An invariant set bifurcation theory for nonautonomous nonlinear evolution equations

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Abstract. In this paper we establish an invariant set bifurcation theory for the nonautonomous dynamical system $(\varphi_\lambda, \theta)_{X,\mathcal{H}}$ generated by the evolution equation

$$u_t + Au = \lambda u + p(t, u), \quad p \in \mathcal{H} = \mathcal{H}[f(\cdot, u)] \tag{0.1}$$

on a Hilbert space X, where A is a sectorial operator, λ is the bifurcation parameter, $f(\cdot, u) : \mathbb{R} \to X$ is translation compact, $f(t, 0) \equiv 0$ and $\mathcal{H}[f]$ is the hull of $f(\cdot, u)$. Denote by $\varphi_\lambda := \varphi_\lambda(t, p)u$ the cocycle semiflow generated by the system. Under some other assumptions on f, we show that as the parameter λ crosses an eigenvalue $\lambda_0 \in \mathbb{R}$ of A, the system bifurcates from 0 to a nonautonomous invariant set $B_\lambda(\cdot)$ on one-sided neighborhood of λ_0. Moreover,

$$\lim_{\lambda \to \lambda_0} H_{X^\alpha}(B_\lambda(p), 0) = 0, \quad p \in P,$$

where $H_{X^\alpha}(\cdot, \cdot)$ denotes the Hausdorff semidistance in X^α (here X^α ($\alpha \geq 0$) defined below is the fractional power spaces associated with A).

Our result is based on the pullback attractor bifurcation on the local central invariant manifolds $\mathcal{M}_{\text{loc}}^\lambda(\cdot)$.

Keywords: stability of pullback attractors, local invariant manifolds, nonautonomous invariant set bifurcations.

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1 Introduction

Invariant set bifurcation theory of autonomous dynamical systems has been extremely well developed [1, 6, 16, 17, 19, 23–27, 30–32]. A relatively simpler but important case is that of bifurcations from equilibria, including bifurcation to multiple equilibria (static bifurcation) and to periodic solutions (Hopf bifurcation) (see among others, [6, 27]). Ma and Wang [23] and Sanjurjio [31] developed a local attractor bifurcation theory. Roughly speaking, if the trivial equilibrium e of an autonomous system changes from an attractor to a repeller on the

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local center manifold of the equilibrium when the bifurcation parameter λ crosses a critical value λ_0, then the system bifurcates a compact invariant set K which is an attractor of the system restricted to the center manifold. Chow and Hale [6] started to discuss stability and bifurcation phenomena associated with more general invariant sets, e.g. periodic orbits. Using Conley index theory, Rybakowski [30] and Li and Wang [19] developed global bifurcation theorems to discuss bifurcation phenomena of nonlinear autonomous evolution equations.

The study of invariant set bifurcation for nonautonomous dynamical system has also received a lot of attention. Langa et al. [18] presented a collection of examples to illustrate bifurcation phenomena in nonautonomous ordinary differential equations. Carvalho et al. [4] studied the structure of the pullback attractor for a nonautonomous version of the Chafee–Infante equation, and investigated the bifurcations that this attractor undergoes as bifurcation parameter varies. In [28], Rasmussen introduced various concepts of bifurcation and transition for nonautonomous systems, corresponding to different time domains. And several examples were presented to illustrate these definitions.

The main aim of the paper is to develop a counterpart for the classical autonomous invariant set bifurcation patterns of Ma and Wang [23] and Sanjurjio [31] in the context of nonautonomous invariant set bifurcation. Unlike autonomous dynamical systems for which forward dynamics is studied, pullback dynamics is much more natural than the more familiar forward dynamics for nonautonomous dynamical systems. But this makes it very difficult to extend the invariant set bifurcation theory of autonomous systems to nonautonomous systems when pullback dynamics is considered. Our approach in the paper is to treat a nonautonomous system as a cocycle semiflow over a suitable base space. One of the advantage of a cocycle semiflow approach is that the synchronizing solutions or the other synchronizing behaviors with the nonautonomous driving force can be studied [12,14,15]. Moreover, in the framework of a cocycle semiflow, the base spaces are compact in many important cases. For example, if the nonlinearity f of (0.1) is periodic (resp. quasiperiodic, almost periodic, local almost periodic) in the time variable t, then the base space $\mathcal{H}$ is compact. Based on the compactness of the base spaces, we can establish the equivalence between pullback attraction of cocycle semiflow and forward attraction of the associated autonomous semiflow. This device makes the dynamics of such a nonautonomous system appear like those of an autonomous system.

Without the compactness assumption on the base spaces, the upper semicontinuity of global pullback attractors for nonautonomous systems was obtained in Caraballo and Langa [2]. However, compact forward invariant sets of the perturbed systems are required to guarantee the existence of perturbed pullback attractors. In the paper, we suppose that the base spaces of cocycle semiflows considered are compact. As a result, after introducing the notion of (local) pullback attractors (see Definition 2.6), we can establish a general result on the stability of local pullback attractors as the perturbation parameter is varied. Based on this result, a local pullback attractor bifurcation theory can be developed. This can be regarded as a generalization of autonomous attractor bifurcation theory in [23] for nonautonomous cases. Finally, we study the bifurcation of invariant sets for the cocycle semiflow φ_λ generated by the nonautonomous nonlinear evolution equation (0.1). We first construct a local central invariant manifold $\mathcal{M}_{\text{loc}}^\lambda(\cdot)$ for φ_λ as λ near λ_0. Under further assumptions on f to ensure that 0 is a pullback attractor for φ_{λ_0}, we then restrict φ_λ to $\mathcal{M}_{\text{loc}}^\lambda(\cdot)$ and obtain a pullback attractor bifurcation on $\mathcal{M}_{\text{loc}}^\lambda(\cdot)$ as λ crosses λ_0. It leads to an invariant set bifurcation for φ_λ. It is worth mentioning that if 0 is not an attractor but a repeller for φ_{λ_0}, our result still holds. Denote by

$B_\lambda(\cdot)$ the bifurcated invariant set. We further know that

$$\lim_{\lambda \to \lambda_0} H_{X^\alpha}(B_\lambda(p), 0) = 0, \quad p \in P.$$

This paper is organized as follows. In Section 2, we present respectively some basic facts in autonomous and nonautonomous dynamical systems which will be required in the rest of the work. Section 3 deals with the stability of pullback attractors as bifurcation parameter varies. In Section 4, we establish an invariant set bifurcation theory for (0.1). We illustrate the main results with an example in Section 5. Finally, Section 6 contains the proofs of two propositions presented earlier in the paper.

2 Preliminaries

In this section we introduce some basic definitions and notions [7,8].

Let X be a complete metric space with metric $d(\cdot, \cdot)$. Given $M \subset X$, we denote $\overline{M}$ and $\operatorname{int} M$ the closure and interior of any subset M of X, respectively. A set $U \subset X$ is called a neighborhood of $M \subset X$, if $\overline{M} \subset \operatorname{int} U$. For any $\rho > 0$, denote by

$$B_X(M, \rho) := \{x \in X : d(x, M) < \rho\}$$

the ρ-neighborhood of M in X, where $d(x, M) = \inf_{y \in M} d(x, y)$.

The Hausdorff semidistance in X is defined as

$$H_X(M, N) = \sup_{x \in M} d(x, N), \qquad \forall M, N \subset X.$$

2.1 Semiflows and attractors

Let $\mathbb{R}^+ = [0, \infty)$. A continuous mapping $S : \mathbb{R}^+ \times X \to X$ is called a semiflow on X, if it satisfies

i) $S(0, x) = x$ for all $x \in X$; and

ii) $S(t + s, x) = S(t, S(s, x))$ for all $x \in X$ and $t, s \in \mathbb{R}^+$.

Let S be a given semiflow on X. As usual, we will rewrite $S(t, x)$ as $S(t)x$.

A set $B \subset X$ is called **invariant** (resp. **positively invariant**) under S if $S(t)B = B$ (resp. $S(t)B \subset B$) for all $t \geq 0$.

Let B and C be subsets of X. We say that B **attracts** C under S, if

$$\lim_{t \to \infty} H_X(S(t)C, B) = 0.$$

Definition 2.1. A compact subset $\mathcal{A} \subset X$ is called an **attractor** for S, if it is invariant under S and attracts one of neighborhood of itself.

It is well known that if U is a compact positively invariant set of S, then the omega-limit set $\omega(U) := \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} S(t)U}$ is an attractor of S. The definition of the attraction basin of the attractor and other properties can be found in [10,22,30].

2.2   Cocycle semiflows and pullback attractors

A nonautonomous system consists of a "base flow" and a "cocycle semiflow" that is in some sense driven by the base flow.

A **base flow** $\{\theta_t\}_{t\in\mathbb{R}} := \{\theta(t)\}_{t\in\mathbb{R}}$ is a flow on a metric space P such that $\theta_t P = P$ for all $t \in \mathbb{R}$.

Definition 2.2. A **cocycle semiflow** φ on the phase space X over θ is a continuous mapping $\varphi : \mathbb{R}^+ \times P \times X \to X$ satisfying

- $\varphi(0, p, x) = x$,

- $\varphi(t + s, p, x) = \varphi(t, \theta_s p, \varphi(s, p, x))$ (cocycle property).

Remark 2.3. If we replace $\mathbb{R}^+$ by $\mathbb{R}$ in the above definition, then φ is called a cocycle flow on X.

We usually denote $\varphi(t, p)x := \varphi(t, p, x)$. Then $\{\varphi(t, p)\}_{t\geq 0,\, p\in P}$ can be viewed as a family of continuous mappings on X.

For convenience in statement, a family of subsets $\{B_p\}_{p\in P}$ of X is called a **nonautonomous set** in X. Let $B(\cdot) = \{B_p\}_{p\in P}$ be a nonautonomous set. For convenience, we will rewrite B_p as $B(p)$, called the p-**section** of $B(\cdot)$. We also denote $\mathbb{B}$ the union of the sets $B(p) \times \{p\}$ ($p \in P$), i.e.,

$$\mathbb{B} = \bigcup_{p\in P} B(p) \times \{p\}.$$

Note that $\mathbb{B}$ is a subset of $X \times P$.

A nonautonomous set $B(\cdot)$ is said to be closed (resp. open, compact), if each section $B(p)$ is closed (resp. open, compact) in X. A nonautonomous set $U(\cdot)$ is called a **neighborhood** of $B(\cdot)$, if $\overline{B}(p) \subset \operatorname{int} U(p)$ for each $p \in P$.

A nonautonomous set $B(\cdot)$ is said to be **invariant** (resp. **forward invariant**) under φ if for $t \geq 0$,

$$\varphi(t, p)B(p) = B(\theta_t p), \qquad p \in P.$$

$$(\text{resp. } \varphi(t, p)B(p) \subset B(\theta_t p), \qquad p \in P.)$$

Let $B(\cdot)$ and $C(\cdot)$ be two nonautonomous subsets of X. We say that $B(\cdot)$ **pullback attracts** $C(\cdot)$ under φ if for any $p \in P$,

$$\lim_{t\to\infty} H_X(\varphi(t, \theta_{-t}p)C(\theta_{-t}p),\, B(p)) = 0.$$

Let φ be a given cocycle semiflow on X with driving system θ on base space P. The (autonomous) semiflow $\Phi := \{\Phi(t)\}_{t\geq 0}$ on $Y := P \times X$, given by

$$\Phi(t)(p, x) = (\theta_t p, \phi(t, p)x), \quad t \geq 0,$$

is called the **skew product semiflow** associated to φ.

The following fundamental result studies the relationship between the pullback attraction of φ and attraction of Φ. The proof is given in the Appendixes.

Proposition 2.4. *Let $(\varphi, \theta)_{X,P}$ be a nonautonomous system, and let Φ be the skew-product flow associated to φ. Let $K(\cdot)$ and $B(\cdot)$ be two nonautonomous sets. Suppose P and $K_P := \overline{\bigcup_{p\in P} K(p)} \subset X$ are both compact. Then $K(\cdot)$ pullback attracts $B(\cdot)$ through φ if and only if $\mathbb{K} := \bigcup_{p\in P} K(p) \times \{p\}$ attracts $\mathbb{B} := \bigcup_{p\in P} B(p) \times \{p\}$ through Φ.*

Remark 2.5. The special case that $K(\cdot)$ is a global pullback attractor was considered in Theorem 15.7 and Theorem 15.8 of [5].

Definition 2.6. Let $(\varphi, \theta)_{X,P}$ be a nonautonomous system. A nonautonomous set $A(\cdot)$ is called a **(local) pullback attractor** for φ if it is compact, invariant and pullback attracts a neighborhood $U(\cdot)$ of itself.

The local pullback attractor defined here, very similar to the notion of a past attractor in Rasmussen [29], can be seen as a nature generalization of the local attractor from the autonomous theory. Similar to the case of autonomous systems, if $U(\cdot)$ is a compact forward invariant set of φ, then the omega-limit set $\omega(U)(\cdot)$ defined as

$$\omega(U)(\omega) = \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} \varphi(t, \theta_{-t}\omega)U(\theta_{-t}\omega)}, \quad \omega \in \Omega$$

is a pullback attractor of φ. For instance, consider the following simple system on $X = \mathbb{R}$:

$$x'(t) = -3x + p(t)x^3, \quad p \in \mathcal{H}[h], \tag{2.1}$$

where $h(t) = 2 + \sin t$ and $\mathcal{H}[h]$ is its hull which is the closure for the uniform convergence topology of the set of t-translates of h. The translation map $\theta_t : \mathcal{H} \to \mathcal{H}$ given by $\theta_t p(s) = p(t+s)$ defines a flow on $\mathcal{H}$. Then the unique solution of (2.1) define a cocycle flow on X given by $\varphi(t, p)x_0 = x(t, 0; p; x_0)$. Since

$$\frac{1}{2}\frac{d}{dt}x^2 = -3x^2 + p(t)x^4 \leq -3(x^2 - x^4) < 0$$

provided that $|x| \leq 1/2$. Therefore $[-1/2, 1/2]$ is a forward invariant set of φ and it is pullback attracted by the pullback attractor 0. It is worth noting that 0 is only a local pullback attractor. Indeed,

$$\frac{1}{2}\frac{d}{dt}x^2 = -3x^2 + p(t)x^4 \geq -3x^2 + x^4 > 0$$

provided that $|x| \geq 2$. Thus 0 is only a local pullback attractor of φ.

In general, it is difficult to define the attraction basin of a pullback attractor. Fortunately, under the assumptions of Proposition 2.4, we can define the pullback attraction basin of a pullback attractor $A(\cdot)$. Specifically, we have

Definition 2.7. Let $(\varphi, \theta)_{X,P}$ be a nonautonomous system, and let Φ be the skew-product flow associated to φ. Suppose P is compact. Let $A(\cdot)$ be a pullback attractor of φ such that $A_P := \overline{\bigcup_{p \in P} A(p)}$ is compact. Then the **pullback attraction basin** of $A(\cdot)$ can be given by

$$B(A)(\cdot) = \{x(\cdot) : \mathbb{A} \text{ attracts } \mathbb{x} \text{ under } \Phi\},$$

where $x(\cdot)$ is any singleton nonautonomous set in X and $\mathbb{x} = \bigcup_{p \in P}\{p\} \times x(p)$.

3 Stability of pullback attractors

We now establish a result on the stability of pullback attractors under a small perturbation. In fact, we prove a continuity result with respect to the Hausdorff semi-distance.

Let X be a Banach space with norm $\|\cdot\|$, and let A be a sectorial operator on X. Pick a number $a > 0$ such that

$$\mathrm{Re}\,\sigma(A + aI) > 0.$$

Denote $\Lambda = A + aI$. For each $\alpha \geq 0$, define the fractional power space as $X^\alpha = D(\Lambda^\alpha)$, which is equipped with the norm $\| \cdot \|_\alpha$ defined by

$$\|x\|_\alpha = \|\Lambda^\alpha x\|, \quad x \in X^\alpha.$$

Note that the definition of X^α is independent of the choice of the number a. If A has compact resolvent, the inclusion $X^{\alpha'} \hookrightarrow X^\alpha$ is compact for $\alpha' > \alpha \geq 0$.

Let φ_{λ_0} $(\lambda_0 \in \mathbb{R})$ be a given cocycle semiflow on X with driving system θ on base space P. For $\delta > 0$, denote $I_{\lambda_0}(\delta) := (\lambda_0 - \delta, \lambda_0 + \delta)$. Assume that φ_λ, $\lambda \in I_{\lambda_0}(\delta)$ is a small perturbation of the given flow φ_{λ_0} based on P. Let us make the following assumptions:

(H1): The base space P is compact.

(H2): For every $T > 0$ and compact subset B of X, we have

$$\lim_{\lambda \to \lambda_0} \|\varphi_\lambda(t, p)x - \varphi_{\lambda_0}(t, p)x\|_\alpha = 0, \tag{3.1}$$

uniformly with respect to $(t, x) \in [0, T] \times B$ and $p \in P$.

Under the assumptions **(H1)**, **(H2)**, we can get a result on the stability of pullback attractors.

Theorem 3.1. *Let $A_{\lambda_0}(\cdot) := \{A_{\lambda_0}(p)\}_{p \in P}$ be an attractor of the cocycle semiflow φ_{λ_0} which pullback attracts a neighborhood $U(\cdot)$ of itself. Let*

$$\mathbb{U} := \bigcup_{p \in P} U(p) \times \{p\} \quad and \quad \mathbb{A}_{\lambda_0} := \overline{\bigcup_{p \in P} A_{\lambda_0}(p) \times \{p\}}.$$

*Assume $\mathbb{U}$ is a compact neighborhood of $\mathbb{A}_{\lambda_0}$ in $Y = X \times P$, then under the assumptions **(H1)**, **(H2)**, the following statements hold.*

(a) *There exists a small $\delta > 0$ such that for each $\lambda \in I_{\lambda_0}(\delta)$, φ_λ has a pullback attractor $A_\lambda(\cdot)$ such that*

$$\lim_{\lambda \to \lambda_0} H_X(A_\lambda(p), \bigcup_{p \in \mathcal{H}} A_{\lambda_0}(p)) = 0. \tag{3.2}$$

(b) *In addition, if $U(\cdot)$ is forward invariant, then*

$$\lim_{\lambda \to \lambda_0} H_X(A_\lambda(p), A_{\lambda_0}(p)) = 0. \tag{3.3}$$

Proof. (a) By the compactness of $\mathbb{U}$, we know that $A_{\lambda_0 P} := \overline{\bigcup_{p \in P} A_{\lambda_0}(p)}$ is compact. Since $A_{\lambda_0}(\cdot)$ pullback attracts $U(\cdot)$ and P is compact, by Proposition 2.4, $\mathbb{A}_{\lambda_0}$ attracts $\mathbb{U}$ through Φ_{λ_0}. Since $\mathbb{U}$ is a neighborhood of $\mathbb{A}_{\lambda_0}$, one knows that $\mathbb{A}_{\lambda_0}$ is an attractor of Φ_{λ_0}. By the assumption **(H2)**, for any compact set $B \subset X$, we have that

$$\lim_{\lambda \to \lambda_0} H_Y(\Phi_\lambda(t)(x, p), \Phi_{\lambda_0}(t)(x, p)) = \lim_{\lambda \to \lambda_0} \|\varphi_\lambda(t, p)x - \varphi_{\lambda_0}(t, p)x\|_\alpha = 0 \tag{3.4}$$

uniformly with respect to $t \in [0, T]$ and $(x, p) \in B \times P$. Then by the stability of the autonomous attractors [21, Theorem 4.1], there exists a $\delta > 0$ (independent of $p \in P$) such that for each $\lambda \in I_{\lambda_0}(\delta) := (\lambda_0 - \delta, \lambda_0 + \delta)$, Φ_λ has an attractor $\mathbb{A}_\lambda$ contained in $\mathbb{U}$. Moreover,

$$\lim_{\lambda \to \lambda_0} H_Y(\mathbb{A}_\lambda, \mathbb{A}_{\lambda_0}) = 0. \tag{3.5}$$

Write $\mathbb{A}_\lambda$ as $\bigcup_{p \in P} A_\lambda(p) \times \{p\}$, $\lambda \in I_{\lambda_0}(\delta)$. Using Proposition 2.4 again, we have that $A_\lambda(\cdot)$ pullback attracts $U(\cdot)$ through φ_λ, i.e., $A_\lambda(\cdot)$ is a pullback attractor of φ_λ. (3.2) is a direct consequence of (3.5).

To complete the proof of (b), we shall prove (3.3) by contradiction. Thus, let us assume that there exist $\sigma > 0$ and a sequence $\lambda_j \to \lambda_0$, as $j \to \infty$, $x_j \in A_{\lambda_j}(p)$ such that

$$d_X(x_j, x) > \sigma, \quad \text{for all } x \in A_{\lambda_0}(p). \tag{3.6}$$

Note that

$$x_j = \varphi_{\lambda_j}(n, \theta_{-n}p)x_j^n, \quad \text{for some } x_j^n \in A_{\lambda_j}(\theta_{-n}p).$$

Similar to the argument in (a), we can assume that $A_{\lambda_j}(p) \subset U(p)$, thus $x_j \in U(p)$. By the compactness of $U(p)$, there exists a subsequence of x_j (still denoted by x_j) which converges to some $x_0 \in U(p)$. Now, for each fixed n we have $x_j^n \in U(\theta_{-n}p)$ so that there is a further subsequence of x_j^n (still denoted by x_j^n) which converges to some $x_0^n \in U(\theta_{-n}p)$. On the other hand, for any given $\nu > 0$, we can use the assumption **(H2)** and the continuity of $\varphi(n, \theta_{-n}p)$ to show that for j large enough,

$$3d(\varphi_{\lambda_j}(n, \theta_{-t}p)x_j^n, \varphi_{\lambda_0}(n, \theta_{-t}p)x_0^n)$$
$$\leq d(\varphi_{\lambda_j}(n, \theta_{-t}p)x_j^n, \varphi_{\lambda_0}(n, \theta_{-t}p)x_j^n) + d(\varphi_{\lambda_0}(n, \theta_{-t}p)x_j^n, \varphi_{\lambda_0}(n, \theta_{-t}p)x_0^n)$$
$$\leq \nu + \nu.$$

Then, for each fixed $n \in \mathbb{N}$,

$$x_0 = \lim_{j \to \infty} x_j = \lim_{j \to \infty} \varphi_{\lambda_j}(n, \theta_{-n}p)x_j^n = \varphi_{\lambda_0}(n, \theta_{-n}p)x_0^n.$$

Since $U(p)$ is forward invariant, we have

$$x_0 \in \bigcap_{n \in \mathbb{N}} \varphi_{\lambda_0}(n, \theta_{-n}p)U(\theta_{-n}p) = A_{\lambda_0}(p),$$

which contradicts (3.6). The proof is complete. $\qquad\square$

The main contribution of Theorem 3.1 is the existence of pullback attractor $A_\lambda(\cdot)$ for φ_λ as λ near λ_0, while the argument of the upper semicontinuity of pullback attractors is an adaptation of that of [2].

The conditions of the following results may be easier to be verified in applications.

Corollary 3.2. *Let $A_{\lambda_0}(\cdot) := \{A_{\lambda_0}(p)\}_{p \in P}$ be an attractor of the cocycle semiflow φ_{λ_0} and $U \subset X$ be a compact forward invariant neighborhood of $A_{\lambda_0}(\cdot)$. Then under the assumptions **(H1), (H2)**, there exists a small $\delta > 0$ such that for each $\lambda \in I_{\lambda_0}(\delta)$, φ_λ has a pullback attractor $A_\lambda(\cdot)$ satisfying*

$$\lim_{\lambda \to \lambda_0} H_X(A_\lambda(p), A_{\lambda_0}(p)) = 0.$$

4 Invariant set bifurcation for nonautonomous nonlinear evolution equations

Based on the general result of the stability of pullback attractors, in the section we can establish some results on invariant set bifurcation for nonautonomous dynamical systems.

4.1 Problem and mathematical setting

From now on, we assume X is a Hilbert space with inner product $(\cdot, \cdot)$. We will consider and study invariant set bifurcation of the evolution equation

$$u_t + Au = \lambda u + f(t, u) \tag{4.1}$$

on X, where $\lambda \in \mathbb{R}$ is a bifurcation parameter, the nonlinearity $f : \mathbb{R} \times X^\alpha \to X$ is bounded continuous mapping satisfying

(F1)

$$f(t, u) = o(\|u\|_\alpha), \qquad \text{as } \|u\|_\alpha \to 0 \tag{4.2}$$

uniformly on $t \in \mathbb{R}$. Moreover, there is $\beta > 0$ such that

$$((f(t, u), u) \leq -\beta \cdot \kappa(u) \tag{4.3}$$

for $t \in \mathbb{R}$ and $u \in X^\alpha$, where $\kappa : X \to \mathbb{R}^+$ is a nonnegative function satisfying that $\kappa(u) = 0$ if and only if $u = 0$.

Denote $k(\rho)$ the Lipschitz constant of $f(t, \cdot)$ in $\overline{B}_{X^\alpha}(\rho)$. Then by (4.2),

$$\lim_{\rho \to 0} k(\rho) = 0$$

and

$$\|f(t, u_1) - f(t, u_2)\| \leq k(\rho)\|u_1 - u_2\|_\alpha, \qquad \forall u_1, u_2 \in \overline{B}_{X^\alpha}(\rho). \tag{4.4}$$

Denote $C_b(\mathbb{R}, X)$ the set of bounded continuous functions from $\mathbb{R}$ to X. Equip $C_b(\mathbb{R}, X)$ with either the uniform convergence topology generated by the metric

$$r(h_1, h_2) = \sup_{t \in \mathbb{R}} \|h_1(t) - h_2(t)\|,$$

or the compact-open topology generated by the metric

$$r(h_1, h_2) = \sum_{n=1}^\infty \frac{1}{2^n} \cdot \frac{\max_{t \in [-n,n]} \|h_1(t) - h_2(t)\|}{1 + \max_{t \in [-n,n]} \|h_1(t) - h_2(t)\|}.$$

Let $f(\cdot, u) \in C_b(\mathbb{R}, X)$ be the function in (4.1). Define the hull of $f(\cdot, u)$ as follows

$$\mathcal{H} := \mathcal{H}[f(\cdot, u)] = \overline{\{f(\tau + \cdot, u) : \ \tau \in \mathbb{R}\}}^{C_b(\mathbb{R}, X)}.$$

In application, $f(\cdot, u)$ is often taken as a periodic function, quasiperiodic function, almost periodic function, local almost periodic function [7,20] or uniformly almost automorphic function [33]. In this case, the hull $\mathcal{H}$ is a **compact** metric space. Accordingly, the translation group θ on $\mathcal{H}$ is given by

$$\theta_\tau p(\cdot, u) = p(\tau + \cdot, u), \qquad t \in \mathbb{R}, \ p \in \mathcal{H}.$$

Instead of (3.2), we will consider the more general cocycle system in X^α (where $\alpha \in [0, 1)$):

$$u_t + Au = \lambda u + p(t, u), \qquad p \in \mathcal{H}. \tag{4.5}$$

Proposition 4.1 ([11]). *Let A and p be given as above. Assume that p is locally Hölder continuous in t. Then for each $u_0 \in X^\alpha$, there is a $T > t_0$ such that (4.5) has a unique solution $u(t) = u_\lambda(t, t_0; u_0, p)$ on $[t_0, T)$ satisfying*

$$u(t) = e^{-A(t-t_0)}x_0 + \int_{t_0}^t e^{-A(t-s)}[\lambda u(s) + p(s, u(s))]ds, \quad t \in [t_0, T). \tag{4.6}$$

For convenience, from now on we always assume that the unique solution (4.6) is globally defined. Define

$$\varphi_\lambda(t, p)u := u_\lambda(t, 0; u, p), \qquad u \in X^\alpha, \ p \in \mathcal{H}.$$

Then φ_λ is a **cocycle semiflow** on X^α driven by the base flow θ on $\mathcal{H}$.

Remark 4.2. Note that for each $p \in \mathcal{H}$, $u(t)$ is a p-solution of φ_λ on an interval J *if and only if* it solves the equation (4.5) on J.

4.2 Local invariant manifolds

Let $\lambda_0 \in \mathbb{R}$ be an isolated eigenvalue of A. Suppose that

(F2) there is a $\eta > 0$ such that the spectrum

$$\sigma(A) \cap \{z \in \mathbb{C} : \lambda_0 - \eta < \operatorname{Re} z < \lambda_0 + \eta\} = \lambda_0.$$

Denote $A_\lambda := A - \lambda$. Then for $\lambda \in I_{\lambda_0}(\eta/4) := (\lambda_0 - \eta/4, \lambda_0 + \eta/4)$, the spectrum $\sigma(A_\lambda)$ has a decomposition $\sigma(A_\lambda) = \sigma_c \cup \sigma_+ \cup \sigma_-$, where

$$\sigma_c = \{\lambda_0 - \lambda\}, \quad \sigma_+ = \sigma(A_\lambda) \cap \{\operatorname{Re} \lambda > 0\} \quad \text{and} \quad \sigma_- = \sigma(A_\lambda) \cap \{\operatorname{Re} \lambda < 0\}.$$

Accordingly, the space X has a direct sum decomposition: $X = X_c \oplus X_+ \oplus X_-$. Denote $X_\pm = X_+ \oplus X_-$ and

$$X_i^\alpha := X_i \cap X^\alpha, \qquad i = c, +, -, \pm.$$

Note that X_c^α is finite dimensional.

Under the assumptions on A and f, we can construct a local invariant manifold for φ_λ, $\lambda \in I_{\lambda_0}(\eta/8)$.

Proposition 4.3. *Suppose the assumptions* **(F1), (F2)** *hold. Then there exists $\varrho > 0$ such that the cocycle semiflow φ_λ, $\lambda \in I_{\lambda_0}(\eta/8)$ has a local invariant manifold $\mathcal{M}_{\text{loc}}^\lambda(\cdot) := \{\mathcal{M}_{\text{loc}}^\lambda(p)\}_{p \in \mathcal{H}}$ in X^α which is represented as*

$$\mathcal{M}_{\text{loc}}^\lambda(p) = \{y + \xi_p^\lambda(y) : y \in \overline{B}_{X_c^\alpha}(\varrho)\},$$

where $\xi_p^\cdot(\cdot) : I_{\lambda_0}(\eta/8) \times \overline{B}_{X_c^\alpha}(\varrho) \to X_\pm^\alpha$ is a Lipschitz continuous mapping satisfying that

$$\xi_p^\lambda(0) = 0 \text{ and } \|\xi_p^\lambda(y) - \xi_p^\lambda(z)\|_\alpha \le L_1 \|y - z\|_\alpha \tag{4.7}$$

and

$$\|\xi_p^{\lambda_1}(y) - \xi_p^{\lambda_2}(y)\|_\alpha \le L_2 |\lambda_1 - \lambda_2|, \tag{4.8}$$

where $L_1 > 0$ is independent of $p \in P$ and $\lambda \in I_{\lambda_0}(\eta/8)$, and $L_2 > 0$ is independent of $p \in P$ and $y \in \overline{B}_{X_c^\alpha}(\varrho)$.

The proof of the above proposition is given in the Appendixes.

4.3 Invariant set bifurcation

Firstly, let us restrict the equation (4.5) on the invariant manifold $\mathcal{M}_{loc}^{\lambda}(\cdot)$, $\lambda \in I_{\lambda_0}(\eta/8)$. Specifically, we study the finite dimensional equation

$$y_t + (\lambda_0 - \lambda)y = p(t, y + \xi_{\theta_t p}^{\lambda}(y)), \quad y \in \overline{B}_{X_c^{\alpha}}(\varrho), \, p \in \mathcal{H}. \tag{4.9}$$

Denote ϕ_{λ}, $\lambda \in I_{\lambda_0}(\eta/8)$ the cocycle flow on $\overline{B}_{X_c^{\alpha}}(\varrho)$ with driving system θ on the base space $\mathcal{H}$ generated by (4.9).

We first say that the condition **(H2)** (in Section 3) holds for the cocycle flow ϕ_{λ}, $\lambda \in I_{\lambda_0}(\eta/8)$. Specifically, we have the following result.

Lemma 4.4. *For every $T > 0$, we have*

$$\lim_{\lambda \to \lambda_0} \|\phi_{\lambda}(t, p)y - \phi_{\lambda_0}(t, p)y\|_{\alpha} = 0, \tag{4.10}$$

uniformly with respect to $(t, y) \in [0, T] \times \overline{B}_{X_c^{\alpha}}(\varrho)$ and $p \in P$.

Proof. For $\lambda \in I_{\lambda_0}(\eta/8)$, denote $y_{\lambda}(t) := \phi_{\lambda}(t, p)y$ and $v(t) = y_{\lambda}(t) - y_{\lambda_0}(t)$, then v satisfies

$$v_t + (\lambda_0 - \lambda)y_{\lambda} = p(t, y_{\lambda} + \xi_{\theta_t p}^{\lambda}(y_{\lambda})) - p(t, y_{\lambda_0} + \xi_{\theta_t p}^{\lambda_0}(y_{\lambda_0})). \tag{4.11}$$

Note that $\|y_{\lambda}\| \leq \rho$ and

$$\begin{aligned}
&\|p(t, y_{\lambda} + \xi_{\theta_t p}^{\lambda}(y_{\lambda})) - p(t, y_{\lambda_0} + \xi_{\theta_t p}^{\lambda_0}(y_{\lambda_0}))\| \\
&\leq k(\rho)\big((L_1 + 1)\|v\|_{\alpha} + L_2|\lambda - \lambda_0|\big) \\
&\leq C'\left(\|v\|^2 + (\lambda - \lambda_0)^2\right) \text{ for some constant } C',
\end{aligned} \tag{4.12}$$

where $\rho > 0$ is the bound of $u \in \mathcal{M}_{loc}^{\lambda}(\cdot)$, which is independent of λ by (4.6). Taking the inner product of the equation (4.11) with v and using (4.12) to obtain that there is a constant $C > 0$ being independent of λ such that

$$\frac{d}{dt}\|v\|^2 \leq C\left(\|v\|^2 + (\lambda - \lambda_0)^2\right).$$

Applying the classical Gronwall lemma to get that

$$\|v(t)\|^2 \leq \left(e^{Ct} - 1\right)(\lambda - \lambda_0)^2, \qquad \square$$

Lemma 4.5. *Under the assumptions **(F1), (F2)**, $y = 0$ is locally asymptotically stable for ϕ_{λ_0}. Therefore 0 is a pullback attractor of ϕ_{λ_0}.*

Proof. Since X_c^{α} is finite dimensional, all the norms on X_c^{α} are equivalent. Hence for convenience, we equip X_c^{α} the norm $\|\cdot\|$ of X in the following argument.

Note that ϕ_{λ_0} is generated by the equation

$$y_t = p(t, y + \xi_{\theta_t p}^{\lambda_0}(y)), \quad y \in \overline{B}_{X_c^{\alpha}}(\varrho), \, p \in \mathcal{H}. \tag{4.13}$$

Taking the inner product of the equation (4.13) with $y + \xi_{\theta_t p}^{\lambda_0}(y)$ in X, using the fact that $(y, \xi_{\theta_t p}^{\lambda_0}(y)) = 0$ and the assumption **(F1)**, it yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|y\|^2 &= \left(p\left(t, y + \xi_{\theta_t p}^{\lambda_0}(y)\right), y + \xi_{\theta_t p}^{\lambda_0}(y)\right) \\
&\leq -\beta \cdot \kappa\left(y + \xi_{\theta_t p}^{\lambda_0}(y)\right).
\end{aligned} \tag{4.14}$$

It is clear that $\kappa\left(y + \xi_{\theta_t p}^{\lambda_0}(y)\right) = 0$ if and only if $y = 0$. Therefore $\lim_{t \to \infty} \|y\| = 0$. The proof is complete. $\qquad\square$

Henceforth we will suppose that

(F3) The hull $\mathcal{H}$ is a compact metric space.

We then obtain a pullback attractor bifurcation theory for ϕ_λ as λ crosses λ_0.

Theorem 4.6. *Under the assumptions* **(F1)**, **(F2)** *and* **(F3)**, *the cocycle semiflow ϕ_λ bifurcates from $(0, \lambda_0)$ a pullback attractor $A_\lambda(\cdot)$ for $\lambda > \lambda_0$, and*

$$\lim_{\lambda \to \lambda_0^+} H_{X_c^\alpha}(A_\lambda(p), \{0\}) = 0. \tag{4.15}$$

Proof. Recall from Lemma 4.5 that 0 is a pullback attractor for ϕ_{λ_0} and it pullback attracts $\overline{B}_{X_c^\alpha}(\varrho)$ for sufficiently small $\varrho > 0$. The bounded set $\overline{B}_{X_c^\alpha}(\varrho) \subset X_c^\alpha$ is compact due to X_c^α being finite dimensional. Moreover, $\overline{B}_{X_c^\alpha}(\varrho)$ is forward invariant under ϕ_{λ_0}. Then by Theorem 3.1, there is a $\eta' \in (0, \eta/8)$ such that for each $\lambda \in I_{\lambda_0}(\eta')$, the cocycle semiflow ϕ_λ has a pullback attractor $A_\lambda(\cdot)$ and (4.15) holds.

In the following, we prove that $0 \notin A_\lambda(\cdot)$ for $\lambda \in I_{\lambda_0}^+(\eta') := (\lambda_0, \lambda_0 + \eta')$, which completes the proof.

Let $\lambda \in I_{\lambda_0}^+(\eta')$ be fixed, and let $w(t) = y(-t)$. Then $w(t)$ satisfies

$$w_t - (\lambda_0 - \lambda)w = -p(-t, w + \xi_{\theta_{-t} p}^\lambda(w)). \tag{4.16}$$

Taking the inner product of the equation (4.16) with w in X^α, we have

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 - (\lambda_0 - \lambda)\|w\|^2 = -(p(t, w + \xi_{\theta_{-t} p}^\lambda(w)), w). \tag{4.17}$$

Since

$$\|p(t, u)\| \le k(\|u\|_\alpha)\|u\|_\alpha \quad \text{and} \quad \|\xi_{\theta_{-t} p}^\lambda(w)\|_\alpha \le L_1\|w\|_\alpha,$$

we have

$$
\begin{aligned}
\|p(-t, w + \xi_{\theta_{-t} p}^\lambda(w))\| &\le k(\|w + \xi_{\theta_{-t} p}^\lambda(w)\|_\alpha)\|w + \xi_{\theta_{-t} p}^\lambda(w)\|_\alpha \\
&\le k(\|w + \xi_{\theta_{-t} p}^\lambda(w)\|_\alpha)\left(\|w\|_\alpha + L_1\|w\|_\alpha\right) \\
&\le k(\|w + \xi_{\theta_{-t} p}^\lambda(w)\|_\alpha) \cdot (1 + L_1)\|w\|_\alpha \\
&\le [(1 + L_1)ck(\|w + \xi_{\theta_{-t} p}^\lambda(w)\|_\alpha)] \cdot \|w\| \\
&\le \frac{1}{2}(\lambda - \lambda_0)\|w\|, \text{ for sufficiently small } \|w\|_\alpha.
\end{aligned}
\tag{4.18}
$$

We get from (4.17) and (4.18) that

$$\frac{d}{dt}\|w\|^2 \le -(\lambda - \lambda_0)\|w\|^2$$

for sufficiently small $\|w\|_\alpha$, which shows for fixed $\lambda \in I_{\lambda_0}^+(\eta')$, 0 locally asymptotically stable for the cocycle flow generated by the equation (4.16). In other words, 0 is a repeller of ϕ_λ when $\lambda \in I_{\lambda_0}^+(\eta')$ and repels a neighborhood of 0 in X_c^α. This implies that $0 \notin A_\lambda(\cdot)$, $\lambda \in I_{\lambda_0}^+(\eta')$. The proof is complete. $\qquad\square$

We are now in position to give and prove the main result of this paper.

Theorem 4.7. *Under the assumptions* **(F1)**, **(F2)** *and* **(F3)**, *the cocycle semiflow φ_λ bifurcates from $(0, \lambda_0)$ an invariant compact set $B_\lambda(\cdot)$ for $\lambda > \lambda_0$, and for each $p \in P$,*

$$\lim_{\lambda \to \lambda_0^+} H_X(B_\lambda(p), \{0\}) = 0. \tag{4.19}$$

Proof. Let $A_\lambda(\cdot)$ be the bifurcated attractor obtained in Theorem 4.6. Define $B_\lambda(\cdot)$ by

$$B_\lambda(p) = \{y + \xi_p^\lambda(y) : y \in A_\lambda(p)\}, \quad p \in \mathcal{H}. \tag{4.20}$$

We know from Theorem 4.6 that $0 \notin B_\lambda(\cdot)$ and $B_\lambda(\cdot) \subset \mathcal{M}_{loc}^\lambda(\cdot)$. Based on the compactness of $A_\lambda(p)$ and the continuity of $\xi_p^\lambda(y)$ in y, we can directly derive the compactness of $B_\lambda(p)$. So $B_\lambda(\cdot)$ is compact.

We claim that $B_\lambda(\cdot)$ is invariant under φ_λ. Indeed, let $p \in P$ and $y + \xi_p^\lambda(y) \in B_\lambda(p)$. Since $\phi_\lambda(t, p)y \in A_\lambda(\theta_t p)$, $t \geq 0$, by the invariance of $\mathcal{M}_{loc}^\lambda(\cdot)$, we have

$$\varphi_\lambda(t, p)(y + \xi_p^\lambda(y)) = \phi_\lambda(t, p)y + \xi_{\theta_t p}^\lambda(\phi_\lambda(t, p)y) \in B_\lambda(\theta_t p),$$

which shows

$$\varphi_\lambda(t, p)B_\lambda(p) \subset B_\lambda(\theta_t p), \quad t \geq 0.$$

On the other hand, for any $y + \xi_{\theta_t p}^\lambda(y) \in B_\lambda(\theta_t p)$, $t \geq 0$. Using the invariance of $A_\lambda(\cdot)$, there is a $y' \in A_\lambda(p)$ such that $y = \phi_\lambda(t, p)y'$. Then

$$\begin{aligned}
y + \xi_{\theta_t p}^\lambda(y) &= \phi_\lambda(t, p)y' + \xi_{\theta_t p}^\lambda(\phi_\lambda(t, p)y') \\
&= \varphi_\lambda(t, p)(y' + \xi_{\theta_t p}^\lambda(y')) \in \varphi(t, p)B_\lambda(p),
\end{aligned}$$

which shows

$$B_\lambda(\theta_t p) \subset \varphi_\lambda(t, p)B_\lambda(p), \quad t \geq 0.$$

Therefore $B_\lambda(\cdot)$ is invariant under φ_λ.

Finally, (4.19) is an immediately consequence of (4.15) and (4.7). □

We now give a result which parallels Theorem 4.7.

Corollary 4.8. *Let the assumptions* **(F1)**, **(F2)**, **(F3)** *hold, but replace* (4.3) *by the assumption that*

$$(f(t, u), u) \geq \beta \cdot \kappa(u).$$

Then the cocycle semiflow φ_λ bifurcates from $(0, \lambda_0)$ an invariant compact set $B_\lambda(\cdot)$ for $\lambda < \lambda_0$, and for each $p \in P$,

$$\lim_{\lambda \to \lambda_0^-} H_X(B_\lambda(p), \{0\}) = 0. \tag{4.21}$$

Proof. Let $\lambda \in I_{\lambda_0}(\eta/8)$. Consider the following equation

$$z_t - (\lambda_0 - \lambda)z = -p(-t, z + \xi_{\theta_{-t} p}^\lambda(z)), \quad z \in \overline{B}_{X_c^\alpha}(\varrho), \ p \in \mathcal{H}. \tag{4.22}$$

Denote by ϕ_λ^- be the cocycle flow generated by (4.22). Then ϕ_λ^- be the inverse flow of ϕ_λ.

Repeating the argument of Lemma 4.4, Lemma 4.5 and Theorem 4.6 (replacing ϕ_λ by ϕ_λ^-) to show ϕ_λ^- bifurcates from $(0, \lambda_0)$ a pullback attractor $R_\lambda(\cdot)$ for $\lambda < \lambda_0$, and

$$\lim_{\lambda \to \lambda_0^-} H_{X_c^\alpha}(R_\lambda(p), \{0\}) = 0.$$

It is clear that $R_\lambda(\cdot)$ is also an invariant set of ϕ_λ. Define a set $B_\lambda(\cdot)$ by

$$B_\lambda(p) = \{y + \xi_p^\lambda(y) : y \in R_\lambda(p)\}, \quad p \in \mathcal{H}.$$

Similar to Theorem 4.7, we can show $B_\lambda(\cdot)$ is an invariant set of φ_λ and (4.21) holds. □

5   An example

Consider the nonautonomous system

$$\begin{cases} u_t - \Delta u = \lambda u \pm h(t)u^3, & t > 0, \ x \in \Omega; \\ u = 0, & t > 0, \ x \in \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary, h is a function such that $h(t) \geq \delta > 0$ for some $\delta > 0$.

Denote by A the operator $-\Delta$ associated with the homogeneous Dirichlet boundary condition. Then A is a sectorial operator on $X = L^2(\Omega)$ with compact resolvent, and $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Note that A has eigenvalues $0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots$. Denote $V = H_0^1(\Omega)$. By $(\cdot, \cdot)$ and $|\cdot|$ we denote the usual inner product and norm on H, respectively. The inner product and norm on V, denoted by $((\cdot, \cdot))$ and $\|\cdot\|$, respectively, are defined as

$$((u, v)) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \quad \|u\| = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x \right)^{1/2}$$

for $u, v \in V$.

The system (5.1) can be written into an abstract equation on X:

$$u_t + Au = \lambda u \pm h(t)u^3.$$

Define the hull $\mathcal{H} := \mathcal{H}[h(\cdot)u^3]$. By the assumption on h, it is clear that

$$(p(t, u), u) \geq \delta \int_\Omega u^4 dx, \quad p \in \mathcal{H}.$$

Consider the cocycle system:

$$u_t + Au = \lambda u \pm p(t, u), \qquad p \in \mathcal{H}. \tag{5.2}$$

Denote $\varphi_\lambda^\pm := \varphi_\lambda^\pm(t, p)u$ the cocycle semiflow on $H_0^1(\Omega)$ driven by the base flow (translation group) θ on $\mathcal{H}$.

Since all the hypotheses in the main theorem above are fulfilled, we obtain some interesting results concerning the dynamics of the perturbed system. In particular,

Theorem 5.1. *Suppose $\mathcal{H}$ is compact. Then the cocycle semiflow φ_λ^- (resp. φ_λ^+) bifurcates from $(0, \mu_k)$, $k = 1, 2, \cdots$ an invariant compact set $B_\lambda^-(\cdot)$ for $\lambda > \lambda_0$ (resp. $B_\lambda^+(\cdot)$ for $\lambda < \lambda_0$) and for each $p \in P$,*

$$\lim_{\lambda \to \lambda_0^+} H_X(B_\lambda^-(p), \{0\}) = 0.$$

$$(\text{resp. } \lim_{\lambda \to \lambda_0^-} H_X(B_\lambda^+(p), \{0\}) = 0.)$$

6 Appendixes

6.1 Relationship between the pullback attraction of φ and the attraction of Φ

Proof of Proposition 2.4. Necessity: By the compactness of P, one finds that

$$
\begin{aligned}
\lim_{t\to\infty} H_Y \left(\Phi(t)\mathbb{B}, P \times K_P\right) &= \lim_{t\to\infty} H_X \left(\varphi(t,p)B(p), K_P\right) \\
&\leq \lim_{t\to\infty} \sup_{p\in P} H_X \left(\varphi(t,p)B(p), K_P\right) \\
&= \lim_{t\to\infty} \sup_{p\in P} H_X \left(\varphi(t,\theta_{-t}p)B(\theta_{-t}p), K_P\right) \\
&= 0.
\end{aligned}
$$

This means the compact set $P \times K_P$ attracts $\mathbb{B}$ through Φ. Therefore the omega-limit set $\omega(\mathbb{B})$ of $\mathbb{B}$ exists and attracts $\mathbb{B}$.

In the following, we prove $\omega(\mathbb{B}) \subset \mathbb{K}$, which completes the necessity. For this purpose, define a nonautonomous set $\tilde{B}(\cdot)$ as follows

$$
\tilde{B}(p) := \overline{\bigcup_{s\geq 0} \varphi(s,\theta_{-s}p)B(\theta_{-s}p)}, \quad p \in P.
$$

It is clear that $B(\cdot) \subset \tilde{B}(\cdot)$. We first say $\tilde{B}(\cdot)$ is forward invariant. Indeed, for any $t \geq 0$ and $p \in P$,

$$
\begin{aligned}
\varphi(t,p)\tilde{B}(p) &= \varphi(t,p)\overline{\bigcup_{s\geq 0} \varphi(s,\theta_{-s}p)B(\theta_{-s}p)} \\
&\subset \overline{\bigcup_{s\geq 0} \varphi(t,p) \circ \varphi(s,\theta_{-s}p)B(\theta_{-s}p)} \\
&= \overline{\bigcup_{s\geq 0} \varphi(t+s,\theta_{-(t+s)} \circ \theta_t p)B(\theta_{-(t+s)} \circ \theta_t p)} \\
&\subset \overline{\bigcup_{s\geq 0} \varphi(s,\theta_{-s} \circ \theta_t p)B(\theta_{-s} \circ \theta_t p)} = \tilde{B}(\theta_t p).
\end{aligned}
\tag{6.1}
$$

So $\tilde{B}(\cdot)$ is forward invariant, which implies the omega-limit set $\omega(\tilde{B})(\cdot)$ of $\tilde{B}(\cdot)$ is the maximal invariant set in $\tilde{B}(\cdot)$. Furthermore, for any $p \in P$,

$$
\begin{aligned}
\omega(\tilde{B})(p) &= \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \varphi(t,\theta_{-t}p)\tilde{B}(\theta_{-t}p)} \\
&= \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \varphi(t,\theta_{-t}p)\overline{\bigcup_{s\geq 0}\varphi(s,\theta_{-(s+t)}p)B(\theta_{-(s+t)}p)}} \\
&= \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \varphi(t,\theta_{-t}p)\bigcup_{s\geq 0}\varphi(s,\theta_{-(s+t)}p)B(\theta_{-(s+t)}p)} \\
&= \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau}\bigcup_{s\geq 0}\varphi(t,\theta_{-t}p) \circ \varphi(s,\theta_{-(s+t)}p)B(\theta_{-(s+t)}p)} \\
&= \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau}\bigcup_{s\geq 0}\varphi(t+s,\theta_{-(s+t)}p)B(\theta_{-(s+t)}p)} \\
&= \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau}\varphi(t,\theta_{-t}p)B(\theta_{-t}p)} \\
&= \omega(B)(p),
\end{aligned}
$$

where the third "=" holds since for each fixed $t \geq 0$ and $p \in P$, $\varphi(t, \theta_{-t}p)$ is a continuous map on X. It follows that $\omega(B)(\cdot)$ is the maximal forward invariant set in $\tilde{B}(\cdot)$. Therefore $\mathbb{C} := \bigcup_{p \in P} (\{p\} \times \omega(B)(p))$ is the maximal invariant set in $\tilde{\mathbb{B}} := \bigcup_{p \in P} (\{p\} \times \tilde{B}(p))$. By the forward invariance of $\tilde{B}(\cdot)$,

$$
\begin{aligned}
\varphi(t)\tilde{\mathbb{B}} &= \varphi(t) \bigcup_{p \in P} (\{p\} \times \tilde{B}(p)) \\
&\subset \bigcup_{p \in P} \varphi(t)(\{p\} \times \tilde{B}(p)) \\
&= \bigcup_{p \in P} (\{\theta_t p\} \times \varphi(t,p)\tilde{B}(p)) \\
&\subset \text{(by (6.1))} \subset \bigcup_{p \in P} (\{\theta_t p\} \times \tilde{B}(\theta_t p)) \\
&= \tilde{\mathbb{B}}, \quad t \geq 0,
\end{aligned}
$$

i.e. $\tilde{\mathbb{B}}$ is positively invariant under φ. Then $\omega(\tilde{\mathbb{B}})$ is the maximal invariant set in $\tilde{\mathbb{B}}$. Recall that $\mathbb{C}$ is also the maximal invariant set in $\tilde{\mathbb{B}}$, we have

$$
\omega(\mathbb{B}) \subset \omega(\tilde{\mathbb{B}}) = \mathbb{C}.
$$

Finally, by the assumption that $K(\cdot)$ attracts $B(\cdot)$, one knows that $\omega(B)(\cdot) \subset K(\cdot)$, and thus $\mathbb{C} \subset \mathbb{K}$, which shows

$$
\omega(\mathbb{B}) \subset \mathbb{K}.
$$

Sufficiency: In a very similar way as above, we can prove the sufficiency.

By the compactness of P,

$$
\begin{aligned}
\lim_{t \to \infty} H_X (\varphi(t, \theta_{-t}p)B(\theta_{-t}p), K_P)] &\leq \lim_{t \to \infty} \sup_{p \in P} H_X (\varphi(t, p)B(p), K_P) \\
&= \lim_{t \to \infty} \sup_{p \in P} H_Y (\Phi(t)\mathbb{B}, P \times K_P) \\
&= \lim_{t \to \infty} H_Y (\Phi(t)\mathbb{B}, P \times K_P) \\
&= 0,
\end{aligned}
$$

which implies $\omega(B)(\cdot)$ exists and pullback attracts $B(\cdot)$.

To complete the proof, it suffices to show $\omega(B)(\cdot) \subset K(\cdot)$. We first define a set

$$
\hat{\mathbb{B}} = \overline{\bigcup_{s \geq 0} \Phi(s)\mathbb{B}}.
$$

Then $\hat{\mathbb{B}}$ is positively invariant and

$$
\omega(\hat{\mathbb{B}}) = \omega(\mathbb{B}).
$$

This implies that $\Omega(\mathbb{B})$ is the maximal invariant set in $\hat{\mathbb{B}}$. Write $\omega(\mathbb{B}) := \bigcup_{p \in P}\{p\} \times C(p)$, then $C(\cdot)$ is the maximal invariant set in $\hat{B}(\cdot)$, where $\hat{B}(\cdot)$ is the set defined by $\hat{\mathbb{B}} := \bigcup_{p \in P}\{p\} \times \hat{B}(p)$. By the positive invariance of $\hat{\mathbb{B}}$, one also knows that $\hat{B}(\cdot)$ is forward invariant. This implies $\Omega_{\hat{B}}(\cdot)$ is the maximal invariant set in $\hat{B}(\cdot)$. We then have that $\omega(B)(\cdot) \subset \omega(\hat{B})(\cdot) = C(\cdot)$. We learn from the condition $\omega(\mathbb{B}) \subset \mathbb{K}$ that $C(\cdot) \subset K(\cdot)$. In summary, $\omega(B)(\cdot) \subset K(\cdot)$, which completes the sufficiency. $\square$

6.2   Construction of local invariant manifold

Let $M > 0$. For $\mu \geq 0$, define a Banach space as

$$\mathscr{X}_\mu = \left\{ u \in C(\mathbb{R}; X^\alpha) : \sup_{t \in \mathbb{R}} e^{-\mu|t|} \|x(t)\|_\alpha < M \right\},$$

which is equipped with the norm $\| \cdot \|_{\mathscr{X}_\mu}$,

$$\|x\|_{\mathscr{X}_\mu} = \sup_{t \in \mathbb{R}} e^{-\mu|t|} \|x(t)\|_\alpha, \qquad \forall x \in \mathscr{X}_\mu.$$

Let $A^\lambda = A - \lambda$. Write $\sigma(A_\lambda) = \sigma_- \cup \sigma_c \cup \sigma_+$, where

$$\sigma_c = \{\lambda_0 - \lambda\},$$

$$\sigma_- = \sigma(A_\lambda) \cap \{\operatorname{Re}\lambda < 0\}, \quad \sigma_+ = \sigma(A_\lambda) \cap \{\operatorname{Re}\lambda > 0\}.$$

According to the spectral decomposition, the space X has a direct sum decomposition: $X = X_- \oplus X_c \oplus X_+$. Denote $X_\pm := X_- \bigcup X_+$. Note that each X_i, $i = -, +, \pm, c$ is independent of λ. Let

$$\Pi_i : X \to X_i, \qquad i = -, +, \pm, c$$

be the projection from X to X_i. Denote $A_i^\lambda = A^\lambda|_{X_i}$. By the assumption **(F2)**, we deduce that if $\lambda \in (\lambda_0 - \eta/4, \lambda_0 + \eta/4)$ then for $\alpha \in [0, 1)$,

$$\|A^\alpha e^{-A_-^\lambda t}\| \leq e^{\frac{3\eta}{4} t}, \quad \|e^{-A_-^\lambda t}\| \leq e^{-\frac{3\eta}{4} t}, \qquad t \leq 0, \tag{6.2}$$

$$\|A^\alpha e^{-A_+^\lambda t} \Pi_+ A^{-\alpha}\| \leq e^{-\frac{3\eta}{4} t}, \quad \|A^\alpha e^{-A_+^\lambda t}\| \leq t^{-\alpha} e^{-\frac{3\eta}{4} t}, \qquad t > 0, \tag{6.3}$$

$$\|A^\alpha e^{-A_c^\lambda t}\| \leq e^{\frac{\eta}{4}|t|}, \quad \|e^{-A_c^\lambda t}\| \leq e^{\frac{\eta}{4}|t|}, \qquad t \in \mathbb{R}. \tag{6.4}$$

Proof of Proposition 4.3. The proof of the existence result for a local invariant manifold is an adaptation of the corresponding result in Chicone and Latushkin [9]. Here we give the details for completeness and the reader's convenience. The main aim of the proof is to show the Lipschitz continuity of $\xi_p^\lambda(y)$ in λ and y, respectively.

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\chi(z) = \begin{cases} 1, & |z| \leq 1/2; \\ 0, & |z| \geq 1. \end{cases}$$

For $\rho > 0$, one can then define a smooth function such that

$$p_\rho(t, u) = \chi\left(\frac{\|u\|_\alpha}{\rho}\right) p(t, u).$$

Select suitable χ such that

$$\|p_\rho(t, u) - p_\rho(t, v)\| \leq k(\rho)\|u - v\|, \tag{6.5}$$

where $k(\rho)$ is the local Lipschitz constant of f given in (4.4). Instead of (4.5), we consider the truncated system

$$u_t + Au = \lambda u + p_\rho(t, u), \qquad p \in \mathcal{H}. \tag{6.6}$$

Suppose that ρ is so small that

$$M_\rho := k(\rho) \int_0^\infty \left(2 + \tau^{-\alpha}\right) e^{-\frac{\eta}{4}\tau} d\tau < 1. \tag{6.7}$$

Let $u \in \mathscr{X}_{\eta/2}$. By simple computations, we know that u is the solution of (6.6) if and only if it solves the integral equation

$$
\begin{aligned}
u(t) = {} & e^{-A_c^\lambda t} \Pi_c u(0) + \int_0^t e^{-A_c^\lambda(t-\tau)} \Pi_c p_\rho(\tau, u(\tau)) d\tau \\
& + \int_{-\infty}^t e^{-A_+^\lambda(t-\tau)} \Pi_+ p_\rho(\tau, u(\tau)) d\tau \\
& - \int_t^\infty e^{-A_-^\lambda(t-\tau)} \Pi_- p_\rho(\tau, u(\tau)) d\tau.
\end{aligned}
\tag{6.8}
$$

Take a $\tilde{\varrho} > 0$ small enough so that

$$\tilde{\varrho} < \left(1 - M_\rho\right) M. \tag{6.9}$$

Let $p \in \mathcal{H}$ and $\lambda \in I_{\lambda_0}(\eta/8)$ be fixed. For each $y \in \overline{\mathrm{B}}_{X_c^\alpha}(\tilde{\varrho})$, one can use the righthand side of equation (6.8) to define a contraction mapping $\mathcal{T} := \mathcal{T}_y$ on $\mathscr{X}_{\eta/2}$ as follows:

$$
\begin{aligned}
\mathcal{T}u(t) = {} & e^{-A_c^\lambda t} y + \int_0^t e^{-A_c^\lambda(t-\tau)} \Pi_c p_\rho(\tau, u(\tau)) d\tau \\
& + \int_{-\infty}^t e^{-A_+^\lambda(t-\tau)} \Pi_+ p_\rho(\tau, u(\tau)) d\tau \\
& - \int_t^\infty e^{-A_-^\lambda(t-\tau)} \Pi_- p_\rho(\tau, u(\tau)) d\tau.
\end{aligned}
$$

We first verify that $\mathcal{T}$ maps $\mathscr{X}_{\eta/2}$ into itself.

For notational convenience, we write

$$0 \wedge t = \min\{0, t\}, \quad 0 \vee t = \max\{0, t\}, \quad \text{for } t \in \mathbb{R}.$$

Let $u \in \mathscr{X}_{\eta/2}$. By (6.2)–(6.4) and (6.5) we have

$$
\begin{aligned}
\|\mathcal{T}u(t)\|_\alpha \leq {} & e^{\frac{\eta}{4}|t|} \|y\|_\alpha + \int_{0\wedge t}^{0\vee t} e^{\frac{\eta}{4}|t-\tau|} k(\rho) \|u(\tau)\|_\alpha d\tau \\
& + \int_{-\infty}^t (t-\tau)^{-\alpha} e^{-\frac{3\eta}{4}(t-\tau)} k(\rho) \|u(\tau)\|_\alpha d\tau \\
& + \int_t^\infty e^{\frac{3\eta}{4}(t-\tau)} k(\rho) \|u(\tau)\|_\alpha d\tau.
\end{aligned}
\tag{6.10}
$$

It is trivial to verify that

$$e^{-\frac{\eta}{2}|t|} \int_{0\wedge t}^{0\vee t} e^{\frac{\eta}{4}|t-\tau|} k(\rho) \|u(\tau)\|_\alpha d\tau = \int_{0\wedge t}^{0\vee t} e^{-\frac{\eta}{4}|t-\tau|} \left[e^{-\frac{\eta}{2}|\tau|} k(\rho) \|u(\tau)\|_\alpha\right] d\tau. \tag{6.11}$$

Observing that

$$e^{-\frac{\eta}{2}|t|} = e^{-\frac{\eta}{2}|(t-\tau)+\tau|} \leq e^{\frac{\eta}{2}|t-\tau|} e^{-\frac{\eta}{2}|\tau|},$$

by (6.9), (6.10) and (6.11) we find that

$$
\begin{aligned}
e^{-\frac{\eta}{2}|t|}\|\mathcal{T}x(t)\|_\alpha \le{}& e^{-\frac{\eta}{4}|t|}\|y\|_\alpha + \int_{0\wedge t}^{0\vee t} e^{-\frac{\eta}{4}|t-\tau|}\left[e^{-\frac{\eta}{2}|\tau|}k(\rho)\|u(\tau)\|_\alpha\right]d\tau \\
&+ \int_{-\infty}^{t} (t-\tau)^{-\alpha}e^{\frac{\eta}{2}|t-\tau|}e^{-\frac{3\eta}{4}(t-\tau)}\left[e^{-\frac{\eta}{2}|\tau|}k(\rho)\|u(\tau)\|_\alpha\right]d\tau \\
&+ \int_{t}^{\infty} e^{\frac{\eta}{2}|t-\tau|}e^{\frac{3\eta}{4}(t-\tau)}\left[e^{-\frac{\eta}{2}|\tau|}k(\rho)\|u(\tau)\|_\alpha\right]d\tau \\
={}& e^{-\frac{\eta}{4}|t|}\|y\|_\alpha + \int_{0\wedge t}^{0\vee t} e^{-\frac{\eta}{4}|t-\tau|}\left[e^{-\frac{\eta}{2}|\tau|}k(\rho)\|u(\tau)\|_\alpha\right]d\tau \\
&+ \int_{-\infty}^{t} (t-\tau)^{-\alpha}e^{-\frac{\eta}{4}(t-\tau)}\left[e^{-\frac{\eta}{2}|\tau|}k(\rho)\|u(\tau)\|_\alpha\right]d\tau \\
&+ \int_{t}^{\infty} e^{\frac{\eta}{4}(t-\tau)}\left[e^{-\frac{\eta}{2}|\tau|}k(\rho)\|u(\tau)\|_\alpha\right]d\tau \\
\le{}& \|y\|_\alpha + M_\rho\|u\|_{\mathscr{X}_{\eta/2}} < M, \quad \forall t\in\mathbb{R}.
\end{aligned}
\tag{6.12}
$$

Hence $\mathcal{T}u\in\mathscr{X}_{\eta/2}$.

Next, we check that $\mathcal{T}$ is contractive. Indeed, in a quite similar fashion as above, it can be shown that for any $u,u'\in\mathscr{X}_{\eta/2}$,

$$
\begin{aligned}
e^{-\frac{\eta}{2}|t|}\|\mathcal{T}u(t)-\mathcal{T}u'(t)\|_\alpha \le{}& k(\rho)\int_{0\wedge t}^{0\vee t} e^{-\frac{\eta}{4}|t-\tau|}\left(e^{-\frac{\eta}{2}|\tau|}\|u(\tau)-u'(\tau)\|_\alpha\right)d\tau \\
&+ k(\rho)\int_{-\infty}^{t} (t-\tau)^{-\alpha}e^{-\frac{\eta}{4}(t-\tau)}\left(e^{-\frac{\eta}{2}|\tau|}\|u(\tau)-u'(\tau)\|_\alpha\right)d\tau \\
&+ k(\rho)\int_{t}^{\infty} e^{\frac{\eta}{4}(t-\tau)}\left(e^{-\frac{\eta}{2}|\tau|}\|u(\tau)-u'(\tau)\|_\alpha\right)d\tau \\
\le{}& \left(k(\rho)\int_{0}^{\infty}\left(2+\tau^{-\alpha}\right)e^{-\frac{\eta}{4}\tau}d\tau\right)\|u-u'\|_{\mathscr{X}_{\eta/2}} \\
:={}& M_\rho\|u-u'\|_{\mathscr{X}_{\eta/2}}, \qquad \forall t\in\mathbb{R}.
\end{aligned}
\tag{6.13}
$$

Thus

$$
\|\mathcal{T}u-\mathcal{T}u'\|_{\mathscr{X}_{\eta/2}} \le M_\rho\|u-u'\|_{\mathscr{X}_{\eta/2}}.
$$

The conditon (6.7) then asserts that $\mathcal{T}$ is contractive.

Thanks to the Banach fixed-point theorem, $\mathcal{T}$ has a unique fixed point $\gamma_{p,\lambda}^{y}\in\mathscr{X}_{\eta/2}$ which is precisely a full solution of (4.5) with $\Pi_c\gamma_{p,\lambda}^{y}(0)=y$ and solves the integral equation

$$
\begin{aligned}
\gamma_{p,\lambda}^{y}(t) ={}& e^{-A_c^\lambda t}y + \int_{0}^{t} e^{-A_c^\lambda(t-\tau)}\Pi_c p_\rho(\tau,\gamma_{p,\lambda}^{y}(\tau))d\tau \\
&+ \int_{-\infty}^{t} e^{-A_+^\lambda(t-\tau)}\Pi_+ p_\rho(\tau,\gamma_{p,\lambda}^{y}(\tau))d\tau \\
&- \int_{t}^{\infty} e^{-A_-^\lambda(t-\tau)}\Pi_- p_\rho(\tau,\gamma_{p,\lambda}^{y}(\tau))d\tau.
\end{aligned}
\tag{6.14}
$$

For $y, z \in \overline{B}_{X_c^\alpha}(\tilde{\varrho})$ and $t \in \mathbb{R}$, similarly to (6.13), by (6.14) we have

$$e^{-\frac{\eta}{2}|t|}\|\gamma_{p,\lambda}^y(t) - \gamma_{p,\lambda}^z(t)\|_\alpha$$

$$\leq e^{-\frac{\eta}{4}|t|}\|y - z\|_\alpha + k(\rho)\int_{0 \wedge t}^{0 \vee t} e^{-\frac{\eta}{4}|t - \tau|}\left(e^{-\frac{\eta}{2}|\tau|}\|\gamma_{p,\lambda}^y(\tau) - \gamma_{p,\lambda}^z(\tau)\|_\alpha\right)d\tau$$

$$+ k(\rho)\int_{-\infty}^{t}(t - \tau)^{-\alpha}e^{-\frac{\eta}{4}(t - \tau)}\left(e^{-\frac{\eta}{2}|\tau|}\|\gamma_{p,\lambda}^y(\tau) - \gamma_{p,\lambda}^z(\tau)\|_\alpha\right)d\tau$$

$$+ k(\rho)\int_{t}^{\infty}e^{\frac{\eta}{4}(t - \tau)}\left(e^{-\frac{\eta}{2}|\tau|}\|\gamma_{p,\lambda}^y(\tau) - \gamma_{p,\lambda}^z(\tau)\|_\alpha\right)d\tau$$

$$\leq \|y - z\|_\alpha + M_\rho\|\gamma_{p,\lambda}^y - \gamma_{p,\lambda}^z\|_{\mathscr{X}_{\eta/2}}.$$

Hence

$$\|\gamma_{p,\lambda}^y - \gamma_{p,\lambda}^z\|_{\mathscr{X}_{\eta/2}} \leq \frac{M}{1 - M_\rho}\|y - z\|_\alpha,$$

which implies that

$$\|\gamma_{p,\lambda}^y(0) - \gamma_{p,\lambda}^z(0)\|_\alpha \leq \frac{M}{1 - M_\rho}\|y - z\|_\alpha. \tag{6.15}$$

For each $p \in \mathcal{H}$ and $\lambda \in I_{\lambda_0}(\eta/8)$, define a mapping from X_c^α to X_{us}^α as

$$\xi_p^\lambda(y) := \int_{-\infty}^0 e^{A_+^\lambda \tau}\Pi_+ p_\rho(\tau, \gamma_{p,\lambda}^y(\tau))d\tau$$
$$- \int_0^\infty e^{A_-^\lambda \tau}\Pi_- p_\rho(\tau, \gamma_{p,\lambda}^y(\tau))d\tau, \quad y \in \overline{B}_{X_c^\alpha}(\tilde{\varrho}). \tag{6.16}$$

Setting $t = 0$ in (6.14) leads to

$$\gamma_{p,\lambda}^y(0) = y + \xi_p^\lambda(y), \qquad y \in \overline{B}_{X_c^\alpha}(\tilde{\varrho}). \tag{6.17}$$

We conclude from (6.15), (6.16) and (6.17) that $\xi_p^\lambda(\cdot) : \overline{B}_{X_c^\alpha}(\tilde{\varrho}) \to X_{us}^\alpha$ is a Lipschitz continuous mapping uniformly on p and λ. More specifically, let

$$L_1 := \frac{M}{1 - M_\rho} + 1.$$

Then for each $y, z \in \overline{B}_{X_c^\alpha}(\tilde{\varrho})$,

$$\|\xi_p^\lambda(y) - \xi_p^\lambda(z)\|_\alpha \leq \|\gamma_{p,\lambda}^y(0) - \gamma_{p,\lambda}^z(0)\|_\alpha + \|y - z\|_\alpha$$
$$\leq L_1\|y - z\|_\alpha.$$

Since $\gamma_{p,\lambda}^y \equiv 0$ is a full solution of (6.6), we have $\xi_p^\lambda(0) \equiv 0$, and thus

$$\lim_{\|y\|_\alpha \to 0}\|\xi_p^\lambda(y)\|_\alpha = 0$$

uniformly on $p \in \mathcal{H}$ and $\lambda \in I_{\lambda_0}(\eta/8)$.

Take a sufficiently small $\varrho > 0$ such that

$$\|y + \xi_p^\lambda(y)\| \leq \frac{\rho}{2}, \quad y \in \overline{B}_{X_c^\alpha}(\varrho).$$

Define for each $p \in \mathcal{H}$ the p-section as

$$\mathcal{M}_{\text{loc}}^\lambda(p) = \{y + \xi_p^\lambda(y) : y \in \overline{B}_{X_c^\alpha}(\varrho)\}.$$

By the definition of p_ρ, $\mathcal{M}_{\text{loc}}^\lambda(\cdot) := \{\mathcal{M}_{\text{loc}}^\lambda(p)\}_{p\in\mathcal{H}}$ is a local invariant manifold of the cocycle semiflow φ_λ, $\lambda \in I_{\lambda_0}(\eta/8)$ generated by (4.5). And for each $p \in \mathcal{H}$, the section $\mathcal{M}_{\text{loc}}^\lambda(p)$ is homeomorphic to $\overline{B}_{X_c^\alpha}(\varrho)$.

In the last part, we show $\xi_p^\cdot(y) : I_{\lambda_0}(\eta/8) \to X_{us}^\alpha$ is Lipschitz uniformly on $y \in \overline{B}_{X_c^\alpha}(\varrho)$ and $p \in P$. Indeed, for $\lambda_1, \lambda_2 \in I_{\lambda_0}(\eta/8)$ with $\lambda_1 \le \lambda_2$, we have for $t \in \mathbb{R}$ that

$$\|e^{-A_c^{\lambda_1}t} - e^{-A_c^{\lambda_2}t}\| \le \|e^{-A_c^{\lambda_1}t}\| \cdot |1 - e^{-(\lambda_2-\lambda_1)t}|$$
$$\le e^{\frac{\eta}{4}|t|} \cdot |1 - e^{-(\lambda_2-\lambda_1)t}|.$$

Then for $t \in \mathbb{R}$,

$$e^{-\frac{\eta}{2}|t|} \int_0^t \|e^{-A_c^{\lambda_1}(t-\tau)}p(\tau, \gamma_{p,\lambda_1}^y(\tau)) - e^{-A_c^{\lambda_2}(t-\tau)}p_\rho(\tau, \gamma_{p,\lambda_2}^y(\tau))\|d\tau$$
$$\le \int_0^t e^{-\frac{\eta}{4}|t-\tau|}k_1(\rho)\left(e^{-\frac{\eta}{2}|\tau|}\|\gamma_{p,\lambda_1}^y(\tau) - \gamma_{p,\lambda_2}^y(\tau)\|_\alpha\right)d\tau$$
$$+ \int_0^t e^{-\frac{\eta}{4}|t-\tau|} \cdot k_1(\rho)|1 - e^{-(\lambda_2-\lambda_1)(t-\tau)}| \cdot \left(e^{-\frac{\eta}{2}|\tau|}\|\gamma_{p,\lambda_2}^y(\tau)\|_\alpha\right)d\tau \tag{6.18}$$
$$\le k(\rho)\int_0^t e^{-\frac{\eta}{4}|t-\tau|}\left(e^{-\frac{\eta}{2}|\tau|}\|\gamma_{p,\lambda_1}^y(\tau) - \gamma_{p,\lambda_2}^y(\tau)\|_\alpha\right)d\tau$$
$$+ k(\rho)M\int_0^t e^{-\frac{\eta}{4}|t-\tau|}|1 - e^{-(\lambda_2-\lambda_1)(t-\tau)}|d\tau.$$

We can apply very similar arguments to get that

$$e^{-\frac{\eta}{2}|t|} \int_{-\infty}^t \|e^{-A_s^{\lambda_1}(t-\tau)}p_\rho(\tau, \gamma_{p,\lambda_1}^y(\tau)) - e^{-A_s^{\lambda_2}(t-\tau)}p_\rho(\tau, \gamma_{p,\lambda_2}^y(\tau))\|d\tau$$
$$\le k(\rho)\int_{-\infty}^t (t-\tau)^\alpha e^{-\frac{\eta}{4}(t-\tau)}\left(e^{-\frac{\eta}{2}|\tau|}\|\gamma_{p,\lambda_1}^y(\tau) - \gamma_{p,\lambda_2}^y(\tau)\|_\alpha\right)d\tau \tag{6.19}$$
$$+ k(\rho)M\int_{-\infty}^t (t-\tau)^\alpha e^{-\frac{\eta}{4}(t-\tau)}|1 - e^{-(\lambda_2-\lambda_1)(t-\tau)}|d\tau$$

and

$$e^{-\frac{\eta}{2}|t|} \int_t^\infty \|e^{-A_u^{\lambda_1}(t-\tau)}p(\tau, \gamma_{p,\lambda_1}^y(\tau)) - e^{-A_u^{\lambda_2}(t-\tau)}p(\tau, \gamma_{p,\lambda_2}^y(\tau))\|d\tau$$
$$\le k(\rho)\int_t^\infty e^{\frac{\eta}{4}(t-\tau)}\left(e^{-\frac{\eta}{2}|\tau|}\|\gamma_{p,\lambda_1}^y(\tau) - \gamma_{p,\lambda_2}^y(\tau)\|_\alpha\right)d\tau \tag{6.20}$$
$$+ k(\rho)M\int_t^\infty e^{\frac{\eta}{4}(t-\tau)}|1 - e^{-(\lambda_2-\lambda_1)(t-\tau)}|d\tau.$$

By (6.18), (6.19) and (6.20), we derive that

$$e^{-\frac{\eta}{2}|t|}\|\gamma_{p,\lambda_1}^y(t) - \gamma_{p,\lambda_2}^y(t)\|_\alpha$$
$$\le e^{-\frac{\eta}{2}|t|} \int_0^t \|e^{-A_c^{\lambda_1}(t-\tau)}p(\tau, \gamma_{p,\lambda_1}^y(\tau)) - e^{-A_c^{\lambda_2}(t-\tau)}p(\tau, \gamma_{p,\lambda_2}^y(\tau))\|d\tau$$
$$+ e^{-\frac{\eta}{2}|t|} \int_{-\infty}^t \|e^{-A_s^{\lambda_1}(t-\tau)}p(\tau, \gamma_{p,\lambda_1}^y(\tau)) - e^{-A_s^{\lambda_2}(t-\tau)}p(\tau, \gamma_{p,\lambda_2}^y(\tau))\|d\tau$$
$$+ e^{-\frac{\eta}{2}|t|} \int_t^\infty \|e^{-A_u^{\lambda_1}(t-\tau)}p(\tau, \gamma_{p,\lambda_1}^y(\tau)) - e^{-A_u^{\lambda_2}(t-\tau)}p(\tau, \gamma_{p,\lambda_2}^y(\tau))\|d\tau \tag{6.21}$$
$$\le k(\rho)\int_0^\infty (2 + t^{-\alpha})e^{-\frac{\eta}{4}t}dt \cdot \sup_{t\in\mathbb{R}} e^{-\frac{\eta}{2}|t|}\|\gamma_{p,\lambda_1}^y(t) - \gamma_{p,\lambda_2}^y(t)\|_\alpha$$
$$+ k(\rho)M\int_0^\infty (2 + t^{-\alpha})e^{-\frac{\eta}{4}t}\left(e^{(\lambda_2-\lambda_1)t} - 1\right)dt.$$

It follows that

$$
\begin{aligned}
\|\xi_p^{\lambda_1}(y) - \xi_p^{\lambda_2}(y)\|_\alpha &= \|u_{\lambda_1}(0) - u_{\lambda_2}(0)\|_\alpha \\
&\leq \sup_{t\in\mathbb{R}} e^{-\frac{\eta}{2}|t|} \|\gamma_{p,\lambda_1}^y(t) - \gamma_{p,\lambda_2}^y(t)\|_\alpha \\
&\leq \frac{k_1(\rho)M}{1 - M_\rho} \int_0^\infty (2 + t^{-\alpha}) e^{-\frac{\eta}{4}t} \left(e^{(\lambda_2 - \lambda_1)t} - 1 \right) dt \\
&\leq \frac{k_1(\rho)M}{1 - M_\rho} \int_0^\infty (2 + t^{-\alpha}) t\, e^{-[\frac{\eta}{4} - (\lambda_2 - \lambda_1)]t} dt \cdot |\lambda_1 - \lambda_2|,
\end{aligned}
$$

where the differential mean value is applied to $e^{(\lambda_2 - \lambda_1)t} - 1$ to get the last term. It is clear that the integral

$$
\int_0^\infty (2 + t^{-\alpha}) t\, e^{-[\frac{\eta}{4} - (\lambda_2 - \lambda_1)]t} dt = \int_0^\infty (2t + t^{1-\alpha}) e^{-[\frac{\eta}{4} - (\lambda_2 - \lambda_1)]t} dt
$$

converges. Therefore

$$
\xi_p^{\lambda_1}(y) - \xi_p^{\lambda_2}(y)\| \leq L_2 |\lambda_1 - \lambda_2|,
$$

where

$$
L_2 := \frac{k_1(\rho)M}{1 - M_\rho} \int_0^\infty (2t + t^{1-\alpha}) e^{-[\frac{\eta}{4} - (\lambda_2 - \lambda_1)]t} dt,
$$

and thus $\xi_p^\cdot(y)$ is Lipschitz continuous on $I_{\lambda_0}(\eta/8)$ uniformly on $p \in P$ and $y \in \overline{B}_{X_c^\alpha}(\varrho)$. $\qquad\square$

Acknowledgements

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