



Multiple small solutions for Schrödinger equations involving the p -Laplacian and positive quasilinear term

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Abstract. We consider the multiplicity of solutions of a class of quasilinear Schrödinger equations involving the p -Laplacian:

$$-\Delta_p u + V(x)|u|^{p-2}u + \Delta_p(u^2)u = K(x)f(x, u), \quad x \in \mathbb{R}^N,$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, $N \geq 3$, V, K belong to $C(\mathbb{R}^N)$ and f is an odd continuous function without any growth restrictions at large. Our method is based on a direct modification of the indefinite variational problem to a definite one. Even for the case $p = 2$, the approach also yields new multiplicity results.

Keywords: quasilinear Schrödinger equations, variational methods, Brezis–Kato type estimates.

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1 Introduction

In this study, the multiplicity of solutions for the quasilinear elliptic problem

$$-\Delta_p u + V(x)|u|^{p-2}u + \tau\Delta_p(u^2)u = K(x)f(x, u), \quad x \in \mathbb{R}^N, \quad (1.1)$$

will be analyzed, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplacian, $1 < p < N$, $\tau \in \mathbb{R}$, f is a continuous function and is only p -sublinear in a neighborhood of $u = 0$, V and K belong to $C(\mathbb{R}^N)$, satisfying

(VK) for all $x \in \mathbb{R}^N$, $0 < V_0 \leq V(x)$, $0 < K(x) \leq K_1$ and

$$W(x) := K(x)^{p/(p-q)}V(x)^{q/(q-p)} \in L^1(\mathbb{R}^N) \quad (q \text{ will be defined in } (f_1)).$$

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For $p = 2$, quasilinear Schrödinger equations (QSE) are widely used in non-Newtonian fluids, reaction-diffusion problems and other physical phenomena. It should be noted that the solutions of problem (1.1) are closely related to solutions of the nonlinear Schrödinger equations:

$$i\partial_t z = -\Delta z + \tilde{V}(x)z - l(x, |z|^2)z + \tau[\Delta\rho(|z|^2)]\rho'(|z|^2)z, \quad (1.2)$$

where $z : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{C}$, $K : \mathbb{R}^N \rightarrow \mathbb{R}$ is a given potential, τ is a real constant, ρ is a real function and $l : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$. They have been derived as models of many physical phenomena corresponding to various types of the function ρ . For example, when $\rho(s) = 1$, one has the classical stationary semilinear Schrödinger equation [3, 12]. If $\rho(s) = s$, the equations of fluid mechanics, plasma physics and dissipative quantum mechanics are established [4, 11]. When $\rho(s) = (1 + s)^{1/2}$, the equation models the propagation of a high-irradiance laser in a plasma and the self-channeling of a high-power ultrashort laser in matter [13]; problem (1.2) is related to condensed matter theory. For more information on the physical background, please refer to [4, 5, 18].

In what follows, we discuss the case of $\rho(s) = s$ and $p = 2$. A standing wave of problem (1.2) is a solution of the form $z(x, t) = \exp(-iEt)u(x)$ where $E \in \mathbb{R}$. It is also called stationary waves. It is generally known that z is a standing wave solution for problem (1.2) when and only when u is a solution for the quasilinear elliptic problem (1.1), where $V(x) = \tilde{V}(x) - E$ indicates the new potential.

When $\tau = 0$, equation (1.1) degenerates into a semilinear equation (i.e., the nonlinear Schrödinger equation), which has been widely studied using the variational method for the past 30 years, see [14]. Obviously, if $\tau \neq 0$, the energy functional of the quasilinear term $\tau \int_{\mathbb{R}^N} u^2 |\nabla u|^2 dx$ is not well defined in $H^1(\mathbb{R}^N)$. Therefore, the energy functional I of (1.1) is not a C^1 functional.

When $\tau < 0$, scholars have obtained a large number of existence and multiplicity results for equation (1.1) based on variational methods. For instance, Poppenberg, Schmitt and Wang proved the existence of positive solutions with a constrained minimization argument in [19] for the first time. By utilizing variable substitution and converting the quasilinear problem (1.1) into a semilinear one in an Orlicz space framework, Liu *et al.* in [15] obtained a general existence result. Colin *et al.* in [6] adopted the same method of variable substitution but chose the classical Sobolev space $H^1(\mathbb{R}^N)$. For further results, please refer to [8, 16, 21, 22, 25].

When $\tau > 0$, in [1], Alves *et al.* introduced a substitution of variables $u = G^{-1}(v)$, where

$$g(t) = \begin{cases} \sqrt{1 - \tau t^2} & \text{if } 0 \leq t < \frac{1}{\sqrt{3\tau}}, \\ \frac{1}{3\sqrt{2\tau t}} + \frac{1}{\sqrt{6}} & \text{if } t \geq \frac{1}{\sqrt{3\tau}}, \end{cases}$$

$g(t) = g(-t)$ for all $t \leq 0$ and $G(s) = \int_0^s g(t)dt$. Given a sufficiently small $\tau > 0$, the authors proved that there exists a solution of

$$-\Delta u + V(x)u + \tau\Delta(u^2)u = |u|^{q-2}u, \quad x \in \mathbb{R}^N,$$

where $2 < q < 2^*$. Wang *et al.* [23] investigated the existence of solutions for QSE with critical growth nonlinearities. [2] with potential V vanishing at infinity and the superlinear nonlinearities, [24] with $f(t) = \lambda|t|^{q-2}t + |t|^{i-2}t$ for $q \geq 22^*$, $4 < i < 22^*$ and $\lambda > 0$ small enough, [20] with potential V being large at infinity and nonlinearities being superlinear or asymptotically linear at infinity.

Now, from [1], two natural questions arise:

- (Q₁) Can the appropriate variational framework for problem (1.1) with $\tau = 1$ (not small enough) be established?
- (Q₂) When $\tau = 1$, if the nonlinearity $|t|^{q-2}t$ with $q > 2$ is replaced by $q < 2$ or a more general sublinear term $f(x, t)$ in problem (1.1), will this problem possess infinitely many solutions?

Regarding the question (Q₁), our earlier work [9] studied the existence of a positive solution for problem (1.1) with $\tau = 1$ under a local superlinear growth condition. Our aim in this work is to seek clear answers to question (Q₂). Therefore, we will be mainly interested in the existence of infinitely many solutions for the following general QSE involving local p -sublinear nonlinearities:

$$-\Delta_p u + V(x)|u|^{p-2}u + \Delta_p(u^2)u = K(x)f(x, u), \quad x \in \mathbb{R}^N, \quad (1.3)$$

where $1 < p < N$, $N \geq 3$, V and K satisfy condition (VK). We remark that our results are new also in the case $p = 2$. Next, we suppose that the nonlinearity f is continuous and meets the following conditions that describe its behavior only in a neighborhood of the origin:

- (f₁) there exist $\delta > 0$, $1 \leq q < p$ and $C > 0$ such that $f \in C(\mathbb{R}^N \times [-\delta, \delta], \mathbb{R})$, f is odd in t and

$$|f(x, t)| \leq C|t|^{q-1}, \quad \text{uniformly in } x \in \mathbb{R}^N;$$

- (f₂) there exist $x_0 \in \mathbb{R}^N$ and $r_0 > 0$ such that

$$\liminf_{t \rightarrow 0} \left(\inf_{x \in B_{r_0}(x_0)} \frac{F(x, t)}{|t|^p} \right) > -\infty$$

and

$$\limsup_{t \rightarrow 0} \left(\inf_{x \in B_{r_0}(x_0)} \frac{F(x, t)}{|t|^p} \right) = +\infty,$$

where $B_{r_0}(x_0) \subset \mathbb{R}^N$ and

$$F(x, t) = \int_0^t f(x, s)ds.$$

Remark 1.1. We do not need any growth condition on f at infinity. There exist many functions satisfying (f₁) and (f₂), for example

- (i) $f(x, u) = |u|^{q-1} \text{sgn } u$ with $q \in (1, p)$;
- (ii) $f(x, u) = Q(x)|u|^{q-1} \text{sgn } u + P(x)|u|^{i-1} \text{sgn } u$, where $1 < q < p$, $i \geq p^* := \frac{pN}{N-p}$, $Q(x)$ and $P(x)$ are bounded Hölder continuous functions on \mathbb{R}^N and $Q(x_0) > 0$ at some $x_0 \in \mathbb{R}^N$.

Remark 1.2. Although problem (1.3) is not a standard elliptic equation, we can still give the definition of the weak solution of problem (1.3). Suppose that conditions (VK), (f₁) and (f₂) are satisfied. A weak solution of problem (1.3) is a function $u \in X$ (X will be defined in Section 2) such that

$$\begin{aligned} & \int_{\mathbb{R}^N} (1 - 2^{p-1}|u|^p) |\nabla u|^{p-2} \nabla u \nabla \varphi dx - 2^{p-1} \int_{\mathbb{R}^N} |\nabla u|^p |u|^{p-2} u \varphi dx + \int_{\mathbb{R}^N} V(x)|u|^{p-2} u \varphi dx \\ & = \int_{\mathbb{R}^N} K(x)f(x, u) \varphi dx, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \end{aligned}$$

From a variational perspective, we give a formally Lagrangian functional of (1.3):

$$J(u) = \frac{1}{p} \int_{\mathbb{R}^N} (1 - 2^{p-1}|u|^p)|\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} K(x)F(x, u)dx,$$

which is not well defined in $W^{1,p}(\mathbb{R}^N)$. For this reason, conventional variational methods cannot be applied directly. Problems such as (1.3) become interesting and challenging in this dilemma. First, because of our lack of information about the function f at infinity, the term $\int_{\mathbb{R}^N} K(x)F(x, u)dx$ may not be well defined. Second, the presence of $\int_{\mathbb{R}^N} (1 - 2^{p-1}|u|^p)|\nabla u|^p dx$ makes us unable to work in a classical Sobolev space. Third, ensuring the positiveness of the principal part, i.e., $\int_{\mathbb{R}^N} (1 - 2^{p-1}|u|^p)|\nabla u|^p dx > 0$, is also difficult.

Drawing lessons from the work of Costa and Wang [7], our earlier work [9] and the variant symmetric mountain pass lemma [10, 17], we can obtain infinitely many solutions for a modified functional with modifications made on the nonlinearity and the principal part of the Lagrangian functional J . Then, we obtain Brezis–Kato type estimates for these critical points of the modified functional. After fine estimates of the solutions for the modified problems we can show that some of these solutions for the modified problems give rise to solutions of problem (1.3) with desired properties.

We now proceed to present our main result.

Theorem 1.3. *Suppose that conditions (VK), (f_1) and (f_2) are satisfied. Then problem (1.3) possesses a sequence of weak solutions $u_n \in X$ with $u_n \rightarrow 0$ strongly in X , $u_n \rightarrow 0$ strongly in $L^\infty(\mathbb{R}^N)$ and $J(u_n) \rightarrow 0$.*

Remark 1.4. Since problem (1.3) is not a standard elliptic equation, conventional critical point theory is not directly applicable. Hence, some fundamental results for elliptic equations are not expected. For instance, without the symmetric condition regarding nonlinearity, the existence of solutions for problem (1.3) may not be proved.

The remainder of this paper is arranged as follows. In Section 2, the problem is reformulated. We provide the variational framework for the reformulated problem in Section 3. Section 4 is devoted to discussing the reformulated problem in detail via a cut-off technique, Morse L^∞ -estimation and proving Theorem 1.3.

In what follows, C denotes positive generic constants.

2 Reformulation

Define $X = \{u \in W^{1,p}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)|u|^p dx < \infty\}$ endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p) dx \right)^{1/p}.$$

As usual, the norms of $L^s(\mathbb{R}^N)$ ($s \geq 1$) are denoted by $\|\cdot\|_s$.

For fixed $\delta > 0$ in (f_1) , set $d(t) \in C(\mathbb{R})$ as a cut-off function satisfying :

$$d(t) = \begin{cases} 1, & \text{if } |t| \leq \frac{\delta}{2}, \\ 0, & \text{if } |t| \geq \delta, \end{cases}$$

$d(-t) = d(t)$ and $0 \leq d(t) \leq 1$ for $t \in \mathbb{R}$. Define

$$\tilde{f}(x, t) = d(t)f(x, t), \quad \text{for all } x \in \mathbb{R}^N, t \in \mathbb{R}$$

and

$$\tilde{F}(x, t) = \int_0^t \tilde{f}(x, s) ds.$$

Inspired by [7, 9], a modified QSE can be established:

$$-\operatorname{div}(h^p(u)|\nabla u|^{p-2}\nabla u) + h^{p-1}(u)h'(u)|\nabla u|^p + V(x)|u|^{p-2}u = K(x)\tilde{f}(x, u), \quad x \in \mathbb{R}^N, \quad (2.1)$$

where $h(s) : [0, +\infty) \rightarrow \mathbb{R}$ satisfying

$$h(s) = \begin{cases} (1 - 2^{p-1}s^p)^{1/p} & \text{if } 0 \leq s < (\frac{2^{1-p}}{3})^{1/p}, \\ \frac{1}{s}(\frac{2^{1-p}}{3})^{2/p} + (\frac{2^{1-p}}{3})^{1/p} & \text{if } s \geq (\frac{2^{1-p}}{3})^{1/p}, \end{cases}$$

and $h(s) = h(-s)$ for $s < 0$. It deduces that $h(s) \in C^1(\mathbb{R}, ((\frac{2^{1-p}}{3})^{1/p}, 1])$ and decreases in $[0, +\infty)$. And then, we define

$$H(t) = \int_0^t h(s) ds.$$

Obviously, $H(t)$ is an odd function, and there exists an inverse function $H^{-1}(t)$. Moreover, $H(t)$ has the following attributes, the similar proofs of which can be found in [9].

Lemma 2.1.

- (i) $\lim_{t \rightarrow 0} \frac{H^{-1}(t)}{t} = 1$;
- (ii) $\lim_{t \rightarrow +\infty} \frac{H^{-1}(t)}{t} = (\frac{3}{2^{1-p}})^{1/p}$;
- (iii) $|t| \leq |H^{-1}(t)| \leq (\frac{3}{2^{1-p}})^{1/p}|t|$, for all $t \in \mathbb{R}$;
- (iv) $\frac{t}{h(t)}h'(t) \leq 0$, for all $t \in \mathbb{R}$.

Our goal is proving that (2.1) has a sequence of weak solutions $\{u_n\}$ satisfying $\|u_n\|_{L^\infty} < \min\{\delta/2, (\frac{2^{1-p}}{3})^{1/p}\}$, in this situation

$$h(u_n) = (1 - 2^{p-1}|u_n|^p)^{1/p} \quad \text{and} \quad \tilde{f}(x, u_n) = f(x, u_n).$$

Thus, they are also the weak solutions of (1.3).

To find the weak solutions of (2.1) with desired properties, we focus on a Lagrangian functional defined by

$$\tilde{J}(u) = \frac{1}{p} \int_{\mathbb{R}^N} h^p(u)|\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} K(x)\tilde{F}(x, u) dx. \quad (2.2)$$

Taking the change of variable

$$v = H(u),$$

it is clear that functional \tilde{J} can be written as follows:

$$I(v) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} V(x)|H^{-1}(v)|^p dx - \int_{\mathbb{R}^N} K(x)\tilde{F}(x, H^{-1}(v)) dx. \quad (2.3)$$

From the definition of $\tilde{F}(x, t)$, we deduce

$$|\tilde{F}(x, t)| \leq C|t|^q, \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R},$$

where $1 \leq q < p$. This together with Lemma 2.1 implies that

$$\begin{aligned} \left| \int_{\mathbb{R}^N} K(x) \tilde{F}(x, H^{-1}(v)) dx \right| &\leq C \int_{\mathbb{R}^N} K(x) |H^{-1}(v)|^q dx \\ &\leq C \left(\int_{\mathbb{R}^N} W(x) dx \right)^{\frac{(p-q)}{p}} \left(\int_{\mathbb{R}^N} V(x) |v|^p dx \right)^{\frac{q}{p}} \\ &\leq C \|v\|^q. \end{aligned} \quad (2.4)$$

From the above estimation and Lemma 2.1, we obtain

$I(v)$ is well defined in X .

Then, it is standard to see that $I \in C^1(X, \mathbb{R})$ and for all $\varphi \in X$

$$\begin{aligned} I'(v)\varphi &= \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla \varphi dx + \int_{\mathbb{R}^N} V(x) |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} \varphi dx \\ &\quad - \int_{\mathbb{R}^N} K(x) \frac{\tilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \varphi dx. \end{aligned}$$

Lemma 2.2. *Suppose that conditions (VK) and (f_1) are satisfied. If $v \in X$ is a critical point of I , then $u = H^{-1}(v) \in X$ and u is a weak solution for (2.1).*

Proof. From $v \in X$ and Lemma 2.1, we have $u = H^{-1}(v) \in X$. By v being a critical point for I , we deduce that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla \varphi dx + \int_{\mathbb{R}^N} V(x) |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} \varphi dx \\ - \int_{\mathbb{R}^N} K(x) \frac{\tilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \varphi dx, \quad \text{for all } \varphi \in X. \end{aligned}$$

Taking $\varphi = h(u)\psi$ as the test function, where $u = H^{-1}(v)$ and $\psi \in C_0^\infty(\mathbb{R}^N)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla u h'(u) \psi dx + \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla \psi h(u) dx + \int_{\mathbb{R}^N} V(x) |u|^{p-2} u \psi dx \\ - \int_{\mathbb{R}^N} K(x) \tilde{f}(x, u) \psi dx = 0. \end{aligned}$$

or

$$\int_{\mathbb{R}^N} \left(-\operatorname{div}(h^p(u) |\nabla u|^{p-2} \nabla u) + h^{p-1}(u) h'(u) |\nabla u|^p + V(x) |u|^{p-2} u - K(x) \tilde{f}(x, u) \right) \psi dx = 0.$$

This ends the proof. \square

Therefore, for the weak solutions of (2.1), we only need to discuss the existence of the weak solutions of the following problem:

$$-\Delta_p v + V(x) |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} = K(x) \frac{\tilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))}, \quad x \in \mathbb{R}^N. \quad (2.5)$$

3 Clark's theorem

Denote

$$\Gamma = \{A \subset X \setminus \{0\} \mid A \text{ is closed, } -A = A\}.$$

Let $A \in \Gamma$, define

$$\gamma(A) = \min\{n \in \mathbb{N} \mid \text{there exists an odd, continuous } \phi : A \rightarrow \mathbb{R}^n \setminus \{0\}\},$$

If such a minimum does not exist, then we define $\gamma(A) = +\infty$. Moreover, set $\gamma(\emptyset) = 0$. In order to prove Theorem 1.3, we introduce the following Clark's theorem due to [10].

Proposition 3.1. *Let X be a Banach space and $\Phi \in C^1(X, \mathbb{R})$ be an even functional with $\Phi(0) = 0$. Assume that Φ satisfies the following.*

- (i) Φ is bounded from below and satisfies the Palais–Smale condition.
- (ii) For all $k \in \mathbb{N}$, $\Gamma_k = \{A \in \Gamma \mid \gamma(A) \geq k\}$, there exists a $A_k \in \Gamma_k$ such that $\sup_{v \in A_k} \Phi(v) < 0$.

Then, at least one of the following conclusions holds.

- (i) There exists a critical point sequence $\{v_k\}$ such that $\Phi(v_k) < 0$ and $v_k \rightarrow 0$ strongly in X .
- (ii) There exist two critical point sequences $\{v_k\}$ and $\{w_k\}$ such that $\Phi(v_k) = 0$, $v_k \neq 0$, $v_k \rightarrow 0$ strongly in X , $\Phi(w_k) < 0$, $\lim_{k \rightarrow \infty} \Phi(w_k) = 0$ and $\{w_k\}$ converges to a non-zero limit.

The following lemma plays a fundamental role in verifying Proposition 3.1. In the proof of this lemma, we adapt some arguments of dealing with the Schrödinger–Poisson systems in [26] and the elliptic problem in [10].

Lemma 3.2. *Suppose that (VK), (f_1) and (f_2) hold. Then for all $k \in \mathbb{N}$, there exists $A_k \in \Gamma$ such that genus $\gamma(A_k) = k$ and $\sup_{v \in A_k} I(v) < 0$.*

Proof. Without loss of generality, we may assume that $x_0 = 0$ in condition (f_2) . Let \mathcal{Q} be the cube

$$\mathcal{Q} := \{x = (x_1, x_2, \dots, x_N) \mid |x_i| \leq r_0/2, i = 1, 2, \dots, N\},$$

where r_0 is chosen in condition (f_2) . Obviously, $\mathcal{Q} \subset B_{r_0}(0)$. From (f_2) and Lemma 2.1-(iii), we can find two sequences $\delta_n \rightarrow 0$, $M_n \rightarrow \infty$ ($\delta_n, M_n > 0$) and a positive constant α such that

$$\frac{F(x, t)}{|t|^p} \geq -\alpha, \quad \text{for all } x \in \mathcal{Q} \text{ and } |t| \leq \delta \quad (3.1)$$

and

$$\frac{F(x, H^{-1}(\delta_n))}{|H^{-1}(\delta_n)|^p} \geq M_n \quad \text{for all } x \in \mathcal{Q} \text{ and } n \in \mathbb{N}. \quad (3.2)$$

Next, for any $k \in \mathbb{N}$ fixed, we shall construct a $A_k \in \Gamma$ which satisfies genus $\gamma(A_k) = k$ and $\sup_{v \in A_k} I(v) < 0$.

Firstly, let $k \in \mathbb{N}$ be fixed and $m \in \mathbb{N}$ is the smallest integer that satisfies $m^N \geq k$. Then, by planes parallel to each face of \mathcal{Q} , we can equally divide cube \mathcal{Q} into m^N small cubes. Set them by \mathcal{Q}_i with $1 \leq i \leq m^N$. It is well known that the length of the edge of \mathcal{Q}_i is $d = r_0/m$. Furthermore, for each $1 \leq i \leq k$, let \mathcal{U}_i be a cube in \mathcal{Q}_i such that \mathcal{U}_i has the same center as that of \mathcal{Q}_i , the faces of \mathcal{U}_i and \mathcal{Q}_i are parallel, and the length of the edge of \mathcal{U}_i is $\frac{d}{2}$.

Define a cut-off function $\mu \in C_0^\infty(\mathbb{R})$ such that $0 \leq \mu \leq 1$, $\mu(x) = 1$ for $s \in [-\frac{d}{4}, \frac{d}{4}]$ and $\mu(x) = 0$ for $s \in \mathbb{R} \setminus [-\frac{d}{2}, \frac{d}{2}]$. Denote

$$v(x) := \mu(x_1)\mu(x_2)\dots\mu(x_N), \quad \text{for all } x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N.$$

For each $1 \leq i \leq k$, let

$$v_i(x) = v(x - y_i), \quad \text{for all } x \in \mathbb{R}^N,$$

where $y_i \in \mathbb{R}^N$ is the center of both \mathcal{Q}_i and \mathcal{U}_i . Obviously, for all $1 \leq i \leq k$, we have

$$\text{supp } v_i \subset \mathcal{Q}_i \tag{3.3}$$

and

$$v_i(x) = 1, \quad \text{for all } x \in \mathcal{U}_i, \quad 0 \leq v_i(x) \leq 1, \quad \text{for all } x \in \mathbb{R}^N. \tag{3.4}$$

Denote

$$D_k := \left\{ (e_1, e_2, \dots, e_k) \in \mathbb{R}^k \mid \max_{1 \leq i \leq k} |e_i| = 1 \right\}$$

and

$$L_k := \left\{ \sum_{i=1}^k e_i v_i \mid (e_1, e_2, \dots, e_k) \in D_k \right\}.$$

It is well known that using an odd mapping, D_k is homeomorphic to the unit sphere in \mathbb{R}^k . Thus, $\gamma(D_k) = k$. And then, since the mapping $\mathcal{L} : D_k \rightarrow L_k$ defined by

$$\mathcal{L}(e_1, e_2, \dots, e_k) = \sum_{i=1}^k e_i v_i, \quad \text{for all } (e_1, e_2, \dots, e_k) \in D_k,$$

is an odd homeomorphism, this deduces $\gamma(D_k) = \gamma(L_k) = k$. Due to the compactness of L_k , there exists a constant $C_k > 0$ such that

$$\|v\| \leq C_k, \quad \text{for all } v \in L_k. \tag{3.5}$$

For $v = \sum_{i=1}^k e_i v_i \in L_k$ and any $t \in (0, \frac{1}{2}(\frac{2^{1-p}}{3})^{1/p}\delta)$, by Lemma 2.1-(iii), the definition of \tilde{F} and the fact that $|H^{-1}(te_i v_i)| < \frac{\delta}{2}$ for all $1 \leq i \leq k$, we have

$$\begin{aligned} I(tv) &\leq \frac{t^p}{p} \int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{3 \cdot 2^{p-1} t^p}{p} \int_{\mathbb{R}^N} V(x) |v|^p dx - \sum_{i=1}^k \int_{\mathcal{Q}_i} K(x) \tilde{F}(x, H^{-1}(te_i v_i)) dx \\ &\leq \frac{3 \cdot 2^{p-1} t^p}{p} \|v\|^p - \sum_{i=1}^k \int_{\mathcal{Q}_i} K(x) F(x, H^{-1}(te_i v_i)) dx. \end{aligned} \tag{3.6}$$

From the definition of D_k , there exists $i_v \in [1, k]$ such that $|e_{i_v}| = 1$. Then, we rewrite the term $\sum_{i=1}^k \int_{\mathcal{Q}_i} K(x) F(x, H^{-1}(te_i v_i)) dx$ in (3.6) as follows:

$$\begin{aligned} &\int_{\mathcal{U}_{i_v}} K(x) F(x, H^{-1}(te_{i_v} v_{i_v})) dx + \int_{\mathcal{Q}_{i_v} \setminus \mathcal{U}_{i_v}} K(x) F(x, H^{-1}(te_{i_v} v_{i_v})) dx \\ &\quad + \sum_{i \neq i_v} \int_{\mathcal{Q}_i} K(x) F(x, H^{-1}(te_i v_i)) dx. \end{aligned} \tag{3.7}$$

From Lemma 2.1-(2), (3.1) and (3.4), we deduce

$$\int_{\mathcal{Q}_{i_v} \setminus \mathcal{U}_{i_v}} K(x) F(x, H^{-1}(te_{i_v} v_{i_v})) dx + \sum_{i \neq i_v} \int_{\mathcal{Q}_i} K(x) F(x, H^{-1}(te_i v_i)) dx \geq -\frac{3}{2^{1-p}} \alpha r_0^N K_1 t^p. \tag{3.8}$$

Choosing $t = \delta_n \in (0, \frac{1}{2}(\frac{2^{1-p}}{3})^{1/p}\delta)$ in (3.6), by using $F(x, t)$ is even for $|t| \leq \delta$, Lemma 2.1-(iii), (3.2) and (3.5)–(3.8), we obtain

$$\begin{aligned} I(\delta_n v) &\leq \frac{3 \cdot 2^{p-1}}{p} C_k^p \delta_n^p + \frac{3}{2^{1-p}} \alpha r_0^N K_1 \delta_n^p - \int_{\mathcal{U}_{i_v}} K(x) F(x, H^{-1}(\delta_n e_{i_v} v_{i_v})) dx \\ &\leq \frac{3 \cdot 2^{p-1}}{p} C_k^p \delta_n^p + \frac{3}{2^{1-p}} \alpha r_0^N K_1 \delta_n^p - C \frac{d^N M_n}{2^N} |H^{-1}(\delta_n)|^p \\ &\leq \delta_n^p \left(\frac{3 \cdot 2^{p-1}}{p} C_k^p + \frac{3}{2^{1-p}} \alpha r_0^N K_1 - C \frac{d^N M_n}{2^N} \right). \end{aligned} \quad (3.9)$$

Note that $M_n \rightarrow \infty$ as $n \rightarrow \infty$, there exists an $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, we obtain

$$\frac{3 \cdot 2^{p-1}}{p} C_k^p + \frac{3}{2^{1-p}} \alpha r_0^N K_1 - C \frac{d^N M_n}{2^N} < 0.$$

Choosing

$$A_k := \{\delta_{n_0} v \mid v \in L_k\},$$

we deduce that A_k satisfies

$$\gamma(A_k) = \gamma(L_k) = k \quad \text{and} \quad \sup_{v \in A_k} I(v) < 0. \quad \square$$

Next, we show a compactness result for the functional I .

Lemma 3.3. *Provided that assumptions (VK) and (f_1) hold, then I is bounded from below and satisfies the Palais–Smale condition.*

Proof. Let $v \in X$. Then, from (2.4), we have

$$\left| \int_{\mathbb{R}^N} K(x) \tilde{F}(x, H^{-1}(v)) dx \right| \leq C \|v\|^q.$$

Therefore, we obtain

$$\begin{aligned} I(v) &= \frac{1}{p} \int_{\mathbb{R}^N} |\nabla v|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} V(x) |H^{-1}(v)|^p dx - \int_{\mathbb{R}^N} K(x) \tilde{F}(x, H^{-1}(v)) dx \\ &\geq \frac{1}{p} \|v\|^p - C \|v\|^q. \end{aligned}$$

Note that $1 < q < p$, we can derive that I is bounded from below and I is coercive.

Next, we shall prove that I satisfies the Palais–Smale conditions. For $\{v_n\} \subset X$, such that

$$|I(v_n)| \leq c \quad \text{and} \quad I'(v_n) \rightarrow 0.$$

By I being coercive, we have the sequence $\{v_n\}$ bounded in X . Up to subsequence, we obtain

$$v_n \rightharpoonup v \text{ weakly in } X, \quad v_n \rightarrow v \text{ strongly in } L_{\text{loc}}^q(\mathbb{R}^N) \quad \text{and} \quad v_n \rightarrow v \text{ a.e. on } \mathbb{R}^N.$$

Consider

$$\begin{aligned}
& \langle I'(v_n) - I'(v), v_n - v \rangle \\
&= \int_{\mathbb{R}^N} (|\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v) (\nabla v_n - \nabla v) dx \\
&+ \int_{\mathbb{R}^N} V(x) \left(|H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} - |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} \right) (v_n - v) dx \\
&- \int_{\mathbb{R}^N} K(x) \left(\frac{\tilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\tilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right) (v_n - v) dx \\
&\geq C \int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^p dx \\
&+ \int_{\mathbb{R}^N} V(x) \left(|H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} - |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} \right) (v_n - v) dx \\
&- \int_{\mathbb{R}^N} K(x) \left(\frac{\tilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\tilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right) (v_n - v) dx \\
&:= C \int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^p dx + I_1 - I_2,
\end{aligned} \tag{3.10}$$

where we use the elementary inequalities:

$$(|a|^{p-2}a - |b|^{p-2}b)(a - b) \geq \begin{cases} C(|a| + |b|)^{p-2}|a - b|^2, & \text{for } a, b \in \mathbb{R}^N \text{ if } 1 < p < 2, \\ C|a - b|^p, & \text{for } a, b \in \mathbb{R}^N \text{ if } p \geq 2. \end{cases}$$

Firstly, we will show that

$$I_1 \geq 0. \tag{3.11}$$

In fact, a direct computation shows that second derivative of the function

$$G(t) = |H^{-1}(t)|^p \quad \text{for } t \in \mathbb{R}$$

satisfies the equality

$$G''(t) = \frac{\left((p-1)g(H^{-1}(t)) - \frac{g'(H^{-1}(t))H^{-1}(t)}{g(H^{-1}(t))} \right) |H^{-1}(t)|^{p-2}}{g^2(H^{-1}(t))} > 0 \quad \text{for } t \in \mathbb{R} \setminus \{0\},$$

which implies that G is a convex function. From this, we obtain

$$(G'(t) - G'(s))(t - s) \geq 0, \quad \text{for all } t, s \in \mathbb{R},$$

that is

$$I_1 = \int_{\mathbb{R}^N} V(x) \left(|H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} - |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} \right) (v_n - v) dx \geq 0.$$

Secondly, for any $R > 0$, we estimate I_2 as follows:

$$\begin{aligned}
& \int_{\mathbb{R}^N} K(x) \left| \frac{\tilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\tilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right| |v_n - v| dx \\
& \leq C \int_{\mathbb{R}^N \setminus B_R(0)} K(x) \left(|H^{-1}(v_n)|^{q-1} + |H^{-1}(v)|^{q-1} \right) (|v_n| + |v|) dx \\
& \quad + C \int_{B_R(0)} \left(|v_n|^{q-1} + |v|^{q-1} \right) |v_n - v| dx \\
& \leq C \int_{\mathbb{R}^N \setminus B_R(0)} K(x) (|v_n|^q + |v|^q) dx + C \int_{B_R(0)} \left(|v_n|^{q-1} + |v|^{q-1} \right) |v_n - v| dx \\
& \leq C \|W(x)\|_{L^1(\mathbb{R}^N \setminus B_R(0))}^{(p-q)/p} \left(\|V(x)v_n^p\|_{L^1(\mathbb{R}^N \setminus B_R(0))}^{q/p} + \|V(x)v^p\|_{L^1(\mathbb{R}^N \setminus B_R(0))}^{q/p} \right) \\
& \quad + C \left(\|v_n\|_{L^q(B_R(0))}^{q-1} + \|v\|_{L^q(B_R(0))}^{q-1} \right) \|v_n - v\|_{L^q(B_R(0))} \\
& \leq C \|W(x)\|_{L^1(\mathbb{R}^N \setminus B_R(0))}^{(p-q)/p} + C \|v_n - v\|_{L^q(B_R(0))},
\end{aligned}$$

it follows that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) \left(\frac{\tilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\tilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right) (v_n - v) dx = 0.$$

From the above estimate, (3.10) and (3.11), we get

$$\int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^p dx = o_n(1) \tag{3.12}$$

and

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^N} V(x) \left(|H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} - |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} \right) (v_n - v) dx \\
&= o_n(1).
\end{aligned} \tag{3.13}$$

It is easy to say (3.13) can also be expressed as

$$\begin{aligned}
\int_{\mathbb{R}^N} V(x) |H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} v_n dx &= \int_{\mathbb{R}^N} V(x) |H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} v dx \\
&+ \int_{\mathbb{R}^N} V(x) |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} (v_n - v) dx \\
&+ o_n(1).
\end{aligned}$$

Since $v_n \rightharpoonup v$ weakly in X ,

$$\int_{\mathbb{R}^N} V(x) |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} (v_n - v) dx = o_n(1),$$

and so,

$$\begin{aligned}
\int_{\mathbb{R}^N} V(x) |H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} v_n dx &= \int_{\mathbb{R}^N} V(x) |H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} v dx \\
&+ o_n(1).
\end{aligned} \tag{3.14}$$

Recalling that

$$|H^{-1}(t)| \leq \left(\frac{3}{2^{1-p}}\right)^{1/p} |t| \quad \text{and} \quad \left(\frac{2^{1-p}}{3}\right)^{1/p} < h(H^{-1}(v_n)) \leq 1 \quad \text{for all } t \in \mathbb{R}.$$

From this, we know $V(x) \frac{p-1}{p} |H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))}$ is bounded sequence in $L^{\frac{p}{p-1}}(\mathbb{R}^N)$. Thus,

$$\begin{aligned} & \int_{\mathbb{R}^N} V(x) |H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} v dx \\ &= \int_{\mathbb{R}^N} V(x)^{(p-1)/p} |H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} V(x)^{1/p} v dx \\ &= \int_{\mathbb{R}^N} V(x) |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} v dx + o_n(1). \end{aligned} \quad (3.15)$$

It follows from (3.14) and (3.15) that

$$\begin{aligned} \int_{\mathbb{R}^N} V(x) |H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} v_n dx &= \int_{\mathbb{R}^N} V(x) |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} v dx \\ &+ o_n(1). \end{aligned} \quad (3.16)$$

By Lemma 2.1, we have

$$V(x) |H^{-1}(v_n)|^p \leq \left(\frac{3}{2^{1-p}}\right)^{1/p} V(x) |H^{-1}(v_n)|^{p-2} \frac{H^{-1}(v_n)}{h(H^{-1}(v_n))} v_n.$$

Then, using the above discussions together with Lebesgue's Theorem, we obtain

$$\int_{\mathbb{R}^3} V(x) |H^{-1}(v_n)|^p dx = \int_{\mathbb{R}^3} V(x) |H^{-1}(v)|^p dx + o_n(1). \quad (3.17)$$

On the other hand, by $(H^{-1}(t))' \leq \left(\frac{3}{2^{1-p}}\right)^{1/p}$ (for all $t \in \mathbb{R}$), we obtain

$$\begin{aligned} |H^{-1}(v_n - v)| &= H^{-1}(|v_n - v|) \\ &\leq H^{-1}(|v_n| + |v|) \\ &\leq H^{-1}(|v_n|) + \left(\frac{3}{2^{1-p}}\right)^{1/p} |v|, \end{aligned}$$

which implies

$$V(x) |H^{-1}(v_n - v)|^p \leq 2^p V(x) \left(|H^{-1}(v_n)|^p + \left(\frac{3}{2^{1-p}}\right)^{1/p} |v|^p \right).$$

From the last inequality, (3.17) and Lebesgue's Theorem, we get

$$\int_{\mathbb{R}^N} V(x) |H^{-1}(v_n - v)|^p dx = o_n(1). \quad (3.18)$$

Finally, combing (3.12) and (3.18), we have

$$\begin{aligned} \int_{\mathbb{R}} (|\nabla(v_n - v)|^p + V(x) |v_n - v|^p) dx &\leq \int_{\mathbb{R}} (|\nabla(v_n - v)|^p + V(x) |H^{-1}(v_n - v)|^p) dx \\ &= o_n(1), \end{aligned}$$

which concludes the proof of the lemma. \square

4 Proof of Theorem 1.3

In this section, we firstly study Brezis–Kato type estimates of the critical points of I .

Lemma 4.1. *Assume that $\{v_k\} \subset X$ is a critical point sequence of I satisfying $v_k \rightarrow 0$ strongly in X . Then, $v_k \rightarrow 0$ strongly in $L^\infty(\mathbb{R}^N)$.*

Proof. Let $v \in X$ be a weak solution of (2.5), i.e.,

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla \varphi dx + \int_{\mathbb{R}^N} V(x) |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} \varphi dx \\ - \int_{\mathbb{R}^N} K(x) \frac{\tilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \varphi dx, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \end{aligned} \quad (4.1)$$

Set $T > 0$, and denote

$$v_T = \begin{cases} -T, & \text{if } v \leq -T, \\ v, & \text{if } -T < v < T, \\ T, & \text{if } v \geq T. \end{cases}$$

Taking $\varphi = |v_T|^{p(\eta-1)} v_T$ as the test function, where $\eta > 1$ to be determined later, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |v_T|^{p(\eta-1)} |\nabla v|^{p-2} \nabla v \nabla v_T dx + p(\eta-1) \int_{\mathbb{R}^N} |v_T|^{p(\eta-1)-1} |\nabla v|^{p-2} \nabla v \nabla v_T dx \\ + \int_{\mathbb{R}^N} V(x) |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} |v_T|^{p(\eta-1)} v_T dx = \int_{\mathbb{R}^N} K(x) \frac{\tilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} |v_T|^{p(\eta-1)} v_T dx. \end{aligned}$$

By using the facts

$$\begin{aligned} p(\eta-1) \int_{\mathbb{R}^N} |v_T|^{p(\eta-1)-1} |\nabla v|^{p-2} \nabla v \nabla v_T dx &\geq 0, \\ \int_{\mathbb{R}^N} V(x) |H^{-1}(v)|^{p-2} \frac{H^{-1}(v)}{h(H^{-1}(v))} |v_T|^{p(\eta-1)} v_T dx &\geq 0 \end{aligned}$$

and Lemma 2.1-(iii), we have

$$\frac{1}{\eta^p} \int_{\mathbb{R}^N} |\nabla |v_T|^\eta| dx \leq C \int_{\mathbb{R}^N} \tilde{f}(x, H^{-1}(v)) |v|^{p(\eta-1)+1} dx \leq C \int_{\mathbb{R}^N} |v|^{p\eta+q-p} dx. \quad (4.2)$$

On the other hand, it follows from the Sobolev inequality that

$$\frac{S}{\eta^p} \|v_T\|_{\eta p^*}^{p\eta} \leq \frac{1}{\eta^p} \int_{\mathbb{R}^N} |\nabla |v_T|^\eta| dx, \quad (4.3)$$

where $S = \inf\{\int_{\mathbb{R}^N} |\nabla v|^p dx \mid \int_{\mathbb{R}^N} |v|^{p^*} dx = 1\}$. In what follows, by (4.2) and (4.3), we get

$$\frac{1}{\eta^p} \|v_T\|_{\eta p^*}^{p\eta} \leq C \int_{\mathbb{R}^N} |v|^{p\eta+q-p} dx. \quad (4.4)$$

From Fatou's lemma, sending $T \rightarrow \infty$ in (4.4), it follows that

$$\|v\|_{\eta p^*} \leq (C\eta)^{1/\eta} \|v\|_{\frac{p\eta+q-p}{p\eta}}^{(p\eta+q-p)/p\eta}. \quad (4.5)$$

Let us define $\eta_k = \frac{p^* \eta_{k-1} + p - q}{p}$, where $k = 1, 2, \dots$ and $\eta_0 = \frac{p^* + p - q}{p}$. We show the first step of Moser's iteration as follows:

$$\begin{aligned} \|v\|_{\eta_1 p^*} &\leq (C\eta_1)^{1/\eta_1} \|v\|_{\frac{(p\eta_1 + q - p)/p\eta_1}{p\eta_1 + q - p}} \\ &\leq (C\eta_1)^{1/\eta_1} (C\eta_0)^{1/\eta_0 \cdot (p\eta_1 + q - p)/p\eta_1} \|v\|_{\frac{(p\eta_0 + q - p)/p\eta_0 \cdot (p\eta_1 + q - p)/p\eta_1}{p\eta_0 + q - p}}. \end{aligned} \quad (4.6)$$

Without loss of generality, we may assume $C > 1$. For $i < j$, we have

$$(C\eta_i)^{(p\eta_j + q - p)/(p\eta_j)} \leq C\eta_i. \quad (4.7)$$

From (4.6) and (4.7), we have

$$\|v\|_{\eta_1 p^*} \leq (C\eta_1)^{1/\eta_1} (C\eta_0)^{1/\eta_0} \|v\|_{\frac{(p\eta_0 + q - p)/p\eta_0 \cdot (p\eta_1 + q - p)/p\eta_1}{p\eta_0 + q - p}}.$$

Then by Moser's iteration method we get

$$\|v\|_{p\eta_{k+1} + q - p} \leq \exp\left(\sum_{i=0}^k \frac{\ln(C\eta_i)}{\eta_i}\right) \|v\|_{p^*}^{\mu_k},$$

where $\mu_k = \prod_{i=0}^k \frac{p\eta_i + q - p}{p\eta_i}$. Sending $k \rightarrow \infty$, we deduce that

$$\|v\|_{\infty} \leq \exp\left(\sum_{i=0}^{\infty} \frac{\ln(C\eta_i)}{\eta_i}\right) \|v\|_{p^*}^{\mu},$$

where $\mu = \prod_{i=0}^{\infty} \frac{p\eta_i + q - p}{p\eta_i}$ ($0 < \mu < 1$) and $\exp\left(\sum_{i=0}^{\infty} \frac{\ln(C\eta_i)}{\eta_i}\right)$ is a positive constant. This, together with the Sobolev embedding theorem, shows that if $\{v_k\}$ is a critical point sequence of J satisfying $v_k \rightarrow 0$ strongly in X as $k \rightarrow \infty$, then $v_k \rightarrow 0$ strongly in $L^\infty(\mathbb{R}^N)$. This completes the proof. \square

Proof of Theorem 1.3. It is well known that I is an even functional with $I(0) = 0$. In addition, by Lemma 3.3, Lemma 3.2 and Proposition 3.1, the functional I possesses a sequence of critical points $\{v_n\}$ such that $I(v_n) \rightarrow 0$ and $v_n \rightarrow 0$ strongly in X . Recall that the weak solutions of (2.1) with an L^∞ -norm not more than $\min\{\delta/2, (2^{1-p}/3)^{1/p}\}$ are also weak solutions of problem (1.3). Then, by Lemma 4.1, this $\{v_n\}$ is a sequence of weak solutions for (2.5) with $v_n \rightarrow 0$ strongly in $L^\infty(\mathbb{R}^N)$. Letting $u_n = H^{-1}(v_n)$, from Lemma 2.2, there exists $n^* \in \mathbb{N}$ such that u_n is a weak solution of (1.3) for each $n \geq n^*$. This ends the proof. \square

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