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Controllability and observability for a class of time-varying impulsive systems on time scales

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Abstract. The aim of this paper is to study the controllability and observability for a class of linear time-varying impulsive control systems on time scales. Sufficient and necessary conditions for state controllability and state observability of such systems are established. The corresponding criteria for time-invariant impulsive control systems on time scales are also obtained.

Keywords: time scale, linear impulsive control system, controllability, observability.

AMS Subject Classification: 93B05, 34A37, 34H05, 93B07.

1 Introduction

Differential equations with impulses have a considerable importance in varied applications as physics, engineering, biology, medicine, economics, neuronal networks, social sciences, and so on. Many investigations have been carried out concerning the existence, uniqueness, and asymptotic properties of solutions. We refer to the monographs [7, 11, 29, 40] and the references therein. It is well known that the study of controllability plays an important role in the control theory. In recent years, some research dealing with the study of controllability for impulsive systems [10, 16, 23, 32, 34, 41, 44, 47]. The most dynamical systems are analyzed in either the continuous or discrete time domain. The population dynamical models in continuous time are usually appropriate for organism that have overlapping generations. On other hand, many biological populations are more accurately described by non-overlapping generations. The dynamics of these populations often are

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more appropriately expressed by so-called difference equations. A hybrid model, so-called sequential-continuous dynamical models, was developed by Busenberg and Cooke [17] for models of vertically transmitted diseases (see also [18]). The sequential-continuous systems are characterized by the fact that they, during certain periods of time, are governed by continuous equations, and during the other periods, are governed by sequential equations. A such sequential-continuous model can be formulated by the help of dynamical systems on time scales. For more details and results in this area see [5], [6], [15] and [45]. S. Hilger [24] introduced the theory of time scales in order to create a theory that can unify continuous and discrete analysis. There has been significant growth in the theory of dynamic systems on time scales, covering a variety of different qualitative aspects. We refer to the books [13, 14, 30] and the references therein. We also refer to the papers [1, 3, 19, 27, 28, 36, 42, 43, 46]. Some authors studied impulsive dynamic systems on time scales [4, 11, 12, 26, 31, 33, 35]. The study of stability, controllability and observability for dynamical systems on time scales has been studied in few works [8, 9, 20, 21, 22, 25, 38, 39], but there has been no result about the controllability and observability of piecewise linear time-varying impulsive control systems. The main purpose of this paper is to derive necessary and sufficient criteria for controllability and observability of a class of such systems on time scales.

2 Preliminaries

Let $\mathbb{R}^n$ be the space of n-dimensional column vectors $x = \text{col}(x_1, x_2, ...x_n)$ with a norm $||\cdot||$. A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. The notations $[a, b]$, $[a, b)$, and so on, will denote time scales intervals such as $[a, b] := \{t \in \mathbb{T}; a \leq t \leq b\}$, where $a, b \in \mathbb{T}$. The set of all *rd-continuous* functions $f : \mathbb{T} \to \mathbb{R}^n$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R}^n)$. A function $f : \mathbb{T} \to \mathbb{R}^n$ is *piecewise rd-continuous* (we write $f \in C_{prd}(\mathbb{T}, \mathbb{R}^n)$) if it is regulated and if it is rd-continuous at all, except possibly at finitely many, right-dense points $t \in \mathbb{T}$.

We denote by $C^1_{rd}(\mathbb{T}, \mathbb{R}^n)$ the set of all functions $f : \mathbb{T} \to \mathbb{R}^n$ that are differentiable on $\mathbb{T}$ and its delta-derivative $f^{\Delta}(t) \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$. The set of rd-continuous (respectively rd-continuous and regressive) matrix-valued functions $A : \mathbb{T} \to M_n(\mathbb{R})$ is denoted by $C_{rd}(\mathbb{T}, M_n(\mathbb{R}))$ (respectively by $C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$). We recall that a matrix-valued function A is said to be *regressive* if $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^k$, where I is the $n \times n$ identity matrix. We refer to [13, 14] and also to the paper [1, 2] for more information on analysis on time scales.

Consider the following impulsive dynamical system

$$\begin{cases} x^{\Delta} = A_k(t)x + B_k(t)u, \ \ t \in [t_{k-1}, t_k), \\ x(t_k^+) = (1 + c_k)x(t_k), \ \ k = 1, 2, ..., \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where $\mathbb{T}$ is a unbounded above time scale with bounded graininess, $[t_{k-1}, t_k) \subset \mathbb{T}_0 := [t_0, \infty) \cap \mathbb{T}$, $t_k \in \mathbb{T}$ are right-dense, $0 \leq t_0 < t_1 < t_2 < ... < t_k < ...$, such that $\lim_{k\to\infty} t_k = \infty$, $x(t_k^+) := \lim_{h\to 0^+} x(t_k + h)$, $x(t_k^-) := \lim_{h\to 0^+} x(t_k - h)$ and $c_k \in \mathbb{R}$ are constants. In this paper, we assume that $A_k \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_n(\mathbb{R}))$, $B_k \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_{n\times m}(\mathbb{R}))$, $x \in \mathbb{R}^n$ is the state variable, and $u \in \mathbb{R}^m$ is the control input.

Corresponding to impulsive system (1), consider the following dynamic system on time scales

$$x^{\Delta} = A_k(t)x \tag{2}$$

where $k = 1, 2, ...,$ and $t \in [t_{k-1}, t_k)$.

A matrix $X_{A_k} \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$ is said to be a *matrix solution* of (2) if each column of X_{A_k} satisfies (2) for all $t \in [t_{k-1}, t_k)$. A *fundamental matrix* of (2) is a matrix solution X_{A_k} of (2) such that $\det X_{A_k}(t) \neq 0$ for all $t \in [t_{k-1}, t_k)$. A *transition matrix* of (2) at initial time $\tau \in [t_{k-1}, t_k)$ is a fundamental matrix such that $X_{A_k}(\tau) = I$. The transition matrix of (2) at initial time $\tau \in [t_{k-1}, t_k)$ will be denoted by $\Phi_{A_k}(t, \tau)$. Therefore, the transition matrix of (2) at initial time $\tau \in [t_{k-1}, t_k)$ is the unique solution of the following matrix initial value problem

$$X^{\Delta} = A_k(t)X, \quad X(\tau) = I \tag{3}$$

and $x(t) = \Phi_{A_k}(t, \tau)\eta$ for $\tau \in [t_{k-1}, t_k)$, is the unique solution of initial value problem

$$x^{\Delta} = A_k(t)x, \quad x(\tau) = \eta.$$

If $A_k(t) = A_k$ is a constant matrix, then we use the notation $e_{A_k}(t, \tau)$ instead of $\Phi_{A_k}(t, \tau)$.

Proposition 1 ([13, Theorem 5.24]). *If* $A \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$ *and* $h \in C_{prd}(\mathbb{T}, \mathbb{R}^n)$, *then for each* $(\tau, \eta) \in \mathbb{T} \times \mathbb{R}^n$ *the initial value problem*

$$x^{\Delta} = A(t)x + h(t), \quad x(\tau) = \eta$$

has a unique solution given by

$$x(t) = \Phi_A(t,\tau)\eta + \int_\tau^t \Phi_A(t,\sigma(s))h(s)\Delta s, \quad t \geq \tau. \ \square$$

The following theorem shows that we can express the matrix exponential as a finite sum of powers of the matrix A with infinitely rd-continuous delta differentiable functions as coefficients.

Proposition 2 ([19, Theorem 5.1]). *For the system* (3) *with* $A \in M_n(\mathbb{R})$ *constant, there exist scalar functions* $\gamma_0(t,\tau), ..., \gamma_{n-1}(t,\tau) \in C^\infty_{rd}(\mathbb{T}_+, \mathbb{R})$ *such that the unique solution has representation*

$$e_A(t,\tau) = \sum_{i=0}^{n-1} \gamma_i(t,\tau)A^i. \tag{4}$$

$\square$

Lemma 1. *For any* $t \in (t_{k-1}, t_k]$, $k = 1, 2, ...$, *the solution of the initial value problem(1) is given by*

$$x(t) = \Phi_{A_k}(t, t_{k-1}) \prod_{i=1}^{k-1}(1+c_i) \prod_{i=k-1}^{1} \Phi_{A_i}(t_i, t_{i-1})x_0$$

$$+ \int_{t_{k-1}}^t \Phi_{A_k}(t,\sigma(\tau))B_k(\tau)u(\tau)\Delta\tau + \sum_{i=1}^{k-1}[\prod_{j=i}^{k-1}(1+c_j) \int_{t_{i-1}}^{t_i} \Phi_{A_k}(t, t_{k-1}) \tag{5}$$

$$\times \prod_{r=k-1}^{i+1} \Phi_{A_r}(t_r, t_{r-1})\Phi_{A_i}(t_i,\sigma(\tau))B_i(\tau)u(\tau)\Delta\tau].$$

Proof. If $t \in [t_0, t_1]$, then the unique solution of (1) is given by

$$x(t) = \Phi_{A_1}(t, t_0)x_0 + \int_{t_0}^t \Phi_{A_1}(t,\sigma(\tau))B_1(\tau)u(\tau)\Delta\tau, \ t \in [t_0, t_1].$$

For $t \in (t_1, t_2]$ the initial value problem

$$\begin{cases} x^\Delta = A_2(t)x + B_2(t)u, \\ x(t_1^+) = (1+c_1)x(t_1), \end{cases}$$

has the unique solution

$$x(t) = \Phi_{A_2}(t, t_1)x(t_1^+) + \int_{t_1}^{t} \Phi_{A_2}(t, \sigma(\tau))B_2(\tau)u(\tau)\Delta\tau.$$

Since

$$x(t_1^+) = (1 + c_1)x(t_1) =$$

$$= (1 + c_1)\Phi_{A_1}(t_1, t_0)x_0 + (1 + c_1) \int_{t_0}^{t_1} \Phi_{A_1}(t_1, \sigma(\tau))B_1(\tau)u(\tau)\Delta\tau$$

it follows that

$$x(t) = \Phi_{A_2}(t, t_1)(1 + c_1)\Phi_{A_1}(t_1, t_0)x_0+$$

$$+(1 + c_1) \int_{t_0}^{t_1} \Phi_{A_2}(t, t_1)\Phi_{A_1}(t_1, \sigma(\tau))B_1(\tau)u(\tau)\Delta\tau$$

$$+ \int_{t_1}^{t} \Phi_{A_2}(t, \sigma(\tau))B_2(\tau)u(\tau)\Delta\tau$$

and so, (5) is true for $k = 2$. Next, suppose that (5) is true for $k = p$, that is, for $t \in (t_{p-1}, t_p]$, we have

$$x(t) = \Phi_{A_p}(t, t_{p-1}) \prod_{i=1}^{p-1}(1 + c_i) \prod_{i=p-1}^{1} \Phi_{A_i}(t_i, t_{i-1})x_0$$

$$+ \int_{t_{p-1}}^{t} \Phi_{A_p}(t, \sigma(\tau))B_p(\tau)u(\tau)\Delta\tau + \sum_{i=1}^{p-1}[\prod_{j=i}^{p-1}(1 + c_j) \int_{t_{i-1}}^{t_i} \Phi_{A_p}(t, t_{p-1})$$

$$\times \prod_{r=p-1}^{i+1} \Phi_{A_r}(t_r, t_{r-1})\Phi_{A_i}(t_i, \sigma(\tau))B_i(\tau)u(\tau)\Delta\tau].$$

Then, for $t \in (t_p, t_{p+1}]$, the initial value problem

$$\begin{cases} x^\Delta = A_{p+1}(t)x + B_{p+1}(t)u, \\ x(t_p^+) = (1 + c_p)x(t_p), \end{cases}$$

has the unique solution

$$x(t) = \Phi_{A_{p+1}}(t, t_p)x(t_p^+) + \int_{t_p}^{t} \Phi_{A_{p+1}}(t, \sigma(\tau))B_{p+1}(\tau)u(\tau)\Delta\tau, \;\; t \in (t_p, t_{p+1}].$$

Since

$$x(t_p^+) = (1+c_p)x(t_p) = \prod_{i=1}^{p}(1+c_i)\prod_{i=p}^{1}\Phi_{A_i}(t_i,t_{i-1})x_0$$

$$+(1+c_p)\int_{t_{p-1}}^{t_p}\Phi_{A_p}(t_p,\sigma(\tau))B_p(\tau)u(\tau)\Delta\tau + \sum_{i=1}^{p-1}[\prod_{j=i}^{p}(1+c_j)\int_{t_{i-1}}^{t_i}\Phi_{A_p}(t_p,t_{p-1})$$

$$\times\prod_{r=p-1}^{i+1}\Phi_{A_r}(t_r,t_{r-1})\Phi_{A_i}(t_i,\sigma(\tau))B_i(\tau)u(\tau)\Delta\tau].$$

It follows that

$$x(t) = \Phi_{A_{p+1}}(t,t_p)\prod_{i=1}^{p}(1+c_i)\prod_{i=p}^{1}\Phi_{A_i}(t_i,t_{i-1})x_0$$

$$+\int_{t_p}^{t}\Phi_{A_{p+1}}(t,\sigma(\tau))B_{p+1}(\tau)u(\tau)\Delta\tau + \sum_{i=1}^{p}[\prod_{j=i}^{p}(1+c_j)\int_{t_{i-1}}^{t_i}\Phi_{A_{p+1}}(t,t_p)$$

$$\times\prod_{r=p}^{i+1}\Phi_{A_r}(t_r,t_{r-1})\Phi_{A_i}(t_i,\sigma(\tau))B_i(\tau)u(\tau)\Delta\tau],$$

and thus (5) is true for $k = p+1$. Therefore, by induction, (5) is proved. □

3 Controllability

Definition 1. *The impulsive system (1) is called controllable on* $[t_0, t_f]$*, with* $t_f > t_0$*, if given any initial state* $x_0 \in R^n$ *there exists a piecewise rd-continuous input signal* $u(\cdot) : [t_0, t_f] \to R^m$ *such that the corresponding solution of (1) satisfies* $x(t_f) = 0$.

We consider the following matrices:

$$G_i := G(t_0, t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i}\Psi_i(t_0,\sigma(\tau))B_i(\tau)B_i^T(\tau)\Psi_i^T(t_0,\sigma(\tau))\Delta\tau, \qquad (6)$$

for $i=1,2,...,k-1$, and

$$G_k := G(t_0,t_{k-1},t_f) = \int_{t_{k-1}}^{t_f} \Psi_k(t_0,\sigma(\tau))B_k(\tau)B_k^T(\tau)\Psi_k^T(t_0,\sigma(\tau))\Delta\tau, \quad (7)$$

where $\Psi_1(\tau) := \Psi_1(t_0,\sigma(\tau) = \Phi_{A_1}(t_0,\sigma(\tau))$, for $\tau \in (t_0,t_1]$, and

$$\Psi_i(\tau) := \Psi_i(t_0,\sigma(\tau)) = \prod_{j=1}^{i-1} \Phi_{A_j}(t_{j-1},t_j)\Phi_{A_i}(t_{i-1},\sigma(\tau)), \ \ \tau \in (t_{i-1},t_i], \quad (8)$$

for $i=2,3,...,k$.

If $A_k(t)=A_k$ and $B_k(t)=B_k$ are constant matrices then

$$G_i := G(t_0,t_{i-1},t_i) = \int_{t_{i-1}}^{t_i} \Psi_i(t_0,\sigma(\tau))B_iB_i^T\Psi_i^T(t_0,\sigma(\tau))\Delta\tau, \quad (9)$$

for $i=1,2,...,k-1$ and

$$G_k := G(t_0,t_{k-1},t_f) = \int_{t_{k-1}}^{t_f} \Psi_k(t_0,\sigma(\tau))B_kB_k^T\Psi_k^T(t_0,\sigma(\tau))\Delta\tau, \quad (10)$$

where $\Psi_1(\tau) := \Psi_1(t_0,\sigma(\tau) = e_{A_1}(t_0,\sigma(\tau))$, for $\tau \in (t_0,t_1]$, and

$$\Psi_i(\tau) := \Psi_i(t_0,\sigma(\tau)) = \prod_{j=1}^{i-1} e_{A_j}(t_{j-1},t_j)e_{A_i}(t_{i-1},\sigma(\tau)), \ \ \tau \in (t_{i-1},t_i], \quad (11)$$

for $i=2,3,...,k$.

The Gramian matrix in the case of time scales was defined in [21]. The above definition is adopted from [21] for impulsive case. Now we are formulating the results for controllability.

Theorem 1. *(i) If there exists at least $l \in \{1,2,...,k\}$ such that* $\operatorname{rank}(G_l) = n$*, then the impulsive system (1) is controllable on* $[t_0,t_f]$ $(t_f \in (t_{k-1},t_k])$.

(ii) Assume that $c_i \neq -1$, $i=1,2,...,k-1$. If the impulsive system (1) *is controllable on* $[t_0,t_f]$ $(t_f \in (t_{k-1},t_k])$*, then*

$$\operatorname{rank}(G_0\ G_1\ ...\ G_k) = n. \quad (12)$$

Proof. (i) Let $l \in \{1,2,...,k\}$ be such that $\operatorname{rank}(G_l) = n$, that is, $G(t_0,t_{l-1},t_l)$ is invertible. Then for a given $x_0 \in \mathbb{R}^n$, choose

$$u(t) = \begin{cases} a_l B_l^T(t)\Psi_l^T G_l^{-1}x_0 & \text{if } t \in (t_{l-1},t_l] \\ 0 & \text{if } t \in [t_0,t_f] \setminus (t_{l-1},t_l], \end{cases} \quad (13)$$

where a_l is a constant such that

$$\prod_{i=1}^{k-1}(1+c_i)+a_l\prod_{j=l}^{k-1}(1+c_j)=0.$$

Obviously, the control input $u(\cdot)$ is piecewise rd-continuous on $[t_0,t_f]$. By Lemma 1, we have

$$x(t_f)=\Phi_{A_k}(t_f,t_{k-1})\prod_{i=1}^{k-1}(1+c_i)\prod_{i=k-1}^{1}\Phi_{A_i}(t_i,t_{i-1})x_0+[\prod_{j=l}^{k-1}(1+c_j)a_l\times$$

$$\int_{t_{l-1}}^{t_l}\Phi_{A_k}(t_f,t_{k-1})\prod_{r=k-1}^{l+1}\Phi_{A_r}(t_r,t_{r-1})\Phi_{A_l}(t_l,\sigma(\tau))B_l(\tau)B_l^T(\tau)\Psi_l^T(\tau)G_l^{-1}\Delta\tau]x_0.$$

Since

$$\prod_{r=k-1}^{l+1}\Phi_{A_r}(t_r,t_{r-1})\Phi_{A_l}(t_l,\sigma(\tau))\Psi_l^{-1}(\tau)=\Phi_{A_{k-1}}(t_{k-1},t_{k-2})...\Phi_{A_l}(t_l,\sigma(\tau))$$

$$\times\Phi_{A_l}(\sigma(\tau),t_{l-1})\Phi_{A_{l-1}}(t_{l-1},t_{l-2})...\Phi_{A_1}(t_1,t_0)=\prod_{i=k-1}^{1}\Phi_{A_i}(t_i,t_{i-1}),$$

it follows that

$$x(t_f)=\Phi_{A_k}(t_f,t_{k-1})\prod_{i=1}^{k-1}(1+c_i)\prod_{i=k-1}^{1}\Phi_{A_i}(t_i,t_{i-1})x_0$$

$$+[\prod_{j=l}^{k-1}(1+c_j)a_l\int_{t_{l-1}}^{t_l}\Phi_{A_k}(t_f,t_{k-1})\prod_{r=k-1}^{l+1}\Phi_{A_r}(t_r,t_{r-1})\Phi_{A_l}(t_l,\sigma(\tau))\Psi_l^{-1}(\tau)$$

$$\times\Psi_l(\tau)B_l(\tau)B_l^T(\tau)\Psi_l^T(\tau)G_l^{-1}\Delta\tau]x_0.$$

Therefore, we obtain

$$x(t_f)=\left[\prod_{i=1}^{k-1}(1+c_i)+\prod_{j=l}^{k-1}(1+c_j)a_l\right]\Phi_{A_k}(t_f,t_{k-1})\prod_{i=k-1}^{1}\Phi_{A_i}(t_i,t_{i-1})=0,$$

and so, the impulsive system (1) is controllable on $[t_0, t_f]$.

(ii) Suppose that (1) is controllable on $[t_0, t_f]$ and $\text{rank}(G_0\ G_1\ ...\ G_k) < n$. Then, there exists nonzero $x_\alpha \in \mathbb{R}^n$ such that

$$0 = x_\alpha^T G(t_0, t_{i-1}, t_i) x_\alpha = \int_{t_{i-1}}^{t_i} x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i(\tau) B_i^T(\tau) \Psi_i^T(t_0, \sigma(\tau)) x_\alpha \Delta\tau,$$

for $i = 1, 2, ..., k-1$, and

$$0 = x_\alpha^T G(t_0, t_{k-1}, t_f) x_\alpha = \int_{t_{k-1}}^{t_f} x_\alpha^T \Psi_k(t_0, \sigma(\tau)) B_k(\tau) B_k^T(\tau) \Psi_k^T(t_0, \sigma(\tau)) x_\alpha \Delta\tau.$$

Since $x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i(\tau)$ are rd-continuous functions and

$$x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i(\tau) B_i^T(\tau) \Psi_i^T(t_0, \sigma(\tau)) x_\alpha = \left\| x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i(\tau) \right\|^2,$$

for $\tau \in (t_{i-1}, t_i]$, $i = 1, 2, ..., k$, then from the last equalities we obtain

$$x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i(\tau) = 0, \quad \tau \in (t_{i-1}, t_i],\ i = 1, 2, ..., k. \tag{14}$$

However, the impulsive system (1) is controllable on $[t_0, t_f]$, and so choosing $x_0 = x_\alpha$, there exists a piecewise rd-continuous input $u(\cdot)$ such that

$$\begin{aligned} 0 = x(t_f) &= \Phi_{A_k}(t_f, t_{k-1}) \prod_{i=1}^{k-1}(1 + c_i) \prod_{i=k-1}^{1} \Phi_{A_i}(t_i, t_{i-1}) x_\alpha \\ &+ \textstyle\int_{t_{k-1}}^{t_f} \Phi_{A_k}(t_f, \sigma(\tau)) B_k(\tau) u(\tau) \Delta\tau + \displaystyle\sum_{i=1}^{k-1} [\prod_{j=i}^{k-1} (1 + c_j) \times \\ &\textstyle\int_{t_{i-1}}^{t_i} \Phi_{A_k}(t_f, t_{k-1}) \displaystyle\prod_{r=k-1}^{i+1} \Phi_{A_r}(t_r, t_{r-1}) \Phi_{A_i}(t_i, \sigma(\tau)) B_i(\tau) u(\tau) \Delta\tau]. \end{aligned} \tag{15}$$

Multiplying by $\Phi_{A_1}(t_0, t_1)\Phi_{A_2}(t_1, t_2)...\Phi_{A_k}(t_{k-1}, t_f)$ in (15) we obtain

$$\begin{aligned} \prod_{i=1}^{k-1}(1 + c_i) x_\alpha &= -\sum_{i=1}^{k-1} [\prod_{j=i}^{k-1} (1 + c_j) \Phi_{A_1}(t_0, t_1) \Phi_{A_2}(t_1, t_2) ... \Phi_{A_i}(t_{i-1}, t_i) \\ &\times \int_{t_{i-1}}^{t_i} \Phi_{A_i}(t_i, \sigma(\tau)) B_i(\tau) u(\tau) \Delta\tau] \\ &- \Phi_{A_1}(t_0, t_1) \Phi_{A_2}(t_1, t_2) ... \Phi_{A_k}(t_{k-1}, t_f) \int_{t_{k-1}}^{t_f} \Phi_{A_k}(t_f, \sigma(\tau)) B_k(\tau) u(\tau) \Delta\tau. \end{aligned}$$

Now, using (14) and multiplying by x_α^T to the both side of the above equality, we obtain

$$\prod_{i=1}^{k-1}(1+c_i)x_\alpha^T x_\alpha = -\sum_{i=1}^{k-1}[\prod_{j=i}^{k-1}(1+c_j)\int_{t_{i-1}}^{t_i} x_\alpha^T \Psi_i(t_0,\sigma(\tau))B_i(\tau)u(\tau)\Delta\tau]$$

$$-\int_{t_{k-1}}^{t_f} x_\alpha^T \Psi_k(t_0,\sigma(\tau))B_k(\tau)u(\tau)\Delta\tau = 0.$$

Since $\prod_{j=1}^{k}(1+c_j) \neq 0$, it follows that $x_\alpha x_\alpha^T = 0$. This contradicts $x_\alpha \neq 0$ and so we conclude that $\text{rank}(G_0\ G_1\ ...\ G_k) = n$. □

If $\mathbb{T} = \mathbb{R}$, then we obtain the result of Theorem 1 in [47]. If $A_k(t) = A(t)$, $B_k(t) = B(t)$, then we obtain the Theorem 1 in [36], and the Theorem 3.1 in [23] if $\mathbb{T} = \mathbb{R}$. The version of non impulsive case on time scales ($c_i = -1$) can be found in [8, Theorem 4], [21, Theorem 3.2] and [25, Theorem 3.7].

Theorem 2. *Assume that* $c_i \neq -1$, $i = 1, 2, ..., k-1$, *and* $A_k(t) = A_k$, $B_k(t) = B_k$ *are constant matrices. Then the impulsive system (1) is controllable on* $[t_0, t_f](t_f \in (t_{k-1}, t_k])$ *if and only if*

$$\text{rank}(W_1\ W_2\ ...\ W_k) = n, \tag{16}$$

where $W_i = \Lambda_i(B_i\ A_iB_i\ ...\ A_i^{n-1}B_i)$ for $i = 1, 2, ..., k-1$, $W_k = \Lambda_{k-1}e_{A_k}(t_{k-1}, t_f)$ $(B_k\ A_kB_k\ ...\ A_k^{n-1}B_k)$, *and* $\Lambda_i = e_{A_1}(t_0, t_1)e_{A_2}(t_1, t_2)...e_{A_i}(t_{i-1}, t_i)$.

Proof. Suppose that the impulsive system (1) is controllable on $[t_0, t_f]$. If the rank condition (16) does not hold, then there exists nonzero $x_\alpha \in \mathbb{R}^n$ such that

$$x_\alpha^T \Lambda_i A_i^j B_i = 0,$$

for $i = 1, 2, ..., k$, $j = 0, 1, ..., n-1$. Using (4), (9) and (10), we obtain that

$$x_\alpha^T G(t_0, t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i B_i^T \Psi_i(\tau) \Delta\tau$$

$$= \int_{t_{i-1}}^{t_i} x_\alpha^T \Lambda_{i-1} e_{A_i}(t_{i-1}, \sigma(\tau)) B_i B_i^T \Psi_i(\tau) \Delta\tau$$

$$= \int_{t_{i-1}}^{t_i} x_\alpha^T \Lambda_{i-1} e_{A_i}(t_{i-1}, t_i) e_{A_i}(t_i, \sigma(\tau)) B_i B_i^T \Psi_i(\tau) \Delta\tau$$

$$= \int_{t_{i-1}}^{t_i} x_\alpha^T \Lambda_i e_{A_i}(t_i, \sigma(\tau)) B_i B_i^T \Psi_i(\tau) \Delta\tau$$

$$= \int_{t_{i-1}}^{t_i} \left[\sum_{j=0}^{n-1} \gamma_{ij}(t_i, \sigma(\tau)) x_\alpha^T \Lambda_i A_i^j B_i \right] B_i^T \Psi_i(\tau) \Delta\tau = 0$$

for $i = 1, 2, ..., k-1$. Similarly, $x_\alpha^T G(t_0, t_{k-1}, t_f) = 0$. It follows that rank$(G_0\ G_1\ ...\ G_k) < n$. This contradicts the conclusion (ii) of Theorem 1 and therefore, we can conclude that the condition (16) is true.

Conversely, suppose that (16) holds. If the impulsive system (1) is not controllable on $[t_0, t_f]$ ($t_f \in (t_{k-1}, t_k]$), then it follows from conclusion (i) of Theorem 1 that the matrices $G(t_0, t_{i-1}, t_i)$ ($i = 1, 2, ..., k-1$) and $G(t_0, t_{k-1}, t_f)$ are not invertible. Thus there exists nonzero $x_\alpha \in \mathbb{R}^n$ such that

$$0 = x_\alpha^T G(t_0, t_{i-1}, t_i) x_\alpha = \int_{t_{i-1}}^{t_i} x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i B_i^T \Psi_i^T(t_0, \sigma(\tau)) x_\alpha \Delta\tau,$$

for $i = 1, 2, ..., k-1$, and

$$0 = x_\alpha^T G(t_0, t_{k-1}, t_f) x_\alpha = \int_{t_{k-1}}^{t_f} x_\alpha^T \Psi_k(t_0, \sigma(\tau)) B_k B_k^T \Psi_k^T(t_0, \sigma(\tau)) x_\alpha \Delta\tau.$$

Exactly as in proof of Theorem 1, it follows that

$$0 = x_\alpha^T \Psi_i(t_0, \sigma(\tau)) B_i = x_\alpha^T \Lambda_i e_{A_i}(t_i, \sigma(\tau)) B_i, \ \ \tau \in (t_{i-1}, t_i]$$

and

$$0 = x_\alpha^T \Psi_k(t_0, \sigma(\tau)) B_k = x_\alpha^T \Lambda_k e_{A_k}(t_f, \sigma(\tau)) B_k = 0, \ \ \tau \in (t_{k-1}, t_f].$$

By continuity of $e_{A_i}(t_i,\cdot)$ and density of $\sigma((t_{i-1},t_i])$ in the interval $(\sigma(t_{i-1}),\sigma(t_i)]=(t_{i-1},t_i]$ we obtain that

$$x_\alpha^T\Lambda_i e_{A_i}(t_i,\tau)B_i=0 \text{ for all } \tau\in(t_{i-1},t_i],\ i=1,2,...,k-1. \tag{17}$$

Also, by continuity of $e_{A_k}(t_f,\cdot)$ and density of $\sigma((t_{k-1},t_f])$ in the interval $(\sigma(t_{k-1}),\sigma(t_f)]=(t_{k-1},t_f]$ we obtain that

$$x_\alpha^T\Lambda_k e_{A_k}(t_f,\tau)B_k=0 \text{ for all } \tau\in(t_{k-1},t_f]. \tag{18}$$

In particular, if we take $\tau=t_i$ in (17) and $\tau=t_f$ in (18), then, it follows that $x_\alpha^T\Lambda_i B_i=0$ for $i=1,2,...,k$. Since $e_{A_i}(t_i,\cdot)$ is delta differentiable and $\frac{\partial}{\Delta\tau}e_{A_i}(t_i,\tau)=-e_{A_i}(t_i,\sigma(\tau))A_i$ (see [13, Theorem 5.23]), then subsequent derivatives and the density argument as above, gives

$$(-1)^j x_\alpha^T\Lambda_i e_{A_i}(t_i,\tau)A_i^jB_i=0,\ \tau\in(t_{i-1},t_i] \tag{19}$$

for $j=0,1,...,n-1$ and $i=1,2,...,k-1$. Similarly,

$$(-1)^j x_\alpha^T\Lambda_k e_{A_k}(t_f,\tau)A_k^jB_k=0\ \tau\in(t_{k-1},t_f] \tag{20}$$

for $j=0,1,...,n-1$. If we take $\tau=t_i$ in (19) and $\tau=t_f$ in (20), then it follows that $x_\alpha^T\Lambda_iA_i^jB_i=0$ for $i=1,2,...,k$, $j=0,1,...,n-1$. Therefore,

$$x_\alpha^T\Lambda_i(B_i\ A_iB_i\ ...\ A_i^{n-1}B_i)=0,$$

which implies that the rank condition (16) fails. This contradiction proves that the impulsive system (1) is controllable on $[t_0,t_f]$ ($t_f\in(t_{k-1},t_k]$). □

If $\mathbb{T}=\mathbb{R}$, then we obtain the result of Theorem 2 in [47]. If $A_k(t)=A(t)$, $B_k(t)=B(t)$, then we obtain the Theorem 2 in [36], and the Theorem 3.2 in [23] if $\mathbb{T}=\mathbb{R}$. The version for non impulsive case ($c_i=-1$) of the above theorem can be found in [8, Corollary 3], [21, Theorem 2.7] and [25, Theoem 3.3].

Example 1. Consider the following impulsive system on a time scale $\mathbb{T}$:

$$\begin{cases} x^\triangle(t)=A_k(t)x(t)+B_k(t)u(t),\ t\in[t_{k-1},t_k), \\ x(t_k^+)=\frac{1}{2}x(t_k),\ t=t_k\ :\ k=1,2,3, \\ x(0)=x_0, \end{cases} \tag{21}$$

where

$$A_1 = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix},\ B_1 = \begin{bmatrix} e_3(\sigma(t),0) \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ e_3(\sigma(t),\frac{1}{2}) \end{bmatrix} \tag{22}$$

$$A_3 = \begin{bmatrix} -3 & -2 \\ 3 & 4 \end{bmatrix},\ B_3 = \begin{bmatrix} 0 \\ e_{-2}(\sigma(t),\frac{5}{2}) \end{bmatrix}.$$

Then the exponential matrices corresponding to A_1, A_2, A_3 are given by

$$\begin{aligned} e_{A_1}(0,\sigma(t)) &= \begin{bmatrix} -e_2(0,\sigma(t)) & 0 \\ e_3(0,\sigma(t)) & e_3(0,\sigma(t)) \end{bmatrix} \\ e_{A_2}(0,\sigma(t)) &= \begin{bmatrix} e_1(0,\sigma(t)) & -e_1(0,\sigma(t)) \\ 0 & e_3(0,\sigma(t)) \end{bmatrix} \\ e_{A_3}(0,\sigma(t)) &= \begin{bmatrix} \frac{3}{5}e_{-2}(0,\sigma(t)) & \frac{1}{5}e_{-2}(0,\sigma(t)) \\ -\frac{1}{5}e_3(0,\sigma(t)) & -\frac{2}{5}e_3(0,\sigma(t)) \end{bmatrix} \end{aligned}$$

respectively. We have to compute the following matrices

$$G_i := G(0,t_{i-1},t_i) = \int_{t_{i-1}}^{t_i} \Psi_i(t_0,\sigma(\tau))B_i(\tau)B_i^T(\tau)\Psi_i^T(t_0,\sigma(\tau))\triangle\tau, \tag{23}$$

where

$$\Psi_1(0,\sigma(t)) = e_{A_1}(0,\sigma(t)) \quad t \in (0,t_1],$$

and

$$\Psi_i(0,\sigma(t)) = \prod_{j=1}^{i-1} e_{A_j}(t_{j-1},t_j)e_{A_i}(t_{i-1,}\sigma(t)) \quad t \in (t_{i-1},t_i],\ i=2,3.$$

If $\mathbb{T}=\mathbb{R}$ then $\sigma(t)=t$, $\mu(t)=0$ and $e_p(t,\tau)=e^{p(t-\tau)}$. Next, if we choose $t_k=\frac{4k-3}{2}$, $k=1,2,3$, then we have

$$\Psi_1(0,t)B_1(t)B_1^T(t)\Psi_1^T(0,t) = \begin{pmatrix} e^{2t} & -e^t \\ -e^t & 1 \end{pmatrix}, \tag{24}$$

$$\Psi_2(0,t)B_2(t)B_2^T(t)\Psi_2^T(0,t) = \begin{pmatrix} e^{4t-4} & e^{2t-7/2}-e^{4t-9/2} \\ e^{2t-7/2}-e^{4t-9/2} & e^{4t-5}-2e^{2t-4}+e^{-3} \end{pmatrix}, \tag{25}$$

and

$$\Psi_3(0,t)B_3(t)B_3^T(t)\Psi_3^T(0,t) = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{26}$$

where

$$\begin{aligned}
a &= \frac{1}{25}(4e^{19-10t} + 4e^{13/2-5t} + e^{-6}) \\
b &= \frac{1}{25}(2e^{2-5t} - 4e^{6-5t} + 4e^{29/2-10t} - 4e^{37/2-10t} - e^{-13/2}) \\
c &= \frac{1}{25}(4e^{10-10t} - 8e^{14-10t} + 4e^{18-10t} - 4e^{3/2-5t} + 4e^{11/2-5t} + e^{-7}).
\end{aligned}$$

Substituting (24), (25) and (26) in (23), we obtain

$$G_1 = \begin{pmatrix} \frac{1}{2}e - \frac{1}{2} & 1 - e^{1/2} \\ 1 - e^{1/2} & \frac{1}{2} \end{pmatrix},$$

$$G_2 = \begin{pmatrix} \frac{1}{4}e^6 - \frac{1}{4}e^{-2} & \frac{1}{2}e^{3/2} - \frac{1}{4}e^{-5/2} - \frac{1}{4}e^{11/2} \\ \frac{1}{2}e^{3/2} - \frac{1}{4}e^{-5/2} - \frac{1}{4}e^{11/2} & \frac{11}{4}e^{-3} - e + \frac{1}{4}e^5 \end{pmatrix},$$

and

$$G_3 = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

where

$$\begin{aligned}
a &= \frac{2}{125}(8e^{-6} - 2e^{-16} - e^{-26}) \\
b &= \frac{2}{125}(2e^{-21/2} - 8e^{-13/2} + 2e^{-33/2} - e^{-41/2} + e^{-53/2} - e^{-61/2}) \\
c &= \frac{2}{125}(8e^{-7} - 4e^{-11} + e^{-15} - 2e^{-17} + 2e^{-21} - e^{-27} + 2e^{-31} - e^{-35}).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\det G_3 &\approx 1.371\,2 \times 10^{-12} \\
\det G_2 &\approx 5.051\,8 \\
\det G_1 &\approx 8.7324 \times 10^{-3}.
\end{aligned}$$

It follows that $\text{rank}(G_i) = 2$, $i = 1, 2, 3$.

Further, if we choose $\mathbb{T} = \mathbb{P}_{1,1} = \bigcup_{k=0}^{\infty}[2k, 2k+1]$, then $e_p(t,t_0) = (1+p)^j e^{p(t-t_0)} e^{-pj}$ for $t_0 \in [2i, 2i+1)$, $t \in [2(i+j), 2(i+j)+1]$ with $j \geq 0$. In this case, $\mu(t) = 0$ if $t \in \bigcup_{k=0}^{\infty}[2k, 2k+1)$ and $\mu(t) = 1$ if $t \in \bigcup_{k=0}^{\infty}\{2k+1\}$. Then it follows that

$$\Psi_1(0,t)B_1(t)B_1^T(t)\Psi_1^T(0,t) = \begin{pmatrix} e^{2t} & -e^t \\ -e^t & 1 \end{pmatrix} t \in (0, \frac{1}{2}], \tag{27}$$

$$\Psi_2(0,t)B_2(t)B_2^T(t)\Psi_2^T(0,t) =$$

$$= \begin{cases} \begin{pmatrix} e^{4t-4} & e^{2t-7/2} - e^{4t-9/2} \\ e^{2t-7/2} - e^{4t-9/2} & e^{4t-5} - 2e^{2t-4} + e^{-3} \end{pmatrix}, & t \in (\frac{1}{2}, 1] \\ \begin{pmatrix} 4e^{4t-8} & 2e^{2t-11/2} - 4e^{4t-17/2} \\ 2e^{2t-11/2} - 4e^{4t-17/2} & 4e^{4t-9} - 4e^{2t-6} + e^{-3} \end{pmatrix}, & t \in [2, \frac{5}{2}], \end{cases} \tag{28}$$

and

$$\Psi_3(0,t)B_3(t)B_3^T(t)\Psi_3^T(0,t) = \begin{cases} \begin{pmatrix} a & b \\ b & c \end{pmatrix}, & t \in (\frac{5}{2}, 3] \\ \begin{pmatrix} d & e \\ e & f \end{pmatrix}, & t \in [4, \frac{9}{2}] \end{cases} \tag{29}$$

where

$$\begin{aligned} a &= \frac{1}{25}(e^{21-10t} - e^{17/2-5t} + \frac{1}{4}e^{-4}) \\ b &= \frac{1}{25}(\frac{1}{4}e^{6-5t} + e^{8-5t} - \frac{1}{2}e^{37/2-10t} - e^{41/2-10t} - \frac{1}{4}e^{-9/2}) \\ c &= \frac{1}{25}(\frac{1}{4}e^{16-10t} + e^{18-10t} + e^{20-10t} - \frac{1}{2}e^{11/2-5t} - e^{15/2-5t} + \frac{1}{4}e^{-5}) \\ d &= \frac{1}{100}(\frac{1}{9}e^{31-10t} - \frac{2}{3}e^{27/2-5t} + e^{-4}) \\ e &= \frac{1}{100}(\frac{1}{6}e^{11-5t} + \frac{2}{3}e^{13-5t} - \frac{1}{18}e^{57/2-10t} - \frac{1}{9}e^{61/2-10t} - e^{-9/2}) \\ f &= \frac{1}{100}(\frac{1}{36}e^{26-10t} + \frac{1}{9}e^{28-10t} + \frac{1}{9}e^{30-10t} - \frac{1}{3}e^{21/2-5t} - \frac{2}{3}e^{25/2-5t} + e^{-5}). \end{aligned}$$

Substituting (27), (28) and (29) in (23) we obtain

$$G_1 = \begin{pmatrix} \frac{1}{2}e - \frac{1}{2} & 1 - e^{1/2} \\ 1 - e^{1/2} & \frac{1}{2} \end{pmatrix},$$

$$G_2 = \begin{pmatrix} e^2 - \frac{1}{4}e^{-2} - \frac{3}{4} & \frac{7}{4}e^{-1/2} - \frac{1}{2}e^{-3/2} - e^{3/2} - \frac{1}{4}e^{-5/2} \\ \frac{7}{4}e^{-1/2} - \frac{1}{2}e^{-3/2} - e^{3/2} - \frac{1}{4}e^{-5/2} & e - \frac{11}{4}e^{-1} + e^{-2} + \frac{7}{4}e^{-3} \end{pmatrix},$$

and

$$G_3 = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

where

$$\begin{aligned} a &= -\frac{1}{9000}(-54e^{-4} + 23e^{-9} + e^{-14} - 60e^{-13/2}) \\ b &= -\frac{1}{18000}(120e^{-7} + 30e^{-9} + 108e^{-9/2} - 46e^{-19/2} - 29e^{-23/2} - \\ &\quad -2e^{-29/2} - e^{-33/2}) \\ c &= \frac{1}{36000}(216e^{-5} + 36e^{-9} - 92e^{-10} - 116e^{-12} - 35e^{-14} - 4e^{-15} - 4e^{-17} \\ &\quad -e^{-19} + 240e^{-15/2} + 120e^{-19/2}). \end{aligned}$$

Then

$$\begin{aligned} \det G_3 &\approx 1.4581 \times 10^{-11} \\ \det G_2 &\approx 0.12274 \\ \det G_1 &\approx 8.7324 \times 10^{-3}. \end{aligned}$$

It follows that $\operatorname{rank}(G_i) = 2$, $i = 1, 2, 3$. Therefore, the impulsive system (21) is controllable in the both cases.

4 Observability

Consider the following impulsive dynamical system

$$\begin{cases} x^{\Delta} = A_k(t)x + B_k(t)u, \ \ t \in [t_{k-1}, t_k), \\ x(t_k^+) = (1 + c_k)x(t_k), \ \ k = 1, 2, ..., \\ y(t) = C_k(t)x + D_k(t)u, \\ x(t_0) = x_0, \end{cases} \tag{30}$$

where $\mathbb{T}$ is a unbounded above time scale, $[t_{k-1}, t_k) \subset \mathbb{T}_0 := [t_0, \infty) \cap \mathbb{T}$, $t_k \in \mathbb{T}_0$ are right-dense, $0 \leq t_0 < t_1 < t_2 < ... < t_k < ...$, such that $\lim_{k\to\infty} t_k = \infty$, $x(t_k^+) := \lim_{h\to 0^+} x(t_k + h)$, $x(t_k^-) := \lim_{h\to 0^+} x(t_k - h)$ and $c_k \in \mathbb{R}$ are constants. Also, we assume that $A_k \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_n(\mathbb{R}))$, $B_k \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_{n\times m}(\mathbb{R}))$, $C_k \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_{p\times n}(\mathbb{R}))$, $D_k \in C_{rd}\mathcal{R}(\mathbb{T}_0, M_{p\times m}(\mathbb{R}))$, $x \in \mathbb{R}^n$ is the sate variable, $u \in \mathbb{R}^m$ is the control input, and $y \in \mathbb{R}^p$ is the output.

Definition 2. *The impulsive system (30) is called state observable on* $[t_0, t_f]$ $(t_f > t_0)$ *if any initial state* $x(t_0) = x_0 \in R^n$ *is uniquely determined by the corresponding system input* $u(t)$ *and system output* $y(t)$ *for* $t \in [t_0, t_f]$.

Theorem 3. *Assume that* $1 + c_i \geq 0$, $i = 1, 2, ..., k-1$. *Then the impulsive system (30) is observable on* $[t_0, t_f](t_f \in (t_{k-1}, t_k])$ *if and only if the matrix*

$$M(t_0, t_f) := M(t_0, t_0, t_1) + \sum_{i=2}^{k-1} \prod_{j=1}^{i-1} (1+c_j) M(t_0, t_{i-1}, t_i) + \prod_{j=1}^{k-1} (1+c_j) M(t_0, t_{k-1}, t_f)$$

is invertible, where

$$M(t_0, t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} \Omega_i^T(\tau, t_0) C_i^T(\tau) C_i(\tau) \Omega_i(\tau, t_0) \Delta\tau,\ i = 1, 2, ..., k-1,$$

$$M(t_0, t_{k-1}, t_f) = \int_{t_{k-1}}^{t_f} \Omega_k^T(\tau, t_0) C_k^T(\tau) C_k(\tau) \Omega_k(\tau, t_0) \Delta\tau,$$

and

$$\Omega_i(\tau, t_0) = \Phi_{A_i}(\tau, t_{i-1}) \Phi_{A_{i-1}}(t_{i-1}, t_{i-2}) ... \Phi_{A_1}(t_1, t_0)$$

for $\tau \in (t_{i-1}, t_i]$ *and* $i = 1, 2, ..., k$.

Proof. Suppose that $M(t_0, t_f)$ is invertible. From (5) and (30) we obtain

$$y(t) = C_1(t)\Phi_{A_1}(t, t_0)x_0 + C_1(t) \int_{t_0}^{t} \Phi_{A_1}(t, \sigma(\tau)) B_1(\tau) u(\tau) \Delta\tau + D_1(t) u(t) \tag{31}$$

for $t \in [t_0, t_1]$ and

$$y(t) = C_l(t)x(t) + D_l(t)u(t)$$

$$= C_l(t)\Phi_{A_l}(t, t_{l-1}) \prod_{i=1}^{l-1}(1 + c_i) \prod_{i=l-1}^{1} \Phi_{A_i}(t_i, t_{i-1})x_0+$$

$$C_l(t) \sum_{i=1}^{l-1} \left[\prod_{j=i}^{l-1}(1 + c_j) \int_{t_{i-1}}^{t_i} \Phi_{A_l}(t, \sigma(\tau)) \prod_{r=l-1}^{i+1} \Phi_{A_r}(t_r, t_{r-1})\Phi_{A_i}(t_i, \sigma(\tau))B_i(\tau)u(\tau)\Delta\tau\right]$$

$$+C_l(t) \int_{t_{l-1}}^{t} \Phi_{A_l}(t, \sigma(\tau))B_l(\tau)u(\tau)\Delta\tau + D_l(t)u(t), \tag{32}$$

for $t \in (t_{l-1}, t_l]$, $l = 2, 3, ...k$. It is easy to see from the Definition 2 that the observability of system (30) is equivalent to the observability of $y(t)$ given by

$$y(t) = \begin{cases} C_1(t)\Phi_{A_1}(t, t_0)x_0, & t \in [t_0, t_1] \\ \prod_{i=1}^{l-1}(1 + c_i)C_l(t)\Omega_l(t, t_0)x_0, & t \in (t_{l-1}, t_l],\ l = 1, 2, ..., k, \end{cases} \tag{33}$$

as $u(t) = 0$. Now, multiplying $\Omega_l^T(t, t_0)C_l^T(t)$ to both sides of (33) and integrating with respect to t from t_0 to t_f, we have

$$\int_{t_0}^{t_f} \Omega_l^T(\tau, t_0)C_l^T(\tau)y(\tau)\Delta\tau = \left[\int_{t_0}^{t1} \Phi_{A_1}^T(\tau, t_0)C_1^T(\tau)C_1(\tau)\Phi_{A_1}(\tau, t_0)\Delta\tau\right.$$

$$+ \sum_{i=2}^{k-1} \prod_{j=1}^{i-1}(1 + c_j) \int_{t_{i-1}}^{t_i} \Omega_i^T(\tau, t_0)C_i^T(\tau)C_l(\tau)\ \Omega_i(\tau, t_0)\Delta\tau$$

$$\left.+ \prod_{j=1}^{k-1}(1 + c_j) \int_{t_{k-1}}^{t_f} \Omega_k^T(\tau, t_0)C_k^T(\tau)C_k(\tau)\Omega_k(\tau, t_0)\Delta\tau\right] x_0$$

and so,

$$\int_{t_0}^{t_f} \Omega_l^T(\tau,t_0)C_l^T(\tau)y(\tau)\Delta\tau$$

$$= [M(t_0,t_0,t_1) + \sum_{i=2}^{k-1}\prod_{j=1}^{i-1}(1+c_j)M(t_0,t_{i-1},t_i) \tag{34}$$

$$+ \prod_{j=1}^{k-1}(1+c_j)M(t_0,t_{k-1},t_f)]x_0.$$

Obviously, the left-hand side of (34) depend on $y(t),\ t \in [t_0,t_f]$. Since the matrix $M(t_0,t_f)$ is invertible, so from linear algebraic equation (34) we deduce that $x(t_0) = x_0$ is uniquely determined by the corresponding system output $y(t)$ for $t \in [t_0,t_f]$.

Conversely, if we suppose that the matrix $M(t_0,t_f)$ is not invertible, then there exist nonzero $x_\alpha \in \mathbb{R}^n$ such that $x_\alpha^T M(t_0,t_f)x_\alpha = 0$. Since $1+c_i \geq 0$, $i = 1,2.....,k$, $M(t_0,t_{i-1},t_i)$, $i = 1,2,....k-1$, and $M(t_0,t_{k-1},t_f)$ are positive semidefinite matrices, we have

$$\begin{aligned} &x_\alpha^T M(t_0,t_{i-1},t_i)x_\alpha = 0,\ i = 0,1,...,k-1 \\ &x_\alpha^T M(t_0,t_{k-1},t_f)x_\alpha = 0. \end{aligned} \tag{35}$$

Choose $x_0 = x_\alpha$. Then, from (33) and (35), it follows that

$$\int_{t_0}^{t_f} y^T(\tau)y(\tau)\Delta\tau = \sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} y^T(\tau)y(\tau)\Delta\tau + \int_{t_{k-1}}^{t_f} y^T(\tau)y(\tau)\Delta\tau$$

$$= \int_{t_0}^{t_1} x_\alpha^T \Phi_{A_1}^T(\tau,t_0)C_1^T(\tau)C_1(\tau)\Phi_{A_1}(\tau,t_0)x_\alpha\Delta\tau$$

$$+ \sum_{i=2}^{k-1}\left[\prod_{j=1}^{i-1}(1+c_j)\right]^2 \int_{t_{i-1}}^{t_i} x_\alpha^T\Omega_i^T(\tau,t_0)C_i^T(\tau)C_i(\tau)\Omega_i(\tau,t_0)x_\alpha\Delta\tau$$

$$+ \left[\prod_{j=1}^{k-1}(1+c_j)\right]^2 \int_{t_{k-1}}^{t_f} x_\alpha^T\Omega_k^T(\tau,t_0)C_k^T(\tau)C_k(\tau)\Omega_k(\tau,t_0)x_\alpha\Delta\tau.$$

Further, we have

$$\int_{t_0}^{t_f} y^T(\tau)y(\tau)\Delta\tau$$

$$= x_\alpha^T M(t_0,t_0,t_1)x_\alpha + \sum_{i=1}^{k-1}\left[\prod_{j=1}^{i}(1+c_j)\right]^2 x_\alpha^T M(t_0,t_{i-1},t_i)x_\alpha$$

$$+ \left[\prod_{j=1}^{k-1}(1+c_j)\right]^2 x_\alpha^T M(t_0,t_{k-1},t_f)x_\alpha = 0$$

and so,

$$\int_{t_0}^{t_f} \|y(\tau)\|^2 \Delta\tau = 0.$$

It follows that

$$0 = y(t) = \begin{cases} C_1(t)\Phi_{A_1}(t,t_0)x_0, \quad t\in[t_0,t_1], \\ \prod_{j=1}^{l-1}(1+c_j)C_l(t)\Omega_l(t,t_0)x_\alpha, \quad t\in(t_{l-1},t_l],\ l=1,2,...k-1, \\ \prod_{j=1}^{k-1}(1+c_j)C_k(t)\Omega_k(t,t_0)x_\alpha, \quad t\in(t_{k-1},t_f]. \end{cases}$$

The last equality implies, by Definition 2, that the impulsive system is not observable on $[t_0,t_f]$ $(t_f\in(t_{k-1},t_k])$. $\square$

If $\mathbb{T}=\mathbb{R}$, then we obtain the result of Theorem 3 in [47]. If $A_k(t)=A(t)$, $B_k(t)=B(t)$, then we obtain the Theorem 3 in [36], and the Theorem 3.3 in [23] if $\mathbb{T}=\mathbb{R}$. The version of non impulsive case on time scales $(c_i=-1)$ can be found in [21, Theorem 3.2] and [25, Theorem 3.7].

In the following, we consider the sufficient and necessary criterion for time-invariant case. For impulsive system (30), we denote

$$S = \begin{bmatrix} V_1 \\ \cdot \\ \cdot \\ \cdot \\ V_k \end{bmatrix} \text{ and } V_i = \begin{bmatrix} C_i \\ C_iA_i \\ \cdot \\ \cdot \\ \cdot \\ C_iA_i^{n-1} \end{bmatrix} \Upsilon_i \tag{36}$$

where $\Upsilon_i = e_{A_i}(t_i, t_{i-1})...e_{A_2}(t_2, t_1)e_{A_1}(t_1, t_0)$ if $i = 1, 2, ...k$.

Theorem 4. *Assume that $1 + c_i \geq 0$, $i = 1, 2, ..., k$ and $A_k(t) = A_k$, $C_k(t) = C_k$ are constant matrices. Then impulsive system (30) is observable on $[t_0, t_f]$ ($t_f \in (t_{k-1}, t_k]$) if and only if $rank(S) = n$.*

Proof. Suppose $rank(S) = n$ and we have to show that system (30) is observable on $[t_0, t_f](t_f \in (t_{k-1}, t_k])$. If otherwise, namely, system(30) is not observable then, by Theorem 3, it follows that the matrix $M(t_0, t_f)$ is not invertible. Hence there exists a nonzero vector x_α such that $x_\alpha^T M(t_0, t_f)x_\alpha = 0$. Similar to the proof of Theorem 3, we obtain

$$x_\alpha^T M(t_0, t_{i-1}, t_i)x_\alpha = \int_{t_{i-1}}^{t_i} [x_\alpha^T \Omega_i^T(\tau, t_0)C_i^T][C_i\Omega_i(\tau, t_0)x_\alpha]\Delta\tau$$

$$= \int_{t_{i-1}}^{t_i} [C_i\Omega_i(\tau, t_0)x_\alpha]^T[C_i\Omega_i(\tau, t_0)x_\alpha]\Delta\tau = 0, \;\; i = 1, 2, ..., k-1$$

and

$$x_\alpha^T M(t_0, t_{k-1}, t_f)x_\alpha = \int_{t_{k-1}}^{t_f} [C_i\Omega_k(\tau, t_0)x_\alpha]^T[C_i\Omega_k(\tau, t_0)x_\alpha]\Delta\tau = 0.$$

Since $\Omega_i(\tau, t_0) = e_{A_i}(\tau, t_{i-1})...e_{A_2}(t_2, t_1)e_{A_1}(t_1, t_0)$ for $i = 1, 2, ..., k$, it follows that

$$C_ie_{A_i}(\tau, t_{i-1})...e_{A_2}(t_2, t_1)e_{A_1}(t_1, t_0)x_\alpha = 0 \tag{37}$$

for $\tau \in (t_{i-1}, t_i]$, $i = 1, 2, ..., k-1$, and

$$C_ke_{A_k}(\tau, t_{k-1})...e_{A_2}(t_2, t_1)e_{A_1}(t_1, t_0)x_\alpha = 0 \tag{38}$$

for $\tau \in (t_{k-1}, t_f]$. Obviously, at $\tau = t_{i-1}$, we have $C_i\Upsilon_{i-1}x_\alpha = 0$, $i = 1, 2, ..., k$, and differentiating in (37) and (38) j times and evaluating the result at $\tau = t_{i-1}$, $i = 1, 2, ..., k$, we obtain

$$C_iA_i^j\Upsilon_{i-1}x_\alpha = 0, \;\; i = 1, 2, ..., k, \;\; j = 0, 1, 2, ...n-1. \tag{39}$$

Therefore, by (36) and (39) it follows that $Sx_\alpha = 0$ and moreover, $x_\alpha \neq 0$ implies that $rank(S) < n$ which leads to a contradiction with the assumption that $rank(S) = n$. The proof of the sufficiency part is finished.

Conversely, we suppose that $rank(S) < n$. Then there exist $x_\alpha \neq 0$ such that $Sx_\alpha = 0$, which leads to (39). By (4) and (39) we have

$$M(t_0, t_{i-1}, t_i)x_\alpha = \int_{t_{i-1}}^{t_i} \sum_{j=0}^{n-1} \gamma_{ij}(\tau, t_{i-1})[C_i\Omega_i(\tau, t_0)]^T[C_iA_i^j\Upsilon_{i-1}]x_\alpha\Delta\tau = 0$$

for $i = 1, 2, ..., k-1$, and

$$M(t_0, t_{k-1}, t_f)x_\alpha = \int_{t_{k-1}}^{t_f} \sum_{j=0}^{n-1} \gamma_{ij}(\tau, t_0)[C_k\Omega_k(\tau, t_0)]^T[C_kA_k^j\Upsilon_{k-1}]x_\alpha\Delta\tau = 0,$$

and so, by (39), we obtain $M(t_0, t_f)x_\alpha = 0$. Since $x_\alpha \neq 0$ the matrix $M(t_0, t_f)x_\alpha$ is not invertible. Hence system (30) is not observable from Theorem 3, and it contradicts with the assumption of observability. The proof is completed. □

If $\mathbb{T} = \mathbb{R}$, then we obtain the result of Theorem 4 in [47]. If $A_k(t) = A(t)$, $B_k(t) = B(t)$, then we obtain the Theorem 4 in [36], and the Theorem 3.4 in [23] if $\mathbb{T} = \mathbb{R}$. The version of non impulsive case on time scales ($c_i = -1$) can be found in [8, Theorem 4], [21, Theorem 3.7] and [25, Theorem 3.9].

Example 2. Consider the following impulsive system on a time scale $\mathbb{T}$:

$$\begin{cases} x^\triangle(t) = A_k(t)x(t) + B_k(t)u(t), \quad t \in [t_{k-1}, t_k), \\ x(t_k^+) = \frac{1}{2}x(t_k),\ k = 1, 2, 3, \\ y(t) = C_k(t)x(t) + D_k(t)u(t), \\ x(0) = x_0, \end{cases} \tag{40}$$

where

$$A_1 = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & e_{-3}(0, t) \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & e_3(\frac{1}{2}, t) \end{bmatrix} \tag{41}$$

$$A_3 = \begin{bmatrix} -3 & -2 \\ 3 & 4 \end{bmatrix}, C_3 = \begin{bmatrix} 0 & e_3(\frac{5}{2}, t) \end{bmatrix}.$$

Then the exponential matrices corresponding to A_1, A_2, A_3 are given by

$$\begin{aligned} e_{A_1}(t,t_0) &= \begin{bmatrix} -e_2(t,0) & 0 \\ e_2(t,0) & e_3(t,0) \end{bmatrix} \\ e_{A_2}(t,t_0) &= \begin{bmatrix} e_1(t,0) & e_3(t,0) \\ 0 & e_3(t,0) \end{bmatrix} \\ e_{A_3}(t,t_0) &= \begin{bmatrix} 2e_{-2}(t,0) & e_3(t,0) \\ -e_{-2}(t,0) & -3e_3(t,0) \end{bmatrix} \end{aligned}$$

respectively. We have to compute the following matrix

$$M(0,\frac{9}{2}) := M(0,0,\frac{1}{2}) + \frac{1}{2}M(0,\frac{1}{2},\frac{5}{2}) + \frac{1}{4}M(0,\frac{5}{2},\frac{9}{2}),$$

where

$$\begin{aligned} M(0,0,\tfrac{1}{2}) &= \int_0^{1/2} \Omega_1^T(\tau,0)C_1^T(\tau)C_1(\tau)\ \Omega_1(\tau,0)\Delta\tau \\ M(0,\tfrac{1}{2},\tfrac{5}{2}) &= \int_{1/2}^{5/2} \Omega_2^T(\tau,0)C_2^T(\tau)C_2(\tau)\ \Omega_2(\tau,0)\Delta\tau \qquad (42) \\ M(0,\tfrac{5}{2},\tfrac{9}{2}) &= \int_{5/2}^{9/2} \Omega_3^T(\tau,0)C_3^T(\tau)C_3(\tau)\ \Omega_3(\tau,0)\Delta\tau, \end{aligned}$$

and

$$\Omega_i(s,0) = \Phi_{A_i}(s,t_{i-1})\Phi_{A_{i-1}}(t_{i-1},t_{i-2})...\Phi_{A_1}(t_1,0),\ s \in (t_{i-1},t_i],\quad i = 1,2,3.$$

If $\mathbb{T} = \mathbb{R}$ then

$$M(0,0,\tfrac{1}{2}) = \begin{pmatrix} -\frac{1}{10}\left(-e^5+1\right) & -\frac{1}{11}\left(-e^{11/2}+1\right) \\ -\frac{1}{11}\left(-e^{11/2}+1\right) & -\frac{1}{12}\left(-e^6+1\right) \end{pmatrix},$$

$$M(0,\tfrac{1}{2},\tfrac{5}{2}) = \begin{pmatrix} 2e^2 & 2e^{5/2} \\ 2e^{5/2} & 2e^3 \end{pmatrix}$$

and

$$M(0,\tfrac{5}{2},\tfrac{9}{2})=\begin{pmatrix} a_1 & a_2\\ a_2 & a_3\end{pmatrix}$$

where

$$\begin{aligned}
a_1 &= -\frac{1}{10}\left(12e^{4}+e^{-6}-e^{6}-2e^{-10}+14e^{10}+e^{-14}-193e^{14}-12\right)\\
a_2 &= -\frac{1}{10}\left(-6e^{1/2}+12e^{9/2}+e^{-11/2}-e^{-19/2}+7e^{21/2}-193e^{29/2}\right)\\
a_3 &= \frac{1}{10}e^{-5}\left(-12e^{10}+193e^{20}-1\right).
\end{aligned}$$

We obtain

$$\det M(0,\frac{9}{2})\approx -1.7799\times 10^{9}.$$

Further, if $\mathbb{T}=\mathbb{P}_{1,1}=\bigcup\limits_{k=0}^{\infty}[2k,2k+1]$, then

$$M(0,0,\tfrac{1}{2})=\begin{pmatrix} -\frac{1}{10}\left(-e^{5}+1\right) & -\frac{1}{11}\left(-e^{11/2}+1\right)\\ -\frac{1}{11}\left(-e^{11/2}+1\right) & -\frac{1}{12}\left(-e^{6}+1\right)\end{pmatrix},$$

$$M(0,\tfrac{1}{2},\tfrac{5}{2})=\begin{pmatrix} e^{2} & e^{5/2}\\ e^{5/2} & e^{3}\end{pmatrix}$$

and

$$M(0,\frac{5}{2},\frac{9}{2})=\begin{pmatrix} a_1 & a_2\\ a_2 & a_3\end{pmatrix}$$

where

$$
\begin{aligned}
a_1 &= \frac{33}{10}e^3 - \frac{9}{10}e - \frac{1}{10}e^{-2} - \frac{3}{8}e^{-1} + \frac{1}{10}e^{-4} + \frac{2}{5}e^4 - \frac{1}{40}e^{-6} - \frac{56}{5}e^6 \\
&\quad + \frac{824}{5}e^8 + 12e^{7/2} - 24e^{11/2} \\
a_2 &= 6e^4 - 24e^6 - \frac{1}{10}e^{-\frac{3}{2}} - \frac{9}{20}e^{\frac{3}{2}} + \frac{1}{20}e^{-7/2} + \frac{33}{10}e^{7/2} - \\
&\quad - \frac{28}{5}e^{13/2} + \frac{824}{5}e^{17/2} \\
a_3 &= \frac{33}{10}e^4 - \frac{1}{10}e^{-1} + \frac{824}{5}e^9 - 24e^{13/2}.
\end{aligned}
$$

We obtain

$$\det M(0, \frac{9}{2}) \approx -9.4 \times 10^5.$$

Therefore, the system (40) is observable in the both cases.

5 Applications

5.1. Consider the following application to population growth model with impulse

$$
\begin{cases}
N^{\triangle}(t) = r_k N(t) + c_k U(t), & t \neq t_k, \\
N(t_k^+) = (r_{k+1} - r_k) N(t_k), & t = t_k, \\
N(0) = N_0,
\end{cases}
$$

where $N(t)$ is the number of population at the time t, r_k is the rate of population growth between two consecutive impulsive points and $U(t)$ is a control input. Such model can be describe the evaluation of cicada *magicicada septendecim*. In this case is need to consider the time scale $\mathbb{T} = \mathbb{P}_{1,1}$ (see [13, Example 1.39]) Using the Theorem 2 it is easy to see that the system is controllable.

5.2. Next application is a impulsive model in Nonelectronic [44, Example

11.1.1], that is

$$\begin{cases} \theta^{\triangle}(t) = -\frac{\gamma}{\pi}\theta(t) + \gamma(a - b\cos t), & t \neq t_k, \\ \theta(t_k^+) = -3\pi, & t = t_k, \\ \theta(0) = \theta_0, \\ |\theta(0)| < \pi. \end{cases}$$

Using the Theorem 2, with $A = -\frac{\gamma}{\pi}$, $B = \gamma$ and $n = 1$, it is easy to see that the system is controllable if $\gamma \neq 0$ and $\gamma \neq \pi$. The controllability of this system is independent of the choice of the time scale $\mathbb{T}$.

6 Conclusion

In this paper, the issue on the controllability and observability criteria for linear impulsive time-varying systems on time scales has been addressed. Several sufficient and necessary criteria for state controllability and observability of such systems have been established, respectively, by the variation of parameters for time-varying impulsive systems on time scales. In addition, two examples and two applications have been presented to show the effectiveness of proposed results. As it has been shown that a larger class of systems are considered, the results generalize some known results in [8, 21, 23, 25, 36, 47].

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