 1.3. For $a, b > 0$, let $\alpha(s) = a + bs$ for $s \in \mathbb{R}_+$, and for $\gamma \in (0,2)$, define $g(s) = \sqrt{2s^2+1}$,

$$f(s) = \frac{\sqrt{2}}{4}\sqrt{2s^2+1}\left|\sqrt{2}s\sqrt{2s^2+1} + \ln\left(\sqrt{2}s + \sqrt{2s^2+1}\right)\right|^{2\gamma}\left(\sqrt{2}s\sqrt{2s^2+1}\right.$$
$$\left. + \ln\left(\sqrt{2}s + \sqrt{2s^2+1}\right)\right) \times \ln\left|\sqrt{2}s\sqrt{2s^2+1} + \ln\left(\sqrt{2}s + \sqrt{2s^2+1}\right)\right|,$$

$\phi(s) = \frac{\sqrt{2}}{4}\sqrt{2s^2+1}\left[\sqrt{2}s\sqrt{2s^2+1} + \ln\left(\sqrt{2}s + \sqrt{2s^2+1}\right)\right]^5$ for $s \in \mathbb{R}$. Then (α, g, f) and (α, g, ϕ) belong to $\mathcal{A}$.

Remark 1.4. We know that the critical exponent of equation (1.7) is 6 which has a significant influence on the properties of the solution. The critical exponent of equation (1.1) is different for different g and the critical exponent depends on G^6. This is an interesting phenomenon. For example, when $g(s) = \sqrt{2s^2+1}$ for $s \in \mathbb{R}$, the critical exponent is 12; when $g(s) = s^2 + 1$ for $s \in \mathbb{R}$, the critical exponent is 18.

For any given subdomain D of $\mathbb{R}^3$, the standard norm on $L^p(D)$ is denoted by $|\cdot|_{p,D}$ for $p \in [1, \infty)$. Let $H^1(\Omega_1)$ and $H^1(\Omega_2)$ be the usual Sobolev spaces. Then $H^1(\Omega_1) \times H^1(\Omega_2)$ is also a Sobolev space with the norm

$$\|(u,v)\| = \left(|\nabla u|^2_{2,\Omega_1} + |u|^2_{2,\Omega_1} + |\nabla v|^2_{2,\Omega_2} + |v|^2_{2,\Omega_2}\right)^{1/2}, \qquad (u,v) \in H^1(\Omega_1) \times H^1(\Omega_2). \tag{1.8}$$

Our analysis is based on the following Sobolev space

$$E = \{(u,v) \in H^1(\Omega_1) \times H^1_\Gamma(\Omega_2) : u = v \text{ on } \Sigma\},$$

where

$$H^1_\Gamma(\Omega_2) = \{v \in H^1(\Omega_2) : v = 0 \text{ on } \Gamma\}.$$

In [21] Ma and Muñoz Rivera established the following lemma which gave the definition of norm for the Sobolev space E.

Lemma 1.5 ([21, Lemma 1])**.** *E is a closed subspace of $H^1(\Omega_1) \times H^1(\Omega_2)$, and*

$$\|(u,v)\|_E = \left(|\nabla u|^2_{2,\Omega_1} + |\nabla v|^2_{2,\Omega_2}\right)^{1/2}, \qquad (u,v) \in E,$$

defines also a norm on E, which is equivalent to the standard norm (1.8).

Remark 1.6. From Lemma 1.5, we know that the space E is embedded into $L^p(\Omega_1) \times L^q(\Omega_2)$ for all $p, q \in [1,6]$, and these embeddings are compact for all $p, q \in [1,6)$. In particular, for each $p = q \in [1,6]$, there exists $\nu_p > 0$ such that

$$|(u,v)|_p := \left(|u|^p_{p,\Omega_1} + |v|^p_{p,\Omega_2}\right)^{1/p} \leqslant \nu_p \|(u,v)\|_E, \qquad (u,v) \in E. \tag{1.9}$$

In order to solve the transmission problem (1.1), due to the appearance of nonlocal terms $\int_{\Omega_1} g^2(u)|\nabla u|^2$ and $\int_{\Omega_2} g^2(v)|\nabla v|^2$, the potential working space seems to be

$$E_0 = \left\{(u,v) \in E : \int_{\Omega_1} g^2(u)|\nabla u|^2 < \infty, \int_{\Omega_2} g^2(v)|\nabla v|^2 < \infty\right\}.$$

Obviously, E_0 may not be a linear space under the assumed condition of (G). To avoid this drawback, we gave a change of variables,

$$(u,v) = \left(G^{-1}(u_1), G^{-1}(v_1)\right), \qquad (u_1,v_1) \in E,$$

which is motivated by [9, 25]. According to the properties of $g, G,$ and G^{-1} which will be given in Section 2, if $(u_1,v_1) \in E$, then $(u,v) = (G^{-1}(u_1), G^{-1}(v_1)) \in E$ (see Remark 2.3), $\int_{\Omega_1} g^2(u)|\nabla u|^2 = \int_{\Omega_1} g^2(G^{-1}(u_1))|\nabla G^{-1}(u_1)|^2 = |\nabla u_1|_2^2 < \infty$, and $\int_{\Omega_2} g^2(v)|\nabla v|^2 = \int_{\Omega_2} g^2(G^{-1}(v_1))|\nabla G^{-1}(v_1)|^2 = |\nabla v_1|_2^2 < \infty$. Thus, it follows from the change of variables that E can be used as the working space and the transmission problem (1.1) turns into

$$\begin{cases} -\alpha\left(\int_{\Omega_1} |\nabla u_1|^2\right) g(G^{-1}(u_1))\Delta u_1 = f(G^{-1}(u_1)) + \lambda\phi(G^{-1}(u_1)), & \text{in } \Omega_1, \\ -\beta\left(\int_{\Omega_2} |\nabla v_1|^2\right) g(G^{-1}(v_1))\Delta v_1 = h(G^{-1}(v_1)) + \lambda\psi(G^{-1}(v_1)), & \text{in } \Omega_2, \\ v_1 = 0, & \text{on } \Gamma, \\ u_1 = v_1, & \text{on } \Sigma, \\ \alpha\left(\int_{\Omega_1} |\nabla u_1|^2\right)\frac{\partial u_1}{\partial \nu} = \beta\left(\int_{\Omega_2} |\nabla v_1|^2\right)\frac{\partial v_1}{\partial \nu}, & \text{on } \Sigma. \end{cases} \tag{1.10}$$

Furthermore, we can prove that if $(u_1,v_1) \in E \cap \left(H^2_{\text{loc}}(\Omega_1) \times H^2_{\text{loc}}(\Omega_2)\right)$ is a strong solution to the equation (1.10), then $(u,v) = (G^{-1}(u_1), G^{-1}(v_1)) \in E \cap \left(H^2_{\text{loc}}(\Omega_1) \times H^2_{\text{loc}}(\Omega_2)\right)$ is a strong solution to the equation (1.1). Here, we call that $(u,v) \in E \cap \left(H^2_{\text{loc}}(\Omega_1) \times H^2_{\text{loc}}(\Omega_2)\right)$ is a strong solution to the transmission problem (1.10) or (1.1) if the first two equations in (1.10) or (1.1) hold in the sense of almost everywhere. Actually, we only need to verify that for any an open bounded set $D \subset \mathbb{R}^3$ if $u_1 \in H^2(D)$, then $G^{-1}(u_1) \in H^2(D)$(see Lemma 4.2). Moreover, because of the continuity of $g, G,$ and G^{-1}, to obtain a strong solution to the transmission problem (1.10), it suffices to seek for the weak solution to the following transmission problem

$$\begin{cases} -\alpha\left(\int_{\Omega_1} |\nabla u_1|^2\right)\Delta u_1 = \dfrac{f(G^{-1}(u_1))}{g(G^{-1}(u_1))} + \lambda\dfrac{\phi(G^{-1}(u_1))}{g(G^{-1}(u_1))}, & \text{in } \Omega_1, \\ -\beta\left(\int_{\Omega_2} |\nabla v_1|^2\right)\Delta v_1 = \dfrac{h(G^{-1}(v_1))}{g(G^{-1}(v_1))} + \lambda\dfrac{\psi(G^{-1}(v_1))}{g(G^{-1}(v_1))}, & \text{in } \Omega_2, \\ v_1 = 0, & \text{on } \Gamma, \\ u_1 = v_1, & \text{on } \Sigma, \\ \alpha\left(\int_{\Omega_1} |\nabla u_1|^2\right)\dfrac{\partial u_1}{\partial \nu} = \beta\left(\int_{\Omega_2} |\nabla v_1|^2\right)\dfrac{\partial v_1}{\partial \nu}, & \text{on } \Sigma. \end{cases} \tag{1.11}$$

In fact, if $(u_1,v_1) \in E$ is a weak solution to the transmission problem (1.11), then it should satisfy, for all $(w_1,z_1) \in E$,

$$\begin{aligned} &\alpha\left(|\nabla u_1|^2_{2,\Omega_1}\right)\int_{\Omega_1} \nabla u_1 \cdot \nabla w_1 + \beta\left(|\nabla v_1|^2_{2,\Omega_2}\right)\int_{\Omega_2} \nabla v_1 \cdot \nabla z_1 \\ &\quad = \int_{\Omega_1} \frac{f(G^{-1}(u_1))}{g(G^{-1}(u_1))}w_1 + \int_{\Omega_2} \frac{h(G^{-1}(v_1))}{g(G^{-1}(v_1))}z_1 + \lambda\int_{\Omega_1} \frac{\phi(G^{-1}(u_1))}{g(G^{-1}(u_1))}w_1 + \lambda\int_{\Omega_2} \frac{\psi(G^{-1}(v_1))}{g(G^{-1}(v_1))}z_1. \end{aligned}$$

Hence, $u_1 \in H^1(\Omega_1)$ weakly solves the equation

$$-\alpha\left(|\nabla u_1|^2_{2,\Omega_1}\right)\Delta u_1 = a(x)(1+u_1), \quad \text{in } \Omega_1,$$

with

$$a(x) = \frac{1}{1+u_1(x)}\left(\frac{f(G^{-1}(u_1))}{g(G^{-1}(u_1))} + \lambda\frac{\phi(G^{-1}(u_1))}{g(G^{-1}(u_1))}\right) =: \frac{1}{1+u_1(x)}\left(\widetilde{f}(u_1) + \lambda\widetilde{\phi}(u_1)\right),$$

where $\widetilde{f}(s) := \frac{f(G^{-1}(s))}{g(G^{-1}(s))}$ and $\widetilde{\phi}(s) := \frac{\phi(G^{-1}(s))}{g(G^{-1}(s))}$ for $s \in \mathbb{R}$. The condition (F_2) implies that $a \in L^{3/2}_{\mathrm{loc}}(\Omega_1)$. By the Brézis–Kato theorem, see also [26, Lemma B.3, p. 244] , we know that $u_1 \in L^q_{\mathrm{loc}}(\Omega_1)$ for any $q \in [1,\infty)$. Theorem 8.8 in [11, p. 183] shows that $u_1 \in H^1(\Omega_1) \cap H^2_{\mathrm{loc}}(\Omega_1)$ and

$$-\alpha\left(|\nabla u_1|^2_{2,\Omega_1}\right)\Delta u_1 = \widetilde{f}(u_1) + \lambda\widetilde{\phi}(u_1), \quad \text{a.e. } x \in \Omega_1.$$

Similarly, we can prove that $v_1 \in H^1_\Gamma(\Omega_2) \cap H^2_{\mathrm{loc}}(\Omega_1)$ such that

$$-\beta\left(|\nabla v_1|^2_{2,\Omega_2}\right)\Delta v_1 = \widetilde{h}(v_1) + \lambda\widetilde{\psi}(v_1), \quad \text{a.e. } x \in \Omega_2,$$

where $\widetilde{h}(s) = \frac{h(G^{-1}(s))}{g(G^{-1}(s))}$ and $\widetilde{\psi}(s) = \frac{\psi(G^{-1}(s))}{g(G^{-1}(s))}$ for $s \in \mathbb{R}$. So the problem (1.11) holds in the sense of almost everywhere and $(u_1, v_1) \in \left(H^1(\Omega_1) \cap H^2_{\mathrm{loc}}(\Omega_1)\right) \times \left(H^1_\Gamma(\Omega_2) \cap H^2_{\mathrm{loc}}(\Omega_2)\right)$ is a strong solution to the equation. Here, let $(u, v) = (G^{-1}(u_1), G^{-1}(v_1))$. Then (u, v) is a strong solution to the transmission problem (1.1). For the convenience, removing the subscripts of u_1, v_1 , we rewrite (1.11) as the following transmission problem

$$\begin{cases} -\alpha\left(\int_{\Omega_1}|\nabla u|^2\right)\Delta u = \dfrac{f(G^{-1}(u))}{g(G^{-1}(u))} + \lambda\dfrac{\phi(G^{-1}(u))}{g(G^{-1}(u))}, & \text{in } \Omega_1, \\ -\beta\left(\int_{\Omega_2}|\nabla v|^2\right)\Delta v = \dfrac{h(G^{-1}(v))}{g(G^{-1}(v))} + \lambda\dfrac{\psi(G^{-1}(v))}{g(G^{-1}(v))}, & \text{in } \Omega_2, \\ v = 0, & \text{on } \Gamma, \\ u = v, & \text{on } \Sigma, \\ \alpha\left(\int_{\Omega_1}|\nabla u|^2\right)\dfrac{\partial u}{\partial \nu} = \beta\left(\int_{\Omega_2}|\nabla v|^2\right)\dfrac{\partial v}{\partial \nu}, & \text{on } \Sigma. \end{cases} \tag{1.12}$$

In the following, we make our efforts to find the weak solution to the transmission problem (1.12). To this end, we define the energy functional $I : E \to \mathbb{R}$ associated with the transmission problem (1.12)

$$\begin{aligned} I_\lambda(u, v) = {} & \frac{1}{2}A\left(|\nabla u|^2_{2,\Omega_1}\right) + \frac{1}{2}B\left(|\nabla v|^2_{2,\Omega_2}\right) - \int_{\Omega_1} F(G^{-1}(u)) - \int_{\Omega_2} H(G^{-1}(v)) \\ & - \lambda\int_{\Omega_1}\Phi(G^{-1}(u)) - \lambda\int_{\Omega_2}\Psi(G^{-1}(v)), \qquad (u, v) \in E, \end{aligned}$$

where $A(s) = \int_0^s \alpha(t)\mathrm{d}t, B(s) = \int_0^s \beta(t)\mathrm{d}t$ for $s \in \mathbb{R}_+$, and $H(s) = \int_0^s h(t)\mathrm{d}t, \Phi(s) = \int_0^s \phi(t)\mathrm{d}t$, $\Psi(s) = \int_0^s \psi(t)\mathrm{d}t$ for $s \in \mathbb{R}$. It can be verified that I_λ is of class C^1. And for all $(u, v), (w, z) \in E$,

$$\begin{aligned} \langle I'_\lambda(u, v), (w, z)\rangle = {} & \alpha(|\nabla u|^2_{2,\Omega_1})\int_{\Omega_1}\nabla u\cdot\nabla w + \beta(|\nabla v|^2_{2,\Omega_2})\int_{\Omega_2}\nabla v\cdot\nabla z - \int_{\Omega_1}\frac{f(G^{-1}(u))}{g(G^{-1}(u))}w \\ & - \int_{\Omega_2}\frac{h(G^{-1}(v))}{g(G^{-1}(v))}z - \lambda\int_{\Omega_1}\frac{\phi(G^{-1}(u))}{g(G^{-1}(u))}w - \lambda\int_{\Omega_2}\frac{\psi(G^{-1}(v))}{g(G^{-1}(v))}z. \end{aligned}$$

Let $\widetilde{F}(s) = F(G^{-1}(s)), \widetilde{H}(s) = H(G^{-1}(s)), \widetilde{\Phi}(s) = \Phi(G^{-1}(s))$, and $\widetilde{\Psi}(s) = \Psi(G^{-1}(s))$ for $s \in \mathbb{R}$. Then, for all $(u, v), (w, z) \in E$, we have that

$$I_\lambda(u, v) = \frac{1}{2}A\left(|\nabla u|^2_{2,\Omega_1}\right) + \frac{1}{2}B\left(|\nabla v|^2_{2,\Omega_2}\right) - \int_{\Omega_1}\widetilde{F}(u) - \int_{\Omega_2}\widetilde{H}(v) - \lambda\int_{\Omega_1}\widetilde{\Phi}(u) - \lambda\int_{\Omega_2}\widetilde{\Psi}(v),$$

and

$$\langle I'_\lambda(u,v),(w,z)\rangle = \alpha\left(|\nabla u|^2_{2,\Omega_1}\right)\int_{\Omega_1}\nabla u\cdot\nabla w+\beta\left(|\nabla v|^2_{2,\Omega_2}\right)\int_{\Omega_2}\nabla v\cdot\nabla z$$
$$-\int_{\Omega_1}\widetilde{f}(u)w-\int_{\Omega_2}\widetilde{h}(v)z-\lambda\int_{\Omega_1}\widetilde{\phi}(u)w-\lambda\int_{\Omega_2}\widetilde{\psi}(v)z. \tag{1.13}$$

Then we say that $(u,v)\in E$ is a weak solution to the transmission problem (1.12) if and only if (u,v) is a critical point of the functional I_λ in E, i.e., $I'_\lambda(u,v)=0$. To sum up, it suffices to seek a critical point of the functional I_λ in E to achieve a strong solution to the transmission problem (1.1).

Now, we state our main results through the following theorems.

Theorem 1.7. *Assume that $(\alpha,g,f),(\beta,g,h)\in\mathcal{A}$ with $l_f=l_h=0$, $(\alpha,g,\phi),(\beta,g,\psi)\in\mathcal{A}$ with $l_\phi,l_\psi\neq 0$, and $\phi(s)s>0,\psi(s)s>0$ for $s\neq 0$. Then there exists $\lambda_0>0$ such that both the problem (1.12) and (1.1) have a ground-state solution (u_λ,v_λ) for all $\lambda\in[0,\lambda_0)$. Furthermore, it holds that $(u_\lambda,v_\lambda)\to(u_0,v_0)$ in E as $\lambda\to 0$, where (u_0,v_0) is a ground-state solution to the problem (1.1) with $\lambda=0$.*

Corollary 1.8. *Let $\Omega_2=\emptyset,\alpha(s)=1,g(s)=\sqrt{1+2s^2},f(s)=|s|^{q-2}$, and $\phi(s)=|s|^{10}s$ for $s\in\mathbb{R}$. Then the following equation has a ground-state solution u_λ for all $\lambda\in[0,\lambda_0)$,*

$$\begin{cases}-\Delta u-\Delta(u^2)u=|u|^{q-2}u+\lambda|u|^{10}u, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases} \tag{1.14}$$

where $q\in(4,12)$. Furthermore, it holds that $u_\lambda\to u_0$ in $H^1_0(\Omega)$ as $\lambda\to 0$, where u_0 is a ground-state solution to the above problem with $\lambda=0$.

Remark 1.9. According to [8], for a single quasilinear Schrödinger equation (1.14) in a bounded domain in $\mathbb{R}^3$, there exists a suitable energy level c^* such that if $c(\lambda)<c^*$, then the associated energy functional satisfies the $(\mathrm{PS})_{c(\lambda)}$ condition, where $c^*=S^3/6$ and S is the best Sobolev constant for $\mathcal{D}^{1,2}(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$. However, a large amount of calculations is required to prove that $c(\lambda)<c^*$ by verifying

$$\sup_{t\in\mathbb{R}_+} I_\lambda(tu_\epsilon)<c^*,$$

where u_ϵ is a modification of U and U attains the best Sobolev constant S. In this paper to avoid this difficulty, we adopt the perturbation method from [12,32].

Remark 1.10. Let $g(s)=1$ and $\phi(s)=\psi(s)=s^5$ for $s\in\mathbb{R}$. Then by Theorem 1.7, we have that the transmission problem (1.1) also has a ground-state solution, which has been achieved in [16]. Thus, Theorem 1.7 could be regarded as a generalization of Theorem 1.1 in [16].

This paper is organized as follows. We give some preliminaries in Section 2. Theorem 1.7 is proved in Section 3. Throughout this paper we denote C_i for $i\in\mathbb{N}:=\{1,2,\dots\}$ as constants which can be different from line to line.

2 Preliminaries

In this section we first give some properties of the functions $\alpha,g,\widetilde{f}$, and $A,G,G^{-1},\widetilde{F}$ via the following lemmas.

Lemma 2.1.

(i) Assume that α satisfies the condition (A_0). Then $A(s) \geqslant \alpha(0)s$ for $s \in \mathbb{R}_+$.

(ii) Assume that α satisfies the conditions (A_0) and (A_1). Then $[A(s) - \alpha(0)s]/s^{\gamma+1}$, $\alpha(s)s - (\gamma+1)A(s) + \gamma\alpha(0)s$, and $A(s)/s^{\gamma+1}$ are decreasing on $(0,\infty)$. Furthermore, we have that

$$(\gamma+1)A(s) - \alpha(s)s \geqslant \gamma\alpha(0)s, \qquad s \in \mathbb{R}_+, \tag{2.1}$$

and

$$\alpha'(s)s \leqslant \gamma[\alpha(s) - \alpha(0)] < \gamma\alpha(s), \qquad s \in \mathbb{R}_+. \tag{2.2}$$

Lemma 2.2. *The functions g, G, and G^{-1} have the following properties under the assumption of (G):*

(i) G and G^{-1} are both odd, and

$$t \leqslant G(t) \leqslant g(t)t, \quad t \in \mathbb{R}_+, \qquad s/g(G^{-1}(s)) \leqslant G^{-1}(s) \leqslant s, \quad s \in \mathbb{R}_+;$$

(ii) $\lim_{s\to 0} G^{-1}(s)/s = 1$ and $\lim_{s\to\infty} G^{-1}(s)/s = 1/g(\infty)$, where $g(\infty) = \lim_{s\to\infty} g(s)$;

(iii) $G^{-1}(s)/\left[|s|^{2\gamma} s g(G^{-1}(s))\right]$ is nonincreasing on $(0,\infty)$ and nondecreasing on $(-\infty,0)$;

(iv) $[G^{-1}(s)]^2 - G^{-1}(s)s/g(G^{-1}(s))$ is nondecreasing on $(0,\infty)$ and nonincreasing on $(-\infty,0)$;

(v) if f is a continuous function and (F_2) holds, then $f(G^{-1}(s))s/[(2\gamma+2)g(G^{-1}(s))] - F(G^{-1}(s))$ is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$.

Proof. (i), (ii), and (iv) can be derived from [17, (1), (2), and (4) of Lemma 2.2]. As for (iii), because g is even, we need only to prove that the conclusion holds on $(0,\infty)$. In fact, since $[G(t)/t]^{2\gamma+1}g(t)$ is nondecreasing on $(0,\infty)$, $[G(t)]^{2\gamma+1}g(t)/t$ is also nondecreasing on $(0,\infty)$, and then $G^{-1}(s)/[s^{2\gamma+1}g(G^{-1}(s))]$ is nonincreasing on $(0,\infty)$.

Finally, we prove that (v) holds. Indeed, since $f(t)/[g(t)|G(t)|^{2\gamma}G(t)]$ is nondecreasing on $(0,\infty)$ and nonincreasing on $(-\infty,0)$, according to [17, Lemma A.1], $f(t)G(t)/[(2\gamma+2)g(t)] - F(t)$ is nondecreasing on $(0,\infty)$ and nonincreasing on $(-\infty,0)$, and then $f(G^{-1}(s))s/[(2\gamma+2)g(G^{-1}(s))] - F(G^{-1}(s))$ is nondecreasing on $(0,\infty)$ and nonincreasing on $(-\infty,0)$, that is, (v) holds. The proof is complete. □

Remark 2.3. Let $(u,v) \in E$. Then it follows from $g(t) \geqslant 1$ for $t \in \mathbb{R}_+$, and (i) of Lemma 2.2 that $(G^{-1}(u), G^{-1}(v)) \in E$.

Lemma 2.4. *Assume that g satisfies (G) and f satisfies (F_0), (F_1), and (F_2). Let $\widetilde{f}(s) = \frac{f(G^{-1}(s))}{g(G^{-1}(s))}$ for $s \in \mathbb{R}$. Then $\widetilde{f}$ has the following properties:*

(F_0') *$\widetilde{f} \in C^1(\mathbb{R})$ and $\lim_{s\to 0} \widetilde{f}(s)/s = 0$;*

(F_1')

$$\lim_{|s|\to\infty} \frac{\widetilde{f}(s)}{s^5} = l_f;$$

(F_2') *$\widetilde{f}(s)/(|s|^{2\gamma}s)$ is nondecreasing on $(0,\infty)$ and nonincreasing on $(-\infty,0)$, and*

$$\lim_{|s|\to\infty} \widetilde{F}(s)/|s|^{2\gamma+2} = \infty.$$

From [16], the function $\widetilde{f}$ possesses some other properties as mentioned in the following Remark 2.5. With those properties, we know that Lemmas 2.6–2.8 hold.

Remark 2.5. It follows from (F_2') and [17, Lemma A.1] that $\widetilde{f}(s)s - 2(\gamma+1)\widetilde{F}(s)$ is nondecreasing on $\mathbb{R}_+$ and nonincreasing $(-\infty, 0]$, and then

$$\widetilde{f}(s)s - 2(\gamma+1)\widetilde{F}(s) \geqslant 0, \qquad s \in \mathbb{R}, \tag{2.3}$$

and

$$\widetilde{f}'(s)s - (2\gamma+1)\widetilde{f}(s) \geqslant 0, \qquad s \in \mathbb{R}_+. \tag{2.4}$$

Lemma 2.6. *Suppose that f satisfies the conditions (F_0) and (F_1) and g satisfies the conditions (G). Then for each $u \in H^1(\Omega)$, one has that*

$$\lim_{t\to 0} \int_{\Omega_1} \frac{\widetilde{f}(tu)u}{t} = 0.$$

Lemma 2.7. *Suppose that f satisfies the conditions (F_0) and (F_1) and g satisfies the conditions (G). If $u_n \rightharpoonup u \neq 0$ in $H^1(\Omega)$ and $|t_n| \to \infty$, then*

$$\lim_{n\to\infty} \int_{\Omega} \frac{\widetilde{f}(t_n u_n)u_n}{|t_n|^{2\gamma} t_n} = \infty.$$

Lemma 2.8. *Suppose that f satisfies the conditions (F_0) and (F_1) and g satisfies the conditions (G). Then for each $u \in H^1(\Omega)$ and $u \neq 0$, it holds that*

$$\lim_{|t|\to\infty} \int_{\Omega_1} \frac{\widetilde{f}(tu)u}{|t|^{2\gamma} t} = \infty.$$

3 Existence and convergence of ground-state solutions

In this section, assuming that the all conditions of Theorem 1.7 hold, we will establish the existence of ground-state solutions to the problems (1.12) and complete the proof of Theorem 1.7. First, we verify that the functional I_λ has a mountain pass geometric structure and the functional I_0 satisfies the Palais–Smale (PS for short) condition.

For each $\lambda \in \mathbb{R}_+$, let

$$\Gamma_\lambda = \{\gamma \in C([0,1], E) : \gamma(0) = 0, I_\lambda(\gamma(1)) < 0\}$$

and define

$$c(\lambda) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)).$$

Lemma 3.1. *$\Gamma_\lambda \neq \emptyset$ and $c(\lambda) > 0$ for $\lambda \in \mathbb{R}_+$.*

Proof. For any given $\varepsilon \in \left(0, \left[2(1+\lambda)\nu_2^2\right]^{-1} \min\{\alpha(0), \beta(0)\}\right)$ and $p \in (2\gamma+2, 6]$, we obtain from (F_0') and (F_1') that there exists $C_{\varepsilon,p}, C_\varepsilon > 0$ such that

$$|\widetilde{f}(s)|, |\widetilde{h}(s)| \leqslant \varepsilon \left[|s| + |s|^5\right] + C_{\varepsilon,p}|s|^{p-1}, \qquad s \in \mathbb{R}, \tag{3.1}$$

$$|\widetilde{F}(s)|, |\widetilde{H}(s)| \leqslant \varepsilon \left(s^2 + s^6\right) + C_{\varepsilon,p}|s|^p, \qquad s \in \mathbb{R},$$

$$|\widetilde{\phi}(s)|, |\widetilde{\psi}(s)| \leqslant \varepsilon|s| + C_\varepsilon|s|^5, \qquad s \in \mathbb{R}, \tag{3.2}$$

$$|\widetilde{\Phi}(s)|, |\widetilde{\Psi}(s)| \leqslant \varepsilon s^2 + C_\varepsilon s^6, \qquad s \in \mathbb{R},$$

where $\widetilde{F}(s)=\int_0^s \widetilde{f}(t)\mathrm{d}t$, $\widetilde{H}(s)=\int_0^s \widetilde{h}(t)\mathrm{d}t$, $\widetilde{\Phi}(s)=\int_0^s \widetilde{\varphi}(t)\mathrm{d}t$, and $\widetilde{\Psi}(s)=\int_0^s \widetilde{\psi}(t)\mathrm{d}t$ for $s\in\mathbb{R}$. Then it follows from the Sobolev inequality (1.11) that for $(u,v)\in E$,

$$\left|\int_{\Omega_1}\widetilde{F}(u)+\int_{\Omega_2}\widetilde{H}(v)\right|\leqslant \varepsilon\nu_2^2\|(u,v)\|_E^2+\varepsilon\nu_6^6\|(u,v)\|_E^6+\nu_p^pC_{\varepsilon,p}\|(u,v)\|_E^p,$$

and

$$\left|\int_{\Omega_1}\widetilde{\Phi}(u)+\int_{\Omega_2}\widetilde{\Psi}(v)\right|\leqslant \varepsilon\nu_2^2\|(u,v)\|_E^2+\nu_6^6C_\varepsilon\|(u,v)\|_E^6.$$

Thus, combining this and (i) of Lemma 2.1, we have that for $(u,v)\in E$,

$$\begin{aligned}
&I_\lambda(u,v)\\
&=\frac{1}{2}A\left(|\nabla u|_{2,\Omega_1}^2\right)+\frac{1}{2}B\left(|\nabla v|_{2,\Omega_2}^2\right)-\int_{\Omega_1}\widetilde{F}(u)-\int_{\Omega_2}\widetilde{H}(v)-\lambda\left[\int_{\Omega_1}\widetilde{\Phi}(u)+\int_{\Omega_2}\widetilde{\Psi}(v)\right]\\
&\geqslant\frac{1}{2}\left[\alpha(0)|\nabla u|_{2,\Omega_1}^2+\beta(0)|\nabla v|_{2,\Omega_2}^2\right]-(1+\lambda)\varepsilon\nu_2^2\|(u,v)\|_E^2\\
&\quad-\nu_p^pC_{\varepsilon,p}\|(u,v)\|_E^p-(\varepsilon+\lambda C_\varepsilon)\nu_6^6\|(u,v)\|_E^6\\
&\geqslant\left(\frac{1}{2}\min\{\alpha(0),\beta(0)\}-(1+\lambda)\varepsilon\nu_2^2\right)\|(u,v)\|_E^2-\nu_p^pC_{\varepsilon,p}\|(u,v)\|_E^p-(\varepsilon+\lambda C_\varepsilon)\nu_6^6\|(u,v)\|_E^6.
\end{aligned}$$

Hence, letting $\rho>0$ small enough, it is easy to see that $\inf\{I_\lambda(u,v):\|(u,v)\|_E=\rho\}>0$.

Next, for each $(u,v)\in E\setminus\{0\}$, according to (ii) of Lemma 2.1, the following limits exist

$$a_\infty:=\lim_{t\to\infty}\frac{A\left(t^2|\nabla u|_{2,\Omega_1}^2\right)}{2t^{2\gamma+2}}\in\mathbb{R}_+,\qquad b_\infty:=\lim_{t\to\infty}\frac{B\left(t^2|\nabla v|_{2,\Omega_2}^2\right)}{2t^{2\gamma+2}}\in\mathbb{R}_+.$$

For any given $M>(a_\infty+b_\infty)\left[(1+\lambda)\left(|u|_{2\gamma+2,\Omega_1}^{2\gamma+2}+|v|_{2\gamma+2,\Omega_2}^{2\gamma+2}\right)\right]^{-1}$, it follows from (F_2') and (F_0') that there exists $C>0$ such that

$$\widetilde{F}(s),\widetilde{H}(s),\widetilde{\Phi}(s),\widetilde{\Psi}(s)\geqslant M|s|^{2\gamma+2}-C,\qquad s\in\mathbb{R}.$$

Thus, we have that

$$\begin{aligned}
I_\lambda(t(u,v))&\leqslant\frac{1}{2}A\left(t^2|\nabla u|_{2,\Omega_1}^2\right)+\frac{1}{2}B\left(t^2|\nabla v|_{2,\Omega_2}^2\right)-M(1+\lambda)t^{2\gamma+2}\left[|u|_{2\gamma+2,\Omega_1}^{2\gamma+2}+|v|_{2\gamma+2,\Omega_2}^{2\gamma+2}\right]\\
&\quad+C(1+\lambda)[|\Omega_1|+|\Omega_2|]\\
&=t^{2\gamma+2}\left[\frac{A\left(t^2|\nabla u|_{2,\Omega_1}^2\right)}{2t^{2\gamma+2}}+\frac{B\left(t^2|\nabla v|_{2,\Omega_2}^2\right)}{2t^{2\gamma+2}}-M(1+\lambda)\left[|u|_{2\gamma+2,\Omega_1}^{2\gamma+2}+|v|_{2\gamma+2,\Omega_2}^{2\gamma+2}\right]\right.\\
&\qquad\left.+\frac{C(1+\lambda)}{t^{2\gamma+2}}[|\Omega_1|+|\Omega_2|]\right]\\
&\to-\infty,\quad t\to\infty.
\end{aligned}$$

The proof is complete. □

Lemma 3.2. *For each $\lambda\in\mathbb{R}_+$, any PS sequence of the functional I_λ is always bounded. Particularly, for $\lambda=0$, the functional I_0 satisfies the PS condition.*

Proof. As for the boundedness of PS sequence, one only needs to observe that (2.1) and (2.3) imply the AR condition. Here for the completeness, we sketch out the proof. Assume that $\lambda \in \mathbb{R}_+, c \in \mathbb{R}$, and $\{(u_n, v_n)\}$ is a $(\mathrm{PS})_c$ sequence of I_λ. Then according to (2.1) and (2.3), for sufficiently large n we have that

$$\begin{aligned}
c + 1 + \|(u_n, v_n)\|_E &\geqslant I_\lambda(u_n, v_n) - \frac{1}{2\gamma+2}\langle I'_\lambda(u_n, v_n), (u_n, v_n)\rangle \\
&= \frac{1}{2}A(|\nabla u_n|^2_{2,\Omega_1}) - \frac{1}{2\gamma+2}\alpha(|\nabla u_n|^2_{2,\Omega_1})|\nabla u_n|^2_{2,\Omega_1} \\
&\quad + \frac{1}{2}B(|\nabla v_n|^2_{2,\Omega_2}) - \frac{1}{2\gamma+2}\beta(|\nabla v_n|^2_{2,\Omega_2})|\nabla v_n|^2_{2,\Omega_2} \\
&\quad + \int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{f}(u_n)u_n - \widetilde{F}(u_n)\right] + \int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{h}(v_n)v_n - \widetilde{H}(v_n)\right] \\
&\quad + \lambda\int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{\phi}(u_n)u_n - \widetilde{\Phi}(u_n)\right] + \lambda\int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{\psi}(v_n)v_n - \widetilde{\Psi}(v_n)\right] \\
&\geqslant \frac{\gamma\alpha(0)}{2\gamma+2}|\nabla u_n|^2_{2,\Omega_1} + \frac{\gamma\beta(0)}{2\gamma+2}|\nabla v_n|^2_{2,\Omega_2} \\
&\geqslant \frac{\gamma}{2\gamma+2}\min\{\alpha(0), \beta(0)\}\|(u_n, v_n)\|^2_E. \qquad (3.3)
\end{aligned}$$

It follows that $\{(u_n, v_n)\}$ is bounded in E.

Now, we illustrate that the functional I_0 satisfies the PS condition. In fact, let $\{(u_n, v_n)\}$ be a PS sequence of I_0. First, from the above conclusion we can get the boundedness of $\{(u_n, v_n)\}$ in E . Without loss of generality, there exists $(u, v) \in E$ such that $(u_n, v_n) \rightharpoonup (u, v)$ as $n \to \infty$. Owing to (3.1) and the compact embedding $E \hookrightarrow L^p(\Omega_1) \times L^p(\Omega_2)$ for $p \in [1, 6)$, we can derive that

$$\lim_{n\to\infty}\int_{\Omega_1}\widetilde{f}(u_n)(u_n - u) = 0, \qquad \lim_{n\to\infty}\int_{\Omega_2}\widetilde{h}(v_n)(v_n - v) = 0. \qquad (3.4)$$

Thus, similarly to Lemma 3.2 in [16], we can prove that $\|(u_n - u, v_n - v)\|^2_E \to 0$. The proof is complete. □

It follows from the mountain pass theorem that the following corollary holds.

Corollary 3.3.

$$K_{c(0)} := \{(u, v) \in E : I'_0(u, v) = 0, I_0(u, v) = c(0)\} \neq \emptyset. \qquad (3.5)$$

Define

$$N_\lambda = \left\{(u, v) \in E \setminus \{0\} : \langle I'_\lambda(u, v), (u, v)\rangle = 0\right\}, \qquad d(\lambda) = \inf_{N_\lambda} I_\lambda. \qquad (3.6)$$

We now prove that $N_\lambda \neq \emptyset$ and provide some properties of the mapping $d(\cdot)$.

Lemma 3.4. *Let $(u, v) \in E \backslash \{0\}$.*

(i) For each $\lambda \in \mathbb{R}_+$, there exists a unique $t(\lambda) > 0$ such that $t(\lambda)(u, v) \in N_\lambda$, $\langle I'_\lambda(t(u, v)), t(u, v)\rangle > 0$ for $t \in (0, t(\lambda))$, $\langle I'_\lambda(t(u, v)), t(u, v)\rangle < 0$ for $t \in (t(\lambda), \infty)$, and $I_\lambda(t(\lambda)(u, v)) = \max_{t\in\mathbb{R}_+} I_\lambda(t(u, v))$.

(ii) The function $t(\cdot) : \mathbb{R}_+ \to (0, \infty)$ is continuously differentiable and

$$t'(\lambda) = \frac{\int_{\Omega_1}\widetilde{\phi}(t(\lambda)u)t(\lambda)u + \int_{\Omega_2}\widetilde{\psi}(t(\lambda)v)t(\lambda)v}{W_1(t(\lambda), (u, v))}, \qquad (3.7)$$

where W_1 is defined by

$$\begin{aligned} W_1(t,(u,v)) = & -2\gamma t\left[\alpha\left(t^2|\nabla u|^2_{2,\Omega_1}\right)|\nabla u|^2_{2,\Omega_1}+\beta(t^2|\nabla v|^2_{2,\Omega_2})|\nabla v|^2_{2,\Omega_2}\right] \\ & +2t^3\left[\alpha'(t^2|\nabla u|^2_{2,\Omega_1})|\nabla u|^4_{2,\Omega_1}+\beta'(t^2|\nabla v|^2_{2,\Omega_2})|\nabla v|^4_{2,\Omega_2}\right] \\ & +(2\gamma+1)\left[\int_{\Omega_1}\widetilde{f}(tu)u+\int_{\Omega_2}\widetilde{h}(tv)v\right]-\int_{\Omega_1}\widetilde{f}'(tu)tu^2-\int_{\Omega_2}\widetilde{h}'(tv)tv^2 \\ & +(2\gamma+1)\lambda\left[\int_{\Omega_1}\widetilde{\varphi}(tu)u+\int_{\Omega_2}\widetilde{\psi}(tv)v\right]-\lambda\int_{\Omega_1}\widetilde{\varphi}'(tu)tu^2-\lambda\int_{\Omega_2}\widetilde{\psi}'(tv)tv^2. \end{aligned}$$

Particularly, $t(\cdot)$ is decreasing on $\mathbb{R}_+$.

Proof. (i) Let $(u,v)\in E\backslash\{0\}$ and $\lambda\in\mathbb{R}_+$ be fixed, and let $w(t)=I_\lambda(t(u,v))$ for $t\in\mathbb{R}_+$. Then $w\in C^1(\mathbb{R}_+)$ and we have that for $t>0$,

$$\begin{aligned} w'(t) &= \langle I'_\lambda(t(u,v)),(u,v)\rangle \\ &= t\alpha\left(t^2|\nabla u|^2_{2,\Omega_1}\right)|\nabla u|^2_{2,\Omega_1}+t\beta\left(t^2|\nabla v|^2_{2,\Omega_2}\right)|\nabla v|^2_{2,\Omega_2}-\int_{\Omega_1}\widetilde{f}(tu)u-\int_{\Omega_2}\widetilde{h}(tv)v \\ &\quad -\lambda\left[\int_{\Omega_1}\widetilde{\varphi}(tu)u+\int_{\Omega_2}\widetilde{\psi}(tv)v\right]. \end{aligned} \tag{3.8}$$

By applying (A_0) and Lemma 2.6, we obtain that $w'(t)>0$ for small $t>0$. And by applying (ii) of Lemma 2.1 and Lemma 2.8, we obtain that $w'(t)<0$ for t large. Thus, there must be some $t(\lambda)>0$ such that $w'(t(\lambda))=0$. Therefore, $t(\lambda)(u,v)\in N_\lambda$.

Furthermore, we can also derive the uniqueness of $t(\lambda)$. In fact, suppose by contradiction there are $t_1,t_2\in(0,\infty)$ with $t_1<t_2$ such that $w'(t_1)=w'(t_2)=0$. Then we have that

$$\begin{aligned} &\left[\frac{\alpha\left(t_1^2|\nabla u|^2_{2,\Omega_1}\right)}{t_1^{2\gamma}}-\frac{\alpha\left(t_2^2|\nabla u|^2_{2,\Omega_1}\right)}{t_2^{2\gamma}}\right]|\nabla u|^2_{2,\Omega_1}+\left[\frac{\beta\left(t_1^2|\nabla v|^2_{2,\Omega_2}\right)}{t_1^{2\gamma}}-\frac{\beta\left(t_2^2|\nabla v|^2_{2,\Omega_2}\right)}{t_2^{2\gamma}}\right]|\nabla v|^2_{2,\Omega_2} \\ &= \int_{\Omega_1}\left[\frac{\widetilde{f}(t_1u)}{t_1^{2\gamma+1}}-\frac{\widetilde{f}(t_2u)}{t_2^{2\gamma+1}}\right]u+\int_{\Omega_2}\left[\frac{\widetilde{h}(t_1v)}{t_1^{2\gamma+1}}-\frac{\widetilde{h}(t_2v)}{t_2^{2\gamma+1}}\right]v \\ &\quad +\lambda\int_{\Omega_1}\left[\frac{\widetilde{\varphi}(t_1u)}{t_1^{2\gamma+1}}-\frac{\widetilde{\varphi}(t_2u)}{t_2^{2\gamma+1}}\right]u+\lambda\int_{\Omega_2}\left[\frac{\widetilde{\psi}(t_1v)}{t_1^{2\gamma+1}}-\frac{\widetilde{\psi}(t_2v)}{t_2^{2\gamma+1}}\right]v, \end{aligned}$$

which is absurd in view of (A_1), (F_2'), and $t_1<t_2$.

(ii) Let us define a function $W(t,\lambda)=\langle I'_\lambda(t(u,v)),(u,v)\rangle$ for $(t,\lambda)\in(-1,\infty)^2$. Then $W(t(\lambda),\lambda)=0$ for $\lambda\in\mathbb{R}_+$ and by calculation we know that for $(t,\lambda)\in(-1,\infty)^2$,

$$\begin{aligned} \frac{\partial W}{\partial t}(t,\lambda) &= \alpha\left(t^2|\nabla u|^2_{2,\Omega_1}\right)|\nabla u|^2_{2,\Omega_1}+\beta\left(t^2|\nabla v|^2_{2,\Omega_2}\right)|\nabla v|^2_{2,\Omega_2} \\ &\quad +2t^2\left[\alpha'\left(t^2|\nabla u|^2_{2,\Omega_1}\right)|\nabla u|^4_{2,\Omega_1}+\beta'\left(t^2|\nabla v|^2_{2,\Omega_2}\right)|\nabla v|^4_{2,\Omega_2}\right] \\ &\quad -\int_{\Omega_1}\widetilde{f}'(tu)u^2-\int_{\Omega_2}\widetilde{h}'(tv)v^2-\lambda\left[\int_{\Omega_1}\widetilde{\varphi}'(tu)u^2+\int_{\Omega_2}\widetilde{\psi}'(tv)v^2\right] \end{aligned} \tag{3.9}$$

and

$$\frac{\partial W}{\partial\lambda}(t,\lambda)=-\int_{\Omega_1}\widetilde{\varphi}(tu)u-\int_{\Omega_2}\widetilde{\psi}(tv)v.$$

Moreover, it follows from (3.9), (3.8), (2.2), and (2.4) that for $\lambda \in \mathbb{R}_+$,

$$
\begin{aligned}
\frac{\partial W}{\partial t}(t(\lambda),\lambda) &= \frac{\partial W}{\partial t}(t(\lambda),\lambda) - \frac{2\gamma+1}{t(\lambda)}W(t(\lambda),\lambda)\\
&= -2\gamma\left[\alpha\left(t^2(\lambda)|\nabla u|_{2,\Omega_1}^2\right)|\nabla u|_{2,\Omega_1}^2 + \beta\left(t^2(\lambda)|\nabla v|_{2,\Omega_2}^2\right)|\nabla v|_{2,\Omega_2}^2\right]\\
&\quad + 2t^2\left[\alpha'\left(t^2|\nabla u|_{2,\Omega_1}^2\right)|\nabla u|_{2,\Omega_1}^4 + \beta'\left(t^2|\nabla v|_{2,\Omega_2}^2\right)|\nabla v|_{2,\Omega_2}^4\right]\\
&\quad + \frac{1}{t(\lambda)}\int_{\Omega_1}\left[(2\gamma+1)\widetilde{f}(t(\lambda)u) - \widetilde{f}'(t(\lambda)u)t(\lambda)u\right]u\\
&\quad + \frac{1}{t(\lambda)}\int_{\Omega_2}\left[(2\gamma+1)\widetilde{h}(t(\lambda)v) - \widetilde{h}'(t(\lambda)u)t(\lambda)v\right]v\\
&\quad + \frac{\lambda}{t(\lambda)}\int_{\Omega_1}\left[(2\gamma+1)\widetilde{\phi}(t(\lambda)u) - \widetilde{\phi}'(t(\lambda)u)t(\lambda)u\right]u\\
&\quad + \frac{\lambda}{t(\lambda)}\int_{\Omega_2}\left[(2\gamma+1)\widetilde{\psi}(t(\lambda)v) - \widetilde{\psi}'(t(\lambda)u)t(\lambda)v\right]v\\
&< 0.
\end{aligned}
$$

Hence, the implicit function theorem and (i) imply that $t(\cdot) : \mathbb{R}_+ \to (0,\infty)$ is continuously differentiable and (3.7) holds. Particularly, recall that $\phi(s)s > 0$ and $\psi(s)s > 0$ for $s \neq 0$, so $t'(\lambda) < 0$ for $\lambda \in \mathbb{R}_+$. Thus, $t(\cdot)$ is decreasing on $\mathbb{R}_+$. □

Lemma 3.5. *For each $\mu > 0$, it holds that $\rho_\mu := \inf_{\lambda\in[0,\mu]} \operatorname{dist}(0, N_\lambda) > 0$.*

Proof. Let $\lambda \in [0,\mu]$ and $(u,v) \in N_\lambda$. Then for each $\varepsilon \in \left(0, \left[(1+\mu)\nu_2^2\right]^{-1}\min\{\alpha(0),\beta(0)\}\right)$, it follows from (3.6), (3.1), and (3.2) with $p = 5$, and the Sobolev embedding theorem that

$$
\begin{aligned}
&\min\{\alpha(0),\beta(0)\}\|(u,v)\|_E^2\\
&\quad\leqslant \alpha(0)|\nabla u|_{2,\Omega_1}^2 + \beta(0)|\nabla v|_{2,\Omega_2}^2\\
&\quad\leqslant \alpha\left(|\nabla u|_{2,\Omega_1}^2\right)|\nabla u|_{2,\Omega_1}^2 + \beta\left(|\nabla v|_{2,\Omega_2}^2\right)|\nabla v|_{2,\Omega_2}^2\\
&\quad= \int_{\Omega_1}\widetilde{f}(u)u + \int_{\Omega_2}\widetilde{h}(v)v + \lambda\left[\int_{\Omega_1}\widetilde{\phi}(u)u + \int_{\Omega_2}\widetilde{\psi}(v)v\right]\\
&\quad\leqslant (1+\lambda)\varepsilon\left[|u|_{2,\Omega_1}^2 + |v|_{2,\Omega_2}^2\right] + (\varepsilon+\lambda C_\varepsilon)\left[|u|_{6,\Omega_1}^6 + |v|_{6,\Omega_2}^6\right] + C_{\varepsilon,p}\left[|u|_{p,\Omega_1}^p + |v|_{p,\Omega_2}^p\right]\\
&\quad\leqslant (1+\mu)\varepsilon\nu_2^2\|(u,v)\|_E^2 + (\varepsilon+\lambda C_\varepsilon + C_{\varepsilon,6})\nu_6^6\|(u,v)\|_E^6.
\end{aligned}
$$

Thus, there exists a positive number σ independent of λ such that $\|(u,v)\|_E \geqslant \sigma$ for $(u,v) \in N_\lambda$. Hence, $\rho_\mu \geqslant \sigma$. □

Subsequently, we will obtain a minimax characterization of $d(\cdot)$ given by the following lemma.

Lemma 3.6. *$d(\lambda) = c(\lambda) = \inf_{(u,v)\in E\setminus\{0\}} \max_{t\in\mathbb{R}_+} I_\lambda(t(u,v))$ for $\lambda \in \mathbb{R}_+$.*

This lemma can be achieved from (i) of Lemma 3.4 and Lemma 3.1. Here we omit the proof, and for the concrete process readers can refer Lemma 3.6 in [16].

According to the above lemma, since $c(\cdot)$ is nonincreasing on $\mathbb{R}_+$, we know that $d(\cdot)$ is nonincreasing on $\mathbb{R}_+$ and $d(\lambda) \leqslant d(0)$ for $\lambda \in \mathbb{R}_+$. Similarly, to establish the right continuity of $c(\cdot)$ at $\lambda = 0$, it suffices to prove that $d(\cdot)$ is continuous at $\lambda = 0$ from the right.

Lemma 3.7. *$\lim_{\lambda\to 0} d(\lambda) = d(0)$.*

Proof. Let $\{\lambda_n\} \subset (0,\mu]$ satisfy $\lambda_n \to 0$ as $n \to \infty$. Then for any given $\varepsilon \in (0, d(0))$ it follows from the definition of $d(\lambda_n)$ that there exists $(u_n, v_n) \in N_{\lambda_n}$ such that for all n,

$$I_{\lambda_n}(u_n, v_n) \leqslant d(\lambda_n) + \varepsilon. \tag{3.10}$$

We note that as in (3.3), for fixed $\lambda \in [0,\mu]$,

$$\frac{\gamma}{2\gamma+2}\min\{\alpha(0),\beta(0)\}\|(u,v)\|_E^2 \leqslant I_\lambda(u,v), \quad (u,v) \in N_\lambda.$$

Then it follows from (3.10) that for all n,

$$\|(u_n,v_n)\|_E^2 \leqslant \frac{2\gamma+2}{\gamma\min\{\alpha(0),\beta(0)\}}\left(d(\lambda_n)+\varepsilon\right) < \frac{4(\gamma+1)d(0)}{\gamma\min\{\alpha(0),\beta(0)\}}.$$

Hence, there exist $(u,v) \in E$ and a subsequence of $\{(u_n,v_n)\}$, still denoted by $\{(u_n,v_n)\}$, satisfying that $(u_n,v_n) \rightharpoonup (u,v)$. Particularly, it holds that $(u,v) \neq 0$. Otherwise, by (1.13), (3.4), and the fact that $\lambda_n \to 0$, one can conclude that

$$\begin{aligned}
&\min\{\alpha(0),\beta(0)\}\|(u_n,v_n)\|_E^2 \\
&\quad\leqslant \alpha\left(|\nabla u_n|_{2,\Omega_1}^2\right)|\nabla u_n|_{2,\Omega_1}^2 + \beta\left(|\nabla v_n|_{2,\Omega_2}^2\right)|\nabla v_n|_{2,\Omega_2}^2 \\
&\quad= \int_{\Omega_1}\widetilde{f}(u_n)u_n + \int_{\Omega_2}\widetilde{h}(v_n)v_n + \lambda_n\left[\int_{\Omega_1}\widetilde{\varphi}(u_n)u_n + \int_{\Omega_2}\widetilde{\psi}(v_n)v_n\right] \to 0.
\end{aligned}$$

This contradicts the fact that $\{\|(u_n,v_n)\|_E\}$ has a positive lower bound which can be derived from Lemma 3.5.

For $(u_n,v_n) \in N_{\lambda_n}$ chosen above, by $\langle I_0'(u_n,v_n),(u_n,v_n)\rangle > \langle I_{\lambda_n}'(u_n,v_n),(u_n,v_n)\rangle = 0$ and (i) of Lemma 3.4, there exists a unique $t_n(0) > 1$ such that $t_n(0)(u_n,v_n) \in N_0$. Therefore,

$$0 \leqslant d(0) - d(\lambda_n) \leqslant I_0(t_n(0)(u_n,v_n)) - I_{\lambda_n}(u_n,v_n) + \varepsilon. \tag{3.11}$$

It follows from Lemma 3.4 that there exists $t_n(\lambda) > 0$ such that $t_n(\lambda)(u_n,v_n) \in N_\lambda$. Let us define $g_n(\lambda) = I_\lambda(t_n(\lambda)(u_n,v_n))$ for $\lambda \in \mathbb{R}_+$. Then the fact $t_n(\lambda)(u_n,v_n) \in N_\lambda$ implies that

$$\begin{aligned}
g_n'(\lambda) &= \langle I_\lambda'(t_n(\lambda)(u_n,v_n)),(u_n,v_n)\rangle t_n'(\lambda) - \left[\int_{\Omega_1}\widetilde{\Phi}(t_n(\lambda)u_n) + \int_{\Omega_2}\widetilde{\Psi}(t_n(\lambda)v_n)\right] \\
&= -\left[\int_{\Omega_1}\widetilde{\Phi}(t_n(\lambda)u_n) + \int_{\Omega_2}\widetilde{\Psi}(t_n(\lambda)v_n)\right], \qquad \lambda \in \mathbb{R}_+.
\end{aligned}$$

Thus, it follows from (ii) of Lemma 3.4 that

$$\begin{aligned}
&I_0(t_n(0)(u_n,v_n)) - I_{\lambda_n}(u_n,v_n) \\
&\quad= g_n(0) - g_n(\lambda_n) \\
&\quad= -\int_0^{\lambda_n} g_n'(s)\mathrm{d}s \\
&\quad= \int_0^{\lambda_n}\left[\int_{\Omega_1}\widetilde{\Phi}(t_n(s)u_n) + \int_{\Omega_2}\widetilde{\Psi}(t_n(s)v_n)\right]\mathrm{d}s \\
&\quad\leqslant \lambda_n\left(t_n^2(0)\left[|u_n|_{2,\Omega_1}^2 + |v_n|_{2,\Omega_2}^2\right] + C_\varepsilon t_n^6(0)\left[|u_n|_{6,\Omega_1}^6 + |v_n|_{6,\Omega_2}^6\right]\right).
\end{aligned} \tag{3.12}$$

By (3.11), (3.12), and the Sobolev embedding theorem, to establish that $d(\lambda) \to d(0)$ as $\lambda \to 0$, it suffices to prove that $\{t_n(0)\}$ is bounded. We assume toward a contradiction that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $s_i := t_{n_i}(0) \to \infty$ as $i \to \infty$. Then by the fact that $t_{n_i}(0)(u_{n_i}, v_{n_i}) \in N_0$ for all i and (1.13), we have that

$$\frac{\alpha\left(s_i^2|\nabla u_{n_i}|^2_{2,\Omega_1}|\right)}{s_i^{2\gamma}}|\nabla u_{n_i}|^2_{2,\Omega_1} + \frac{\beta\left(s_i^2|\nabla v_{n_i}|^2_{2,\Omega_2}|\right)}{s_i^{2\gamma}}|\nabla v_{n_i}|^2_{2,\Omega_2} = \int_{\Omega_1} \frac{\widetilde{f}(s_i u_{n_i})}{s_i^{2\gamma+1}} u_{n_i} + \int_{\Omega_2} \frac{\widetilde{h}(s_i v_{n_i})}{s_i^{2\gamma+1}} v_{n_i}. \quad (3.13)$$

Moreover, it follows from $(u_{n_i}, v_{n_i}) \rightharpoonup (u, v) \neq 0$ and Lemma 2.7 that the right-hand side of (3.13) converges to infinity. This contradicts the fact that the limit superior of the left-hand side is finite by (A_1). Hence, $\{t_n(0)\}$ is bounded. The proof is complete. □

We now establish the existence of ground-state solutions to the problem (1.1). Motivated by [16,32], we first study the distance between any $(PS)_{c(\lambda)}$ sequence of I_λ and a compact set $K_{c(0)}$ defined in (3.5). Here, the existence of a $(PS)_{c(\lambda)}$ sequence can be derived from Lemma 3.1 and a general minimax principle [28, Theorem 2.8, p. 41]. The compactness of $K_{c(0)}$ follows directly from the fact that I_0 satisfies the PS condition.

Lemma 3.8. *For each $\lambda \in \mathbb{R}_+$, let $\{(u_n^\lambda, v_n^\lambda)\}$ be any $(PS)_{c(\lambda)}$* sequence of I_λ. Then

$$\lim_{\lambda\to 0} \limsup_{n\to\infty} \operatorname{dist}\left((u_n^\lambda, v_n^\lambda), K_{c(0)}\right) = 0.$$

Proof. It just needs to repeat the proof of Lemma 3.8 in [16]. □

Finally, we prove Theorem 1.7.

Proof of Theorem 1.7. For each $\lambda \in \mathbb{R}_+$, let $\{(u_n^\lambda, v_n^\lambda)\}$ be a $(PS)_{c(\lambda)}$ sequence of I_λ. We note that $\{(u_n^\lambda, v_n^\lambda)\}$ is bounded by Lemma 3.2. Then there exist a subsequence of $\{(u_n^\lambda, v_n^\lambda)\}$, still denoted by $\{(u_n^\lambda, v_n^\lambda)\}$, and $(u_\lambda, v_\lambda) \in E$ such that $(u_n^\lambda, v_n^\lambda) \rightharpoonup (u_\lambda, v_\lambda)$ as $n \to \infty$. We will try to find a $\lambda_0 > 0$ such that $(u_\lambda, v_\lambda) \neq 0$ for $\lambda \in [0, \lambda_0)$. In fact, since $c(0) \neq 0$ and $K_{c(0)}$ is compact, it holds that

$$\delta_0 := \operatorname{dist}(0, K_{c(0)}) = \min_{(u,v)\in K_{c(0)}} \|(u,v)\|_E > 0.$$

Moreover, according to Lemma 3.8, for any given $\varepsilon_0 \in [0, \delta_0)$, there exists a $\lambda_0 = \lambda(\varepsilon_0)$ such that when $\lambda \in (0, \lambda_0)$, there is some $n_\lambda := n(\lambda)$ such that $\operatorname{dist}((u_n^\lambda, v_n^\lambda), K_{c(0)}) \leqslant \varepsilon_0$ for all $n > n_\lambda$. Fixing $\lambda \in (0, \lambda_0)$ and by the compactness of $K_{c(0)}$, there exists a sequence $\{(w_n^\lambda, z_n^\lambda)\} \subset K_{c(0)}$ such that $\|(u_n^\lambda, v_n^\lambda) - (w_n^\lambda, z_n^\lambda)\|_E \leqslant \varepsilon_0$ for all $n > n_\lambda$. Furthermore, for a subsequence of $\{(w_n^\lambda, z_n^\lambda)\}$, still denoted by $\{(w_n^\lambda, z_n^\lambda)\}$, and some $(w_\lambda, z_\lambda) \in K_{c(0)}$, it holds that $(w_n^\lambda, z_n^\lambda) \to (w_\lambda, z_\lambda)$ as $n \to \infty$. Hence, we have that $(u_n^\lambda, v_n^\lambda) \in B_{\varepsilon_0}(w_\lambda, z_\lambda)$ for sufficiently large n. Thus, $(u_\lambda, v_\lambda) \in \overline{B}_{\varepsilon_0}(w_\lambda, z_\lambda)$ because $\overline{B}_{\varepsilon_0}(w_\lambda, z_\lambda)$ is weakly closed. Therefore, $\|(u_\lambda, v_\lambda)\|_E \geqslant \|(w_\lambda, z_\lambda)\|_E - \varepsilon_0 \geqslant \delta_0 - \varepsilon_0 > 0$, that is, $(u_\lambda, v_\lambda) \neq 0$.

We now prove that $I'_\lambda(u_\lambda, v_\lambda) = 0$ and $I_\lambda(u_\lambda, v_\lambda) = d(\lambda)$, that is, (u_λ, v_λ) is a ground-state solution to the problem (1.1). Without loss of generality, we may assume that the sequence $\{(u_n^\lambda, v_n^\lambda)\}$ satisfies that $\left(|\nabla u_n^\lambda|^2_{2,\Omega_1}, |\nabla v_n^\lambda|^2_{2,\Omega_2}\right) \to (a,b)$ as $n \to \infty$ for some $(a,b) \in \mathbb{R}_+^2 \setminus \{0\}$. For all $(u,v) \in E$, let

$$I_{\lambda,(a,b)}(u,v) = \frac{1}{2}\left[\alpha(a)|\nabla u|^2_{2,\Omega_1} + \beta(b)|\nabla v|^2_{2,\Omega_2}\right] - \int_{\Omega_1} \widetilde{F}(u) - \int_{\Omega_2} \widetilde{H}(v) - \lambda\left[\int_{\Omega_1} \widetilde{\Phi}(u) + \int_{\Omega_1} \widetilde{\Psi}(v)\right].$$

Then $I'_{\lambda,(a,b)}(u_n^\lambda, v_n^\lambda) \to 0$. Hence, $I'_{\lambda,(a,b)}(u_\lambda, v_\lambda) = 0$. We claim that $\left(|\nabla u_\lambda|^2_{2,\Omega_1}, |\nabla v_\lambda|^2_{2,\Omega_2}\right) = (a,b)$. In fact, it would follow from $(u_n^\lambda, v_n^\lambda) \rightharpoonup (u_\lambda, v_\lambda)$ that $|\nabla u_\lambda|^2_{2,\Omega_1} \leqslant a$ and $|\nabla v_\lambda|^2_{2,\Omega_2} \leqslant b$, and then

$$\langle I'_\lambda(u_\lambda, v_\lambda), (u_\lambda, v_\lambda)\rangle \leqslant \langle I'_{\lambda,(a,b)}(u_\lambda, v_\lambda), (u_\lambda, v_\lambda)\rangle = 0.$$

Since $(u_\lambda, v_\lambda) \neq 0$, which is obtained on the above paragraph, by (i) of Lemma 3.4 there exists a unique $t(\lambda) \in (0,1]$ such that $t(\lambda)(u_\lambda, v_\lambda) \in N_\lambda$. Furthermore, the monotonicity obtained in Lemma 2.1 and Remark 2.5, the weak lower continuity of norm, Fatou's lemma, and the choice of $\{(u_n^\lambda, v_n^\lambda)\}$ imply that

$$\begin{aligned}
d(\lambda) &\leqslant I_\lambda(t(\lambda)(u_\lambda, v_\lambda)) - \frac{1}{2\gamma+2}\langle I'_\lambda(t(\lambda)(u_\lambda, v_\lambda)), t(\lambda)(u_\lambda, v_\lambda)\rangle \\
&= \frac{1}{2}A\left(|t(\lambda)\nabla u_\lambda|^2_{2,\Omega_1}\right) - \frac{1}{2\gamma+2}\alpha\left(|t(\lambda)\nabla u_\lambda|^2_{2,\Omega_1}\right)|t(\lambda)\nabla u_\lambda|^2_{2,\Omega_1} \\
&\quad + \frac{1}{2}B\left(|t(\lambda)\nabla v_\lambda|^2_{2,\Omega_2}\right) - \frac{1}{2\gamma+2}\beta\left(|t(\lambda)\nabla v_\lambda|^2_{2,\Omega_2}\right)|t(\lambda)\nabla v_\lambda|^2_{2,\Omega_2} \\
&\quad + \int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{f}(t(\lambda)u_\lambda)t(\lambda)u_\lambda - \widetilde{F}(t(\lambda)u_\lambda)\right] \\
&\quad + \int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{h}(t(\lambda)v_\lambda)t(\lambda)v_\lambda - \widetilde{H}(t(\lambda)v_\lambda)\right] \\
&\quad + \lambda\int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{\varphi}(t(\lambda)u_\lambda)t(\lambda)u_\lambda - \widetilde{\Phi}(t(\lambda)u_\lambda)\right] \\
&\quad + \lambda\int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{\psi}(t(\lambda)v_\lambda)t(\lambda)v_\lambda - \widetilde{\Psi}(t(\lambda)v_\lambda)\right] \\
&\leqslant \frac{1}{2}A\left(|\nabla u_\lambda|^2_{2,\Omega_1}\right) - \frac{1}{2\gamma+2}\alpha\left(|\nabla u_\lambda|^2_{2,\Omega_1}\right)|\nabla u_\lambda|^2_{2,\Omega_1} \\
&\quad + \frac{1}{2}B\left(|\nabla v_\lambda|^2_{2,\Omega_2}\right) - \frac{1}{2\gamma+2}\beta\left(|\nabla v_\lambda|^2_{2,\Omega_2}\right)|\nabla v_\lambda|^2_{2,\Omega_2} \\
&\quad + \int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{f}(u_\lambda)u_\lambda - \widetilde{F}(u_\lambda)\right] + \int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{h}(v_\lambda)v_\lambda - \widetilde{H}(v_\lambda)\right] \\
&\quad + \lambda\int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{\varphi}(u_\lambda)u_\lambda - \widetilde{\Phi}(u_\lambda)\right] + \lambda\int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{\psi}(v_\lambda)v_\lambda - \widetilde{\Psi}(v_\lambda)\right] \\
&\leqslant \liminf_{n\to\infty}\left[\frac{1}{2}A\left(|\nabla u_n^\lambda|^2_{2,\Omega_1}\right) - \frac{1}{2\gamma+2}\alpha\left(|\nabla u_n^\lambda|^2_{2,\Omega_1}\right)|\nabla u_n^\lambda|^2_{2,\Omega_1}\right] \\
&\quad + \liminf_{n\to\infty}\left[\frac{1}{2}B\left(|\nabla v_n^\lambda|^2_{2,\Omega_2}\right) - \frac{1}{2\gamma+2}\beta\left(|\nabla v_n^\lambda|^2_{2,\Omega_2}\right)|\nabla v_n^\lambda|^2_{2,\Omega_2}\right] \\
&\quad + \liminf_{n\to\infty}\int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{f}(u_n^\lambda)u_n^\lambda - \widetilde{F}(u_n^\lambda)\right] + \liminf_{n\to\infty}\int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{h}(v_n^\lambda)v_n^\lambda - \widetilde{H}(v_n^\lambda)\right] \\
&\quad + \lambda\liminf_{n\to\infty}\int_{\Omega_1}\left[\frac{1}{2\gamma+2}\widetilde{\varphi}(u_n^\lambda)u_n^\lambda - \widetilde{\Phi}(u_n^\lambda)\right] + \lambda\liminf_{n\to\infty}\int_{\Omega_2}\left[\frac{1}{2\gamma+2}\widetilde{\psi}(v_n^\lambda)v_n^\lambda - \widetilde{\Psi}(v_n^\lambda)\right] \\
&\leqslant \lim_{n\to\infty}\left[I_\lambda(u_n^\lambda, v_n^\lambda) - \frac{1}{2\gamma+2}\langle I'_\lambda(u_n^\lambda, v_n^\lambda), (u_n^\lambda, v_n^\lambda)\rangle\right] \\
&= c(\lambda) = d(\lambda).
\end{aligned}$$

Thus, there exists a subsequence $\{(u^\lambda_{n_i}, v^\lambda_{n_i})\}$ of $\{(u^\lambda_n, v^\lambda_n)\}$ such that

$$\begin{aligned}
&\frac{1}{2}A\left(|\nabla u_\lambda|^2_{2,\Omega_1}\right) - \frac{1}{2\gamma+2}\alpha\left(|\nabla u_\lambda|^2_{2,\Omega_1}\right)|\nabla u_\lambda|^2_{2,\Omega_1} \\
&\quad = \lim_{i\to\infty}\left[\frac{1}{2}A\left(|\nabla u^\lambda_{n_i}|^2_{2,\Omega_1}\right) - \frac{1}{2\gamma+2}\alpha\left(|\nabla u^\lambda_{n_i}|^2_{2,\Omega_1}\right)|\nabla u^\lambda_{n_i}|^2_{2,\Omega_1}\right] \\
&\quad = \frac{1}{2}A(a) - \frac{1}{2\gamma+2}\alpha(a)a
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}B\left(|\nabla v_\lambda|^2_{2,\Omega_2}\right) - \frac{1}{2\gamma+2}\beta\left(|\nabla v_\lambda|^2_{2,\Omega_2}\right)|\nabla v_\lambda|^2_{2,\Omega_2} \\
&\quad = \lim_{i\to\infty}\left[\frac{1}{2}B\left(|\nabla v^\lambda_{n_i}|^2_{2,\Omega_2}\right) - \frac{1}{2\gamma+2}\beta\left(|\nabla v^\lambda_{n_i}|^2_{2,\Omega_2}\right)|\nabla v^\lambda_{n_i}|^2_{2,\Omega_2}\right] \\
&\quad = \frac{1}{2}B(b) - \frac{1}{2\gamma+2}B(b)b.
\end{aligned}$$

It follows from the monotonicity of $(\gamma+1)A(s) - \alpha(s)s$ and $(\gamma+1)B(s) + \beta(s)s$ that $(|\nabla u_\lambda|^2_{2,\Omega_1}, |\nabla v_\lambda|^2_{2,\Omega_2}) = (a, b)$ holds, and then $(u^\lambda_n, v^\lambda_n) \to (u_\lambda, v_\lambda)$ in E. Moreover, since I_λ is continuously differentiability, one can also conclude that $I'_\lambda(u_\lambda, v_\lambda) = 0$ and $I_\lambda(u_\lambda, v_\lambda) = c(\lambda) = d(\lambda)$. Thus, (u_λ, v_λ) is a ground-state solution to the problem (1.1).

Finally, we will end the proof of Theorem 1.7 by proving that $(u_\lambda, v_\lambda) \to (u_0, v_0)$ as $\lambda \to 0$, where (u_0, v_0) is a ground-state solution to (1.1) with $\lambda = 0$. Actually, let $\{\lambda_n\} \subset [0, \lambda_0)$ such that $\lambda_n \to 0$ as $n \to \infty$. Then as a consequence of the fact that $(u_{\lambda_n}, v_{\lambda_n})$ is a ground-station solution to (1.1) with $\lambda = \lambda_n$, it hold that $I'_{\lambda_n}(u_{\lambda_n}, v_{\lambda_n}) = 0$ and $I_{\lambda_n}(u_{\lambda_n}, v_{\lambda_n}) = c(\lambda_n)$. By Lemma 3.7, similar to (3.3), we have that as $n \to \infty$,

$$\begin{aligned}
c(0) + o(1) &= c(\lambda_n) \\
&= I_{\lambda_n}(u_{\lambda_n}, v_{\lambda_n}) - \frac{1}{2\gamma+2}\langle I'_{\lambda_n}(u_{\lambda_n}, v_{\lambda_n}), (u_{\lambda_n}, v_{\lambda_n})\rangle \\
&\geqslant \frac{\gamma}{2\gamma+2}\min\{\alpha(0), \beta(0)\}\|(u_{\lambda_n}, v_{\lambda_n})\|^2_E.
\end{aligned}$$

This yields that $\{(u_{\lambda_n}, v_{\lambda_n})\}$ is bounded in E. Hence, it follows from the Sobolev embedding theorem that as $n \to \infty$,

$$I_0(u_{\lambda_n}, v_{\lambda_n}) = I_{\lambda_n}(u_{\lambda_n}, v_{\lambda_n}) + \lambda_n\left[\int_{\Omega_1}\widetilde{\Phi}(u_{\lambda_n}) + \int_{\Omega_1}\widetilde{\Psi}(v_{\lambda_n})\right] \to c(0),$$

and

$$\begin{aligned}
\left\langle I'_0(u_{\lambda_n}, v_{\lambda_n}), (w, z)\right\rangle &= \left\langle I'_{\lambda_n}(u_{\lambda_n}, v_{\lambda_n}), (w, z)\right\rangle + \lambda_n\left[\int_{\Omega_1}\widetilde{\phi}(u_{\lambda_n})w + \int_{\Omega_1}\widetilde{\psi}(v_{\lambda_n})z\right] \\
&= \lambda_n\left[\int_{\Omega_1}\widetilde{\phi}(u_{\lambda_n})w + \int_{\Omega_1}\widetilde{\psi}(v_{\lambda_n})z\right] \\
&= o(1)\|(w, z)\|_E, \qquad (w, z) \in E.
\end{aligned}$$

Thus, $\{(u_{\lambda_n}, v_{\lambda_n})\}$ is a $(\mathrm{PS})_{c(0)}$ sequence of I_0 in E, and then by Lemma 3.2, there exists a subsequence of $\{(u_{\lambda_n}, v_{\lambda_n})\}$, still denoted by $\{(u_{\lambda_n}, v_{\lambda_n})\}$, and $(u_0, v_0) \in E$ such that $(u_{\lambda_n}, v_{\lambda_n}) \to (u_0, v_0)$ and $I_0(u_0, v_0) = c(0)$, that is, (u_0, v_0) is a ground-state solution to (1.1) with $\lambda = 0$. The proof is complete. □

4 Appendix

In this section, some regular properties of compound functions will be proved.

Lemma 4.1. *Assume that $f \in C^2(\mathbb{R})$ and $f', f'' \in L^\infty(\mathbb{R})$. If $u \in W^{2,p}(\Omega)$ and $2(N-p) < N$, then $f(u) \in W^{2,p}(\Omega)$ and*

$$D^\alpha(f(u)) = \begin{cases} f'(u)D^\alpha u, & |\alpha| \leqslant 1, \\ f''(u)(D_1 u)^2 + f'(u)D^\alpha u, & \alpha = (2,0,0,\dots), \\ \dots \end{cases}$$

Proof. According to [27, Theorem 2.5.1, p. 70], we have that $f(u) \in W^{1,p}(\Omega)$ and $D(f(u)) = f'(u)Du$. When $|\alpha| \leqslant 1$, we have that $D^\alpha(f(u)) \in L^p(\Omega)$. Assume that $|\alpha| = 2$, without loss of generality, let $\alpha = (2,0,\dots,0)$. Then we could calculate $D_1(f(u))$ as follows. Actually, for any given $\phi \in C_0^\infty(\Omega)$, because $f(u) \in W^{1,p}(\Omega)$ and $D_1\phi \in C_0^\infty(\Omega)$, we have that

$$\int_\Omega f(u)D^\alpha\phi = -\int_\Omega D_1(f(u))D_1\phi = -\int_\Omega f'(u)D_1 u D_1\phi. \tag{4.1}$$

Let $g(s) = f'(s)$ for $s \in \mathbb{R}$. Since $f \in C^2(\mathbb{R})$ and $f'' \in L^\infty(\mathbb{R})$, then $g \in C^1(\mathbb{R})$ and $g' \in L^\infty(\mathbb{R})$. Thus, $g(u) \in W^{1,p}(\Omega)$. It follows from the weak derivative product formula that $g(u)D_1 u \in W^{1,p}(\Omega)$ and $D_1(g(u)D_1 u) = D_1(g(u))D_1 u + g(u)D^\alpha u = g'(u)(D_1 u)^2 + g(u)D^\alpha u = f''(u)(D_1 u)^2 + f'(u)D^\alpha u$. Moreover, (4.1) can be written

$$\int_\Omega f(u)D^\alpha\phi = -\int_\Omega (g(u)D_1 u)D_1\phi = \int_\Omega D_1(g(u)D_1 u)\phi = \int_\Omega \left[f''(u)(D_1 u)^2 + f'(u)D^\alpha u\right]\phi.$$

Thus, $D^\alpha(f(u)) = f''(u)(D_1 u)^2 + f'(u)D^\alpha u$.

Next, we prove $D^\alpha(f(u)) \in L^p(\Omega)$. In fact, because $f', f'' \in L^\infty(\mathbb{R}^N)$ and $D^\alpha u \in L^p(\Omega)$, we need only illustrate $(D_1 u)^2 \in L^p(\Omega)$, that is, $D_1 u \in L^{2p}(\Omega)$. In fact, since $D_1 u \in W^{1,p}(\Omega)$ and $2p < Np/(N-p)$, it follows from the Sobolev embedding theorem that $W^{1,p}(\Omega) \hookrightarrow L^{2p}(\Omega)$, and then $D_1 u \in L^{2p}(\Omega)$. The proof is complete. □

Lemma 4.2. *Assume that there exists $M > 0$ such that $|g'(s)/g^3(s)| \leqslant M$ for $s \in \mathbb{R}$. If $u \in H^2(D)$, then $G^{-1}(u) \in H^2(D)$, where $D \subset \mathbb{R}^3$ is an open domain with $\partial D \in C^1$.*

Proof. Let $f(s) = G^{-1}(s)$ for $s \in \mathbb{R}$. Then the conclusion holds by Lemma 4.1. □

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