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Algebraic traveling waves for the modified Korteweg–de Vries–Burgers equation

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Abstract. In this paper we characterize all traveling wave solutions of the Generalized Korteweg–de Vries–Burgers equation. In particular we recover the traveling wave solutions for the well-known Korteweg–de Vries–Burgers equation.

Keywords: traveling wave, modified Korteweg–de Vries–Burgers equation, Korteweg–de Vries–Burgers equation.

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1 Introduction and statement of the main results

Looking for traveling waves to nonlinear evolution equations has long been the major problem for mathematicians and physicists. These solutions may well describe various phenomena in physics and other fields and thus they may give more insight into the physical aspects of the problems. Many methods for obtaining traveling wave solutions have been established [4–6, 19, 20, 25, 26] with more or less success. When the degree of the nonlinearity is high most of the methods fail or can only lead to a kind of special solution and the solution procedures become very complex and do not lead to an efficient way to compute them.

In this paper we will focus on obtaining algebraic traveling wave solutions to the modified Korteweg–de Vries–Burgers equation (mKdVB) of the form

$$au_{xxx} + bu_{xx} + du^n u_x + u_t = 0 \tag{1.1}$$

where $n = 1, 2$ and a, b, d are real constants with $abd \neq 0$. When $n = 1$ is the well-known Korteweg–de Vries–Burgers equation (KdVB) that has been intensively investigated. When $n = 2$ we will call it modified Korteweg–de Vries–Burgers equation (mKdVB). These equations are widely used in fields as solid-states physics, plasma physics, fluid physics and quantum field theory (see, for instance [12, 31] and the references therein). They mainly appear when seeking the asymptotic behavior of complicated systems governing physical processes in solid and fluid mechanics.

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An special attention is done to the KdVB, often considered as a combination of the Burgers equation and KdV equation since in the limit $a \to 0$, the equation reduces to the Burgers equation (named after its use by Burgers [2] for studying the turbulence in 1939), and taking the limit as $b \to 0$ we get the KdV equation (first suggested by Korteweg and de Vries [18] who used it as a nonlinear model to study the change of forms of long waves advancing in a rectangular channel).

The KdVB equation is the simplest form of the wave equation in which the nonlinear term uu_x, the dispersion u_{xxx} and the dissipation u_{xx} all occur. It arises from many physical context such as the undulant bores in a shallow water [1, 16], the flow of liquids containing gas bubbles [27], the propagation of waves in an elastic tube filled with a viscous fluid [15], weakly nonlinear plasma waves with certain dissipative effects [9, 11], the cascading down process of turbulence [7] and the atmospheric dynamics [17].

It is nonintegrable in the sense that its spectral problem is nonexistent. The existence of traveling wave solutions for the (KdVB) was obtained by the first time in [29] and after that many other papers computing the traveling wave of the KdVB appeared (see for instance [10, 13, 14, 21, 25, 28, 30]), but most of them did not obtain all the possible traveling wave solutions. However, regardless the attention done to the (KdVB), nothing is known for the existence of traveling wave solutions for the (mKdVB). This is due to the presence of high nonlinear terms. In this paper we will fill in this gap. We will use a method that will supply the already known traveling wave solution for the (KdVG) and will allows us to prove that there are no traveling wave solutions for the KdVG (i.e., equation (1.1) with $n = 2$).

As explained above, there are various approaches for constructing traveling wave solutions, but these methods become more and more useless as the degree of the nonlinear terms increase. However, in [8] the authors gave a technique to prove the existence of traveling wave solutions for general n-th order partial differential equations by showing that traveling wave solutions exist if and only if the associated n-dimensional first order ordinary differential equation has some invariant algebraic curve. In this paper we will consider only the case of 2-nd order partial differential equations.

More precisely, consider the 2-nd order partial differential equations of the form

$$\frac{\partial^2 u}{\partial x^2} = F\Big(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\Big), \tag{1.2}$$

where x and t are real variables and F is a smooth map. The traveling wave solutions of system (1.2) are particular solutions of the form $u = u(x,t) = U(x - ct)$ where $U(s)$ satisfies the boundary conditions

$$\lim_{s\to-\infty} U(s) = A \quad \text{and} \quad \lim_{s\to\infty} U(s) = B, \tag{1.3}$$

where A and B are solutions, not necessarily different, of $F(u,0,0) = 0$. Note that $U(s)$ has to be a solution, defined for all $s \in \mathbb{R}$, of the 2-nd order ordinary differential equation

$$U'' = F(U, U', -cU') = \tilde{F}(U, U'), \tag{1.4}$$

where $U(s)$ and the derivatives are taken with respect to s. The parameter c is called the *speed* of the traveling wave solution.

We say that $u(x,t) = U(x - ct)$ is an algebraic traveling wave solution if $U(s)$ is a non-constant function that satisfies (1.3) and (1.4) and there exists a polynomial p such that $p(U(s), U'(s)) = 0$.

As pointed out in [8] the term algebraic traveling wave means that the waves that we will find correspond to the algebraic curves on the phase plane and do not refer to traveling waves that approach to the constant boundary conditions (1.3) algebraically fast. The traveling wave solutions correspond to homoclinic (when $A = B$) or heteroclinic (when $A \neq B$) solutions of the associated two-dimensional system of ordinary differential equations. In many cases the critical points where this invariant manifolds start and end are hyperbolic. To motivate the definition of algebraic traveling wave solutions initiated in [8] and used in the present paper, we recall that when F is sufficiently regular, using normal form theory, in a neighborhood of these critical points, this manifold can be parameterized as $\varphi(e^{\lambda s})$ for some smooth function φ, where λ is one of the eigenvalues of the critical points.

Note that this definition of algebraic traveling wave revives the interest in the well-known and classic problem of finding invariant algebraic curves. Invariant algebraic curves are the main objects used in several subjects with special emphasis in integrability theory. The search and computation of these objects have been intensively investigated. However to determine the properties and number of them for a given planar vector field is very difficult in particular because there is no bound a priori on the degree of such curves. However in the present paper we will be able to characterize completely the algebraic traveling wave solutions of the Korteweg–de Vries–Burgers equation and of the Generalized Korteweg–de Vries–Burgers equation under some additional assumptions on the constants. We recall that for irreducible polynomials we have the following algebraic characterization of invariant algebraic curves: Given an irreducible polynomial of degree n, $g(x,y)$, we have that $g(x,y) = 0$ is an invariant algebraic curve for the system $x' = P(x,y)$, $y' = Q(x,y)$ for $P, Q \in \mathbb{C}[x,y]$, if there exists a polynomial $K = (x,y)$ of degree at most $n-1$, called the cofactor of g such that

$$P(x,y)\frac{\partial g}{\partial x} + Q(x,y)\frac{\partial g}{\partial y} = K(x,y)g. \tag{1.5}$$

The main result that we will use is the following theorem, see [8] for its proof.

Theorem 1.1. *The partial differential equation (1.2) has an algebraic traveling wave solution if and only if the first order differential system*

$$\begin{cases} y_1' = y_2, \\ y_2' = G_c(y_1, y_2), \end{cases}$$

where

$$G_c(y_1, y_2) = \tilde{F}(y_1, y_2)$$

has an invariant algebraic curve containing the critical points $(A, 0)$and $(B, 0)$ and no other critical points between them.

The main result is, with the techniques in [8], obtain all algebraic traveling wave solutions of the (KdVB) and (mKdVB), i.e., all explicit traveling wave solutions of the equation (1.1) when $n = 1$ and when $n = 2$.

Theorem 1.2. *The following holds for system (1.1):*

(i) If $n = 1$ (KdVB), it has the algebraic traveling wave solution

$$u(x,t) = -\frac{12b^2}{25da}\Big(\frac{1}{1 + \kappa_1 e^{b(x-vt)/(5a)}}\Big)^2 + \frac{6b^2}{25da} + \frac{v}{d},$$

where

$$v^2 = \frac{36b^4 - 1250da^3\kappa_2}{625a^2},$$

being κ_1, κ_2 arbitrary constants with $\kappa_1 > 0$.

(ii) If $n = 2$ (mKdVB), it has no algebraic traveling wave solutions.

The proof of Theorem 1.2 is given in Section 3 when $n = 1$ and in Section 4 when $n = 2$. In section 2 we have included some preliminary results that will be used to prove the results in the paper. The technique used in the paper is very powerful and has been used successfully in the papers [23,24].

2 Preliminary results

In this section we introduce some notions and results that will be used during the proof of Theorem 1.2.

The first result based on the previous works of Seidenberg [22] was stated and proved in [3]. In the next theorem we included only the results from [3] that will be used in the paper.

Theorem 2.1. *Let $g(x,y) = 0$ be an invariant algebraic curve of a planar system with corresponding cofactor $K(x,y)$. Assume that $p = (x_0, y_0)$ is one of the critical points of the system. If $g(x_0, y_0) \neq 0$, then $K(x_0, y_0) = 0$. Moreover, assume that λ and μ are the eigenvalues of such critical point. If either $\mu \neq 0$ and λ and μ are rationally independent or $\lambda\mu < 0$, or $\mu = 0$, then either $K(x_0, y_0) = \lambda$, or $K(x_0, y_0) = \mu$, or $K(x_0, y_0) = \lambda + \mu$ (that we write as $K(x_0, y_0) \in \{\lambda, \mu, \lambda + \mu\}$).*

A polynomial $g(x,y)$ is said to be a *weight homogeneous polynomial* if there exist $s = (s_1, s_2) \in \mathbb{N}^2$ and $m \in \mathbb{N}$ such that for all $\mu \in \mathbb{R} \setminus \{0\}$,

$$g(\mu^{s_1}x, \mu^{s_2}y) = \alpha^m g(x,y),$$

where $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{N}$ the set of positive integers. We shall refer to $s = (s_1, s_2)$ to the weight of g, m the weight degree and $x = (x_1, x_2) \mapsto (\alpha^{s_1}x, \alpha^{s_2}y)$ the weight change of variables.

We first note that if there exists a solution of the form $u(x,t) = U(x - ct)$ then substituting in (1.1) and performing one integration yield

$$U'' = -\beta U' - \gamma U^{n+1} + \delta U + \theta,$$

where $\beta = b/a$, $\gamma = d/(a(n+1))$, $\delta = c/a$ and θ is the integration constant. Therefore, we will look for the invariant algebraic curves of the system

$$\begin{aligned} x' &= y, \\ y' &= -\beta y - \gamma x^{n+1} + \delta x + \theta, \end{aligned} \tag{2.1}$$

where $x(s) = U(s)$ and $\beta, \gamma, \delta, \theta \in \mathbb{R}$ with $\beta\gamma\delta \neq 0$.

When $n = 1$, the solution of $\gamma x^2 - \delta x - \theta = 0$, that is,

$$x_{1,2} = \frac{\delta}{2\gamma} \mp \frac{\sqrt{\delta^2 + 4\gamma\theta}}{2\gamma}$$

must be real, otherwise there would be no algebraic traveling wave solutions. Therefore, $\delta^2 + 4\gamma\theta \geq 0$. Set $x = \bar{x} + x_1$, and $y = \bar{y}$. Then we rewrite system (2.1) with $n = 1$ in the variables $(\bar{x}, \bar{y})$ as

$$\begin{aligned} \bar{x}' &= \bar{y}, \\ \bar{y}' &= -\beta\bar{y} - \gamma(\bar{x} + x_1)^2 + \delta(\bar{x} + x_1) + \theta \\ &= -\beta\bar{y} - \gamma\bar{x}^2 - 2\gamma x_1\bar{x} - \gamma x_1^2 + \delta\bar{x} + \delta x_1 + \theta \\ &= -\beta\bar{y} - \gamma\bar{x}^2 + \bar{\delta}\bar{x}, \end{aligned} \tag{2.2}$$

where $\bar{\delta} = \delta - 2\gamma x_1 = \sqrt{\delta^2 + 4\gamma\theta}$.

When $n = 2$, the solution of $\gamma x^3 - \delta x - \theta = 0$ has at least one real solution, that we denote by x_1. Set $x = \bar{x} + x_1$, and $y = \bar{y}$. Then we rewrite system (2.1) with $n = 2$ in the variables $(\bar{x}, \bar{y})$ as

$$\begin{aligned} \bar{x}' &= \bar{y}, \\ \bar{y}' &= -\beta\bar{y} - \gamma(\bar{x} + x_1)^3 + \delta(\bar{x} + x_1) - \theta \\ &= -\beta\bar{y} - \gamma\bar{x}^3 - 3\gamma x_1\bar{x}^2 - 3\gamma x_1^2\bar{x} - \gamma x_1^3 + \delta\bar{x} + \delta x_1 - \theta \\ &= -\beta\bar{y} - \gamma\bar{x}^3 - \bar{\gamma}\bar{x}^2 + \bar{\delta}\bar{x}, \end{aligned} \tag{2.3}$$

where $\bar{\gamma} = 3\gamma x_1$ and $\bar{\delta} = \delta - 3\gamma x_1^2$.

3 Proof of Theorem 1.2 with $n = 1$

In this section we consider system (2.1) with $n = 1$. By the results in Section 2 this is equivalent to work with system (2.2).

Theorem 3.1. *System* (2.2) *has an invariant algebraic curve* $g(\bar{x}, \bar{y}) = 0$ *if and only if*

$$\beta = \pm\frac{5\sqrt{\bar{\delta}}}{\sqrt{6}}.$$

Moreover, if $\beta = 5\sqrt{\bar{\delta}}/\sqrt{6}$ *then*

$$g(\bar{x}, \bar{y}) = \frac{\bar{y}^2}{2} - \frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{\bar{\delta}}}{\gamma}(\bar{\delta} - \gamma\bar{x})\bar{y} + \frac{\bar{x}}{3\gamma}(\bar{\delta} - \gamma\bar{x})^2,$$

and if $\beta = -5\sqrt{\bar{\delta}}/\sqrt{6}$ *then*

$$g(\bar{x}, \bar{y}) = \frac{\bar{y}^2}{2} + \frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{\bar{\delta}}}{\gamma}(\bar{\delta} - \gamma\bar{x})\bar{y} + \frac{\bar{x}}{3\gamma}(\bar{\delta} - \gamma\bar{x})^2.$$

System (2.2) with $\bar{\delta} = \gamma$ is system (15) in [24]. Proceeding exactly as in the proof of Theorem 2 in [24] (with $\bar{\delta}$ instead of γ when needed) we get the proof of Theorem 3.1. So, the proof of Theorem 3.1 will be omitted.

Proof of Theorem 1.2. Consider first the case $\beta = \frac{5\sqrt{\bar{\delta}}}{\sqrt{6}}$. It follows from Theorem 3.1 that the invariant algebraic curve is

$$g(\bar{x}, \bar{y}) = \frac{\bar{y}^2}{2} - \frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{\bar{\delta}}}{\gamma}(\bar{\delta} - \gamma)\bar{y} + \frac{\bar{x}}{3\gamma}(\bar{\delta} - \gamma\bar{x})^2.$$

The branch of $g(x,y)=0$ that contains the origin is

$$y=\frac{\sqrt{2}}{\sqrt{3}\gamma}(\overline{\delta}-\gamma\overline{x})\left(\sqrt{\overline{\delta}}-\sqrt{\overline{\delta}-\gamma\overline{x}}\right).$$

Since $\overline{x}'=\overline{y}$ we obtain

$$\overline{x}'=\frac{\sqrt{2}}{\sqrt{3}\gamma}(\overline{\delta}-\gamma\overline{x})\left(\sqrt{\overline{\delta}}-\sqrt{\overline{\delta}-\gamma\overline{x}}\right)=\frac{\sqrt{2}\overline{\delta}^{3/2}}{\sqrt{3}\gamma}\left(1-\frac{\gamma}{\overline{\delta}}\overline{x}\right)\left(1-\sqrt{1-\frac{\gamma}{\overline{\delta}}\overline{x}}\right).$$

Set $U(s)=x(s)=\overline{x}(s)+x_1$ and take $W(s)=\sqrt{1-\frac{\gamma}{\overline{\delta}}(U(s)-x_1)}$ Then

$$W'(s)=-\frac{\gamma}{\overline{\delta}}\frac{U'(s)}{2\sqrt{1-\frac{\gamma}{\overline{\delta}}(U(s)-x_1)}}=-\frac{\sqrt{\overline{\delta}}}{\sqrt{6}}W(s)(1-W(s)).$$

Its non-constant solutions that are defined for all $s\in\mathbb{R}$ are

$$W(s)=\frac{1}{1+\kappa e^{\sqrt{\overline{\delta}}s/\sqrt{6}}},\quad \kappa>0.$$

Hence,

$$U(s)=x_1+\frac{\overline{\delta}}{\gamma}\left(1-\left(\frac{1}{1+\kappa e^{\sqrt{\overline{\delta}}s/\sqrt{6}}}\right)^2\right),\quad \kappa>0.$$

This, together with the definition x_1, $\overline{\delta}$, δ, γ and β, yields the traveling wave solution in the statement of the theorem.

If we take the branch of $g(\overline{x},\overline{y})=0$ that does not contain the origin then

$$y=\frac{\sqrt{2}}{\sqrt{3}\gamma}(\overline{\delta}-\gamma\overline{x})\left(\sqrt{\overline{\delta}}+\sqrt{\overline{\delta}-\gamma\overline{x}}\right)$$

Proceeding exactly as above we get that

$$W(s)=\frac{1}{1-\kappa e^{\sqrt{\overline{\delta}}s/\sqrt{6}}},\quad \kappa>0,$$

which is not a global solution. So, in this case there are no traveling wave solutions.

Now take $\beta=-\frac{5\sqrt{\delta}}{\sqrt{6}}$. It follows from Theorem 3.1 that the invariant algebraic curve is

$$g(\overline{x},\overline{y})=\frac{\overline{y}^2}{2}+\frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{\overline{\delta}}}{\gamma}(\overline{\delta}-\overline{\gamma})\overline{y}+\frac{\overline{x}}{3\gamma}(\overline{\delta}-\gamma\overline{x})^2.$$

The branch of $g(,\overline{y})=0$ that contains the origin is

$$\overline{y}=-\frac{\sqrt{2}}{\sqrt{3}\gamma}(\overline{\delta}-\gamma\overline{x})\left(\sqrt{\overline{\delta}}-\sqrt{\overline{\delta}-\gamma\overline{x}}\right)$$

Since $\overline{x}'=\overline{y}$ we obtain

$$\overline{x}'=-\frac{\sqrt{2}}{\sqrt{3}\gamma}(\overline{\delta}-\gamma\overline{x})\left(\sqrt{\overline{\delta}}-\sqrt{\overline{\delta}-\gamma\overline{x}}\right)=-\frac{\sqrt{2}\overline{\delta}^{3/2}}{\sqrt{3}\gamma}\left(1-\frac{\gamma}{\overline{\delta}}\overline{x}\right)\left(1-\sqrt{1-\frac{\gamma}{\overline{\delta}}\overline{x}}\right)$$

Set $U(s) = x(s) = \overline{x}(s) + x_1$ and take $W(s) = \sqrt{1 - \frac{\gamma}{\overline{\delta}}(U(s) - x_1)}$. Then

$$W'(s) = \frac{\gamma}{\overline{\delta}} \frac{U'(s)}{2\sqrt{1 - \frac{\gamma}{\overline{\delta}}(U(s) - x_1)}} = \frac{\sqrt{\delta}}{\sqrt{6}} W(s)(1 - W(s)).$$

Its nonconstant solutions that are defined for all $s \in \mathbb{R}$ are

$$W(s) = \frac{1}{1 + \kappa e^{-\sqrt{\delta}s/\sqrt{6}}}, \quad \kappa > 0.$$

Hence

$$U(s) = x_1 + \frac{\overline{\delta}}{\gamma}\left(1 - \left(\frac{1}{1 + \kappa e^{-\sqrt{\overline{\delta}}s/\sqrt{6}}}\right)^2\right), \quad \kappa > 0.$$

This, together with the definition x_1, $\overline{\delta}$, δ, γ and β, yields the traveling wave solution in the statement of the theorem.

If we take the branch of $g(\overline{x}, \overline{y}) = 0$ that does not contain the origin then

$$y = -\frac{\sqrt{2}}{\sqrt{3}\gamma}(\overline{\delta} - \gamma x)\left(\sqrt{\overline{\delta}} + \sqrt{\overline{\delta} - \gamma \overline{x}}\right).$$

Proceeding exactly as above we get that

$$W(s) = \frac{1}{1 - \kappa e^{-\sqrt{\overline{\delta}}s/\sqrt{6}}}, \quad \kappa > 0,$$

which is not a global solution. So, in this case there are no traveling wave solutions and concludes the proof of the theorem. □

4 Proof of Theorem 1.2 with $n = 2$

In this section we consider system (2.1) with $n = 2$. By the results in Section 2 this is equivalent to work with system (2.3).

The proof of Theorem 1.2 with $n = 2$ follows directly from the following theorem that states that system (2.3) has no invariant algebraic curves.

Theorem 4.1. *System (2.3) has no invariant algebraic curve.*

Proof of Theorem 4.1. Let $g = g(\overline{x}, \overline{y}) = 0$ be an invariant algebraic curve of system (2.3) with cofactor K. We write both g and K in their power series in the variable y as

$$K(\overline{x}, \overline{y}) = \sum_{j=0}^{2} K_j(x)\overline{y}^j, \quad g = \sum_{j=0}^{\ell} g_j(\overline{x})\overline{y}^\ell,$$

for some integer ℓ and where K_j is a polynomial in $\overline{x}$ of degree j. Without loss of generality, since $g \neq 0$ we can assume that $g_\ell = g_\ell(\overline{x}) \neq 0$. Moreover, note that if system (2.3) has an invariant algebraic curve then

$$\overline{y}\frac{\partial g}{\partial \overline{x}} - \left(\beta\overline{y} + \gamma\overline{x}^3 + \overline{\gamma x}^2 - \overline{\delta}\overline{x}\right)\frac{\partial g}{\partial \overline{y}} = Kg. \tag{4.1}$$

We compute the coefficient of $\overline{y}^{2+\ell}$ in (4.1) and we get

$$g_\ell K_2 = 0, \quad \text{that is} \quad K_2 = 0$$

because $g_\ell \neq 0$. So, $K(\overline{x}) = K_0(\overline{x}) + K_1(\overline{x})\overline{y}$. Computing the coefficient of $\overline{y}^{\ell+1}$ in (4.1) we obtain

$$g_\ell'(\overline{x}) = K_1 g_\ell$$

which yields $g_\ell = \kappa e^{\int K_1(\overline{x})\, d\overline{x}}$, for $\kappa \in \mathbb{C} \setminus \{0\}$. Since g_ℓ must be a polynomial then $K_1 = 0$. This implies that $K(\overline{x}) = K_0(\overline{x})$ that we write as

$$K(\overline{x}) = K_0(\overline{x}) = \sum_{j=0}^{2} k_j \overline{x}^j, \quad k_j \in \mathbb{R}.$$

Now, equation (1.5) writes as

$$\overline{y}\frac{\partial g}{\partial \overline{x}} - (\beta\overline{y} + \gamma\overline{x}^3 + \overline{\gamma}\overline{x}^2 - \overline{\delta}\overline{x})\frac{\partial g}{\partial \overline{y}} = \sum_{j=0}^{m} k_j \overline{x}^j g.$$

We introduce the weight-change of variables of the form

$$\overline{x} = \mu^{-2}X, \quad \overline{y} = \mu^{-4}Y, \quad t = \mu^2\tau.$$

In this form, system (2.3) becomes

$$\begin{aligned} X' &= Y, \\ Y' &= -\gamma X^3 - \mu^2\beta Y - \mu^2\overline{\gamma}X^2 + \overline{\delta}\mu^4 X, \end{aligned}$$

where the prime denotes derivative in τ. Now let

$$G(X, Y) = \mu^N g(\mu^{-2}X, \mu^{-4}Y)$$

and

$$\overline{K} = \mu^2 K = \mu^2(k_0 + k_1\mu^{-2}X + \mu^{-4}X^2) = \mu^2 k_0 + k_1 X + \mu^{-2}X^2,$$

where N is the highest weight degree in the weight homogeneous components of g in the variables x and y, with weight $(2, 4)$.

We note that $G = 0$ is an invariant algebraic curve of system (2.3) with cofactor $\mu^2 K$. Indeed

$$\frac{dG}{d\tau} = \mu^N \frac{dg}{d\tau} = \mu^N \mu^2 K g = \mu^N \overline{K} G.$$

Assume that $G = \sum_{i=0}^{\ell} G_i$ where G_i is a weight homogeneous polynomial in X, Y with weight degree $\ell - i$ for $i = 0, \ldots, \ell$ and $\ell \geq N$. Obviously

$$g = G|_{\mu=1}.$$

From the definition of invariant algebraic curve we have

$$\begin{aligned} &Y\sum_{i=0}^{\ell} \mu^i \frac{\partial G_i}{\partial X} - \left(\gamma X^3 + \mu^2\beta Y + \mu^2\overline{\gamma}X^2 - \overline{\delta}\mu^4 X\right)\sum_{i=0}^{\ell} \frac{\partial G_i}{\partial Y} \\ &= (\mu^2 k_0 + k_1 X + \mu^{-2}k_2 X^2)\sum_{i=0}^{\ell} \mu^i G_i. \end{aligned} \tag{4.2}$$

Computing the terms with μ^{-2} we get that $k_2 = 0$. Now the terms with μ^0 in (4.2) become

$$L[G_0] = k_1 G_0, \quad L = Y\frac{\partial}{\partial X} - \gamma X^3 \frac{\partial}{\partial Y}. \tag{4.3}$$

The characteristic equations associated with the first linear partial differential equation of system (2.3) are

$$\frac{dX}{dY} = -\gamma \frac{Y}{X^3}.$$

This system has the general solution $u = Y^2/2 + \gamma X^4/4 = \kappa$, where κ is a constant. According with the method of characteristics we make the change of variables

$$u = \frac{Y^2}{2} + \frac{\gamma}{4}X^4, \quad v = X. \tag{4.4}$$

Its inverse transformation is

$$Y = \pm\sqrt{2u - 2\gamma v^4/2}, \quad X = v. \tag{4.5}$$

In the following for simplicity we only consider the case $Y = +\sqrt{2u - \gamma v^4/2}$. Under changes (4.4) and (4.5), equation (4.3) becomes the following ordinary differential equation (for fixed u)

$$\sqrt{2u - \gamma v^4/2}\frac{d\overline{G}_0}{dv} = k_1 \overline{G}_0,$$

where $\overline{G}_0$ is G_0 written in the variables u, v. In what follows we always write $\overline{\theta}$ to denote a function $\theta = \theta(X, Y)$ written in the (u, v) variables, that is, $\overline{\theta} = \overline{\theta}(u, v)$. The above equation has the general solution

$$\overline{G}_0 = u^\ell \overline{F}_0(u) \exp\left(\frac{k_1}{\sqrt{2u}}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{4}, \frac{5}{4}, \frac{\gamma v^4}{4u}\right)\right),$$

where $\overline{F}_0$ is an arbitrary smooth function in the variable u and

$${}_2F_1(a, b, c, y) = \sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)}{b(b+1)\cdots(b+k-1)c(c+1)\cdots(c+k-1)} \frac{x^k}{k!} \tag{4.6}$$

is the hypergeometric function that is well defined if b, c are not negative integers. In particular, it is a polynomial if and only if a is a negative integer. Note that in this case ${}_2F_1$ is never a polynomial. Since

$$G_0(X, Y) = \overline{F}_0(u) = \overline{F}_0(Y^2/2 + \gamma X^4/4)$$

in order that $\overline{G}_0$ is a weight homogeneous polynomial of weight degree ℓ, since X and Y have weight degrees 2 and 4, respectively, we get that G_0 should be of weight degree $N = 8\ell$ and that $k_1 = 0$. Hence,

$$G_0 = a_\ell \left(\frac{Y^2}{2} + \gamma\frac{X^4}{4}\right)^\ell, \quad a_\ell \in \mathbb{R} \setminus \{0\}.$$

Computing the terms with μ in (4.2) using G_0 we get

$$L[G_1] = 0.$$

By the transformations in (4.4) and (4.5) and working in a similar way as we did to solve $\overline{G}_0$ we get the following ordinary differential equation

$$\sqrt{2u - \gamma v^4/2}\frac{d\overline{G}_1}{dv} = 0,$$

that is $\overline{G}_1 = \overline{G}_1(u)$. Since $\overline{G}_1$ is a weight homogeneous polynomial of weight degree $N - 1 = 8\ell - 1$ and u has even weight degree, we must have $\overline{G}_1 = 0$ and so $G_1 = 0$.

Computing the terms with μ^2 in (4.2) using the expression of G_0 and the fact that $G_1 = 0$ we get

$$\begin{aligned} L[G_2] &= \beta a_\ell \ell Y^2 \left(\frac{Y^2}{2} + \gamma\frac{X^4}{4}\right)^{\ell-1} + \overline{\gamma} a_\ell \ell X^2 Y \left(\frac{Y^2}{2} + \gamma\frac{X^4}{4}\right)^{\ell-1} + k_0 a_\ell \left(\frac{Y^2}{2} + \gamma\frac{X^4}{4}\right)^{\ell} \\ &= \beta a_\ell \ell \left(2\left(\frac{Y^2}{2} + \gamma\frac{X^4}{4}\right) - \frac{2}{3}\gamma X^4\right)\left(\frac{Y^2}{2} + \gamma\frac{X^4}{4}\right)^{\ell-1} + \overline{\gamma} a_\ell \ell X^2 Y \left(\frac{Y^2}{2} + \gamma\frac{X^4}{4}\right)^{\ell-1} \\ &\quad + k_0 a_\ell \left(\frac{Y^2}{2} + \gamma\frac{X^4}{4}\right)^{\ell} \\ &= a_\ell(2\beta\ell + k_0)\left(\frac{Y^2}{2} + \gamma\frac{X^4}{4}\right)^{\ell} - \frac{1}{2}\beta a_\ell \ell \gamma X^4 \left(\frac{Y^2}{2} + \gamma\frac{X^4}{4}\right)^{\ell-1} + \overline{\gamma} a_\ell \ell X^2 Y \left(\frac{Y^2}{2} + \gamma\frac{X^4}{4}\right)^{\ell-1}. \end{aligned}$$

By the transformations in (4.4) and (4.5) and working in a similar way to solve $\overline{G}_0$ we get the following ordinary differential equation

$$\sqrt{2u - \gamma v^4/2}\frac{d\overline{G}_2}{dv} = a_\ell(2\beta\ell + k_0)u^\ell - \frac{1}{2}\beta a_\ell \ell \gamma v^4 u^{\ell-1} + \overline{\gamma} a_\ell \ell v^2 \sqrt{2u - \gamma v^4/2}u^{\ell-1}.$$

Integrating this equation with respect to v we get

$$\begin{aligned} \overline{G}_2 = \overline{F}_2(u) &+ \frac{\beta\ell u^{\ell-1}}{6} v\sqrt{2u - \gamma v^4/2} + \frac{\overline{\gamma} a_\ell \ell}{3} v^3 u^{\ell-1} \\ &+ \frac{1}{3\sqrt{2}} u^{\ell-1/2} v(4\beta\ell + 3k_0)\, {}_2F_1\left(\frac{1}{2}, \frac{1}{4}, \frac{5}{4}, \frac{\gamma v^4}{8u}\right), \end{aligned}$$

where $\overline{F}_2$ is a smooth function in the variable u and ${}_2F_1$ is the hypergeometric function introduced in (4.6). Here, ${}_2F_1$ is never a polynomial. Since G_2 should be a polynomial in the variable X we must have that

$$4\beta\ell + 3k_0 = 0, \quad \text{that is} \quad k_0 = -\frac{4\beta\ell}{3}.$$

Now we apply Theorem 2.1. We recall that k_0 is a constant, $k_0 \neq 0$, and that in view of Theorem 2.1, g must vanish in the critical points of system (2.3), which are $(0, 0)$ and $(\psi_+, 0)$ and $(\psi_-, 0)$ where

$$\psi_\pm = \frac{-\overline{\gamma} \pm \sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}}{2\gamma}.$$

Moreover, the critical point $(0, 0)$ has the eigenvalues

$$\lambda^+ = -\frac{\beta}{2} + \frac{\sqrt{\beta^2 + 4\overline{\delta}}}{2} \quad \text{and} \quad \lambda^- = -\frac{\beta}{2} - \frac{\sqrt{\beta^2 + 4\overline{\delta}}}{2},$$

the critical point $(\psi_+, 0)$ has the eigenvalues

$$\mu^+ = -\frac{\beta}{2} + \frac{\sqrt{\beta^2 + 4T_+}}{2} \quad \text{and} \quad \mu^- = -\frac{\beta}{2} - \frac{\sqrt{\beta^2 + 4T_+}}{2}$$

being

$$T_+ = \frac{\left(\overline{\gamma} - \sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\right)\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}}{2\gamma},$$

and the critical point $(\psi_-, 0)$ has the eigenvalues

$$\nu^+ = -\frac{\beta}{2} + \frac{\sqrt{\beta^2 + 4T_-}}{2} \quad \text{and} \quad \nu^- = -\frac{\beta}{2} - \frac{\sqrt{\beta^2 + 4T_-}}{2}$$

being

$$T_- = \frac{\left(-\overline{\gamma} - \sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\right)\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}}{2\gamma}$$

We consider different cases.

Case 1: *$\overline{\delta}\gamma > 0$ and $\gamma < 0$.* In this case both $(\psi_+, 0)$ and $(\psi_-, 0)$ are saddles. In view of Theorem 2.1 we must have that

$$k_0 \in \{\mu^+, \mu^-, \mu^+ + \mu^-\} = \{\mu^+, \mu^-, -\beta\} \quad \text{and} \quad k_0 \in \{\nu^+, \nu^-, \nu^+ + \nu^-\} = \{\nu^+, \nu^-, -\beta\}.$$

Note that if $k_0 = -\beta$ then

$$-\frac{4\beta\ell}{3} = -\beta, \quad \text{that is} \quad \beta\frac{3 - 4\ell}{3} = 0,$$

which is not possible because $\beta \neq 0$ and ℓ is an integer with $\ell \geq 1$. So, $k_0 \in \{\mu^+, \mu^-\}$ and $k_0 \in \{\nu^+, \nu^-\}$. The only possibility is that $\overline{\gamma} = 0$. In this case

$$-\frac{4\beta\ell}{3} = -\frac{\beta}{2} \pm \frac{\sqrt{\beta^2 - 8\overline{\delta}}}{2}$$

which yields

$$\beta = \pm\frac{3\sqrt{-\overline{\delta}}}{\sqrt{14}}.$$

Moreover the eigenvalues on $(0, 0)$ are λ^+ and λ^-. If $\beta^2 + 4\overline{\delta} < 0$ then λ^+ and λ^- would be rationally independent and in view of Theorem 2.1, then $k_0 \in \{\lambda^+, \lambda^-, \lambda^+ + \lambda^-\} = \{\lambda^+, \lambda^-, -\beta\}$. But then this would imply that

$$\sqrt{-\overline{\delta}}(i\sqrt{47} \pm (8\ell + 3)) = 0,$$

which is not possible. Hence, $\beta^2 + 4\overline{\delta} > 0$. However

$$\beta^2 + 4\overline{\delta} = \frac{47\overline{\delta}}{14} < 0$$

and so this case is not possible.

Case 2: $\overline{\delta}\gamma > 0$ and $\gamma > 0$. In this case $(0,0)$ is a saddle. In view of Theorem 2.1 we must have that $k_0 \in \{\lambda^+, \lambda^-, \lambda^+ + \lambda^-\} = \{\lambda^+, \lambda^-, -\beta\}$. As in Case 1 we cannot have $k_0 = -\beta$. So, imposing that $k_0 \in \{\lambda^+, \lambda^-\}$ we conclude that

$$\beta = \pm \frac{3\sqrt{\overline{\delta}}}{2\sqrt{\ell(3+4\ell)}}.$$

Moreover if $\beta^2 + 4T_+ < 0$ we would have that μ^+ and μ^- are rationally independent and so $k_0 \in \{\mu^+, \mu^-, -\beta\}$. However, $\mu^+ = \lambda^+$ (respectively $\mu^- = \lambda^-$) if and only if

$$\overline{\gamma} = \frac{3i\sqrt{\overline{\delta}\gamma}}{\sqrt{2}},$$

which is not possible. So $\beta^2 + 4T_+ > 0$. Equivalently, if $\beta^2 + 4T_- < 0$ we would have that ν^- and ν^- are rationally independent and so $k_0 \in \{\nu^+, \nu^-, -\beta\}$. However, $\nu^+ = \lambda^+$ (respectively $\nu^- = \lambda^-$) if and only if

$$\overline{\gamma} = \frac{3i\sqrt{\overline{\delta}\gamma}}{\sqrt{2}},$$

which is not possible. So $\beta^2 + 4T_- > 0$. This implies that

$$\frac{9\overline{\delta}}{2\ell(3+4\ell)} > \frac{2}{\gamma}\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\left(\overline{\gamma} + \sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\right)$$

and

$$\frac{9\overline{\delta}}{2\ell(3+4\ell)} > \frac{2}{\gamma}\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\left(-\overline{\gamma} + \sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\right)$$

or, in short,

$$\frac{9\overline{\delta}}{2\ell(3+4\ell)} > \frac{2}{\gamma}\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\left(|\overline{\gamma}| + \sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\right) = 8\overline{\delta} + \frac{2}{\gamma}\left(|\overline{\gamma}|\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma} + \overline{\gamma}^2\right),$$

being $|\overline{\gamma}|$ the absolute value of $\overline{\gamma}$. Note that this in particular implies that

$$-\frac{\overline{\delta}(64\ell^2 + 48\ell - 9)}{2\ell(3+4\ell)} > \frac{2}{\gamma}\left(|\overline{\gamma}|\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma} + \overline{\gamma}^2\right) > 0,$$

which is not possible because $\overline{\delta} > 0$ and $\ell \geq 1$. So, this case is not possible.

Case 3: $\overline{\delta}\gamma < 0$ and $\gamma < 0$. In this case $(0,0)$ is a saddle. In view of Theorem 2.1 we must have that $k_0 \in \{\lambda^+, \lambda^-, \lambda^+ + \lambda^-\} = \{\lambda^+, \lambda^-, -\beta\}$. As in case 1 we cannot have $k_0 = -\beta$. So, imposing that $k_0 \in \{\lambda^+, \lambda^-\}$ we conclude that

$$\beta = \pm \frac{3\sqrt{\overline{\delta}}}{2\sqrt{\ell(3+4\ell)}}.$$

Now we assume that $\overline{\gamma} \leq 0$ (otherwise we will do the argument with T_- instead of T_+). Since T_+ is a saddle we must have $k_0 \in \{\mu^+, \mu^-, \mu^+ + \mu^-\} = \{\mu^+, \mu^-, -\beta\}$. Proceeding as in Case 2, we cannot have $k_0 = -\beta$ and equating it to either μ^+ or μ^- we obtain that

$$\overline{\gamma} = \frac{3i\sqrt{\overline{\delta}\gamma}}{\sqrt{2}} = -\frac{3\sqrt{|\overline{\delta}\gamma|}}{\sqrt{2}},$$

Now proceeding as in Case 1 we have that $\mu^+ = \nu^+$ (respectively $\mu^- = \nu^-$) if and only if $\overline{\gamma} = 0$, which in this case is not possible because then $\overline{\delta} = \delta$ and $\delta\gamma \neq 0$. So, $\beta^2 + 4T_- > 0$, otherwise we would have that ν^+ and ν^- would be rationally independent and so $k_0 \in \{\nu^+, \nu^-, -\beta\}$ which we already shown that it is not possible. So, $\beta^2 + 4T_- > 0$. However, using that $\mu^+ = \lambda^+$ and $\mu^- = \lambda^-$ (that is, $T_+ = \overline{\delta}$) we get that

$$\overline{\gamma}\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma} = 2\gamma\overline{\delta} + \overline{\gamma}^2 + 4\overline{\delta}\gamma$$

and so

$$\begin{aligned}\beta^2 + 4T_- &= \frac{9\overline{\delta}}{4(\ell(3+4\ell))} - \frac{4}{2\gamma}(2\overline{\gamma}^2 + 10\overline{\delta}\gamma) = \frac{9\overline{\delta}}{4(\ell(3+4\ell))} + \frac{2}{\gamma}|\overline{\delta}\gamma| \\ &= \frac{9\overline{\delta}}{4(\ell(3+4\ell))} - 2\overline{\delta} = \frac{\overline{\delta}}{4(\ell(3+4\ell))}(9 - 24\ell - 32\ell^2) < 0,\end{aligned}$$

because $\ell \geq 1$. In short, this case is not possible.

Case 4*: $\overline{\delta}\gamma < 0$ and $\gamma > 0$.* We consider the case $\overline{\gamma} \geq 0$ because the case $\overline{\gamma} < 0$ is the same working with T_- instead of T_+. Since $\overline{\gamma} \geq 0$ we have that T_+ is a saddle. In view of Theorem 2.1 we must have that $k_0 \in \{\lambda^+, \lambda^-, \lambda^+ + \lambda^-\} = \{\lambda^+, \lambda^-, -\beta\}$. As in Case 1 we cannot have $k_0 = -\beta$. So, imposing that $k_0 \in \{\lambda^+, \lambda^-\}$ we conclude that

$$\beta = \pm\frac{3\sqrt{T_+}}{2\sqrt{\ell(3+4\ell)}}.$$

Now proceeding as in Case 1, it follows from Theorem 2.1 that we have either $\mu^+ = \nu^+$ (respectively $\mu^- = \nu^-$) in the case in which $\beta^2 + 4T_- < 0$ (because they will be rationally independent), or $\beta^2 + 4T_- > 0$. In the first case, proceeding as in Case 1 we must have $\overline{\gamma} \geq 0$. Assume first that $\overline{\gamma} > 0$. Then,

$$\begin{aligned}\beta^2 + 4T_- &= \frac{1}{4\ell(3+4\ell)}(9T_+ + 16\ell(3+4\ell)T_-) \\ &= \frac{1}{8\gamma\ell(3+4\ell)}\Big(\overline{\gamma}\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}(9 - 16\ell(3+4\ell)) \\ &\quad - \left(\sqrt{\overline{\gamma}^2 + 4\overline{\delta}\gamma}\right)^2(9 + 16\ell(3+4\ell))\Big) < 0,\end{aligned}$$

which is not possible. So, $\overline{\gamma} = 0$. Then

$$\beta = \pm\frac{3\sqrt{-4\overline{\delta}}}{\sqrt{2}\sqrt{\ell(3+4\ell)}}.$$

Note that

$$\beta^2 + 4\overline{\delta} = \frac{9}{2\ell(3+4\ell)}|\overline{\delta}| - 4|\overline{\delta}| = \frac{|\overline{\delta}|}{2\ell(3+4\ell)}(9 - 8\ell(3+4\ell)) < 0.$$

So, again proceeding as in Case 1 we must have $k_0 \in \{\lambda^+, \lambda^-\}$. Imposing it we conclude that $\overline{\delta} = 0$ which is not possible because $\overline{\delta} = \delta \neq 0$ whenever $\overline{\gamma} = 0$. This concludes the proof of the theorem. □

5 Conclusions

In this paper we have characterized completely the algebraic traveling wave solutions of the Korteweg–de Vries–Burgers equation and of the Generalized Korteweg–de Vries–Burgers equation under some additional assumptions on the constants. The importance of this method is that can be used to completely characterize the algebraic traveling wave solutions of other well-known partial differential equations of any order provided that we are able to obtain the so-called Darboux polynomials. We emphasize that all the methods up to moment are not definite in the sense that if they do not work we cannot conclude that the system does not have traveling wave solutions, whereas in this method, if it fails, we can guarantee that there are not.

The cases of the Generalized Korteweg–de Vries–Burgers equation with $n \geq 3$ is unapproachable right now due to the fact that we are not able to compute the resulting Darboux polynomials, so these cases remain open.

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