



On quasi-periodic solutions of forced higher order nonlinear difference equations

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Abstract. Consider the following higher order difference equation

$$x(n+1) = f(n, x(n)) + g(n, x(n-k)) + b(n), \quad n = 0, 1, \dots$$

where $f(n, x), g(n, x) : \{0, 1, \dots\} \times [0, \infty) \rightarrow [0, \infty)$ are continuous functions in x and periodic functions with period ω in n , $\{b(n)\}$ is a real sequence, and k is a nonnegative integer. We show that under proper conditions, every nonnegative solution of the equation is quasi-periodic with period ω . Applications to some other difference equations derived from mathematical biology are also given.

Keywords: difference equations, quasi-periodic solutions, population models.

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1 Introduction


Consider the following nonlinear difference equation of order $k+1$ with forcing term $b(n)$

$$x(n+1) = f(n, x(n)) + g(n, x(n-k)) + b(n), \quad n = 0, 1, \dots \quad (1.1)$$

where $f(n, x), g(n, x) : \{0, 1, \dots\} \times [0, \infty) \rightarrow [0, \infty)$ are continuous functions in x and periodic functions with period ω in n , $\{b(n)\}$ is a real sequence, and k is a nonnegative integer. Our aim in the paper is to study the quasi-periodicity of solutions of Eq. (1.1) in the sense that

Definition 1.1. We say that a solution $\{x(n)\}$ of Eq. (1.1) is quasi-periodic with period ω if there exist sequences $\{p(n)\}$ and $\{q(n)\}$ such that $\{p(n)\}$ is periodic with period ω and $\{q(n)\}$ converges to zero as $n \rightarrow \infty$ and $x(n) = p(n) + q(n)$, $n = 0, 1, \dots$

By using, among others, some methods and ideas related to the linear first-order difference equation, in the next section we show that under proper conditions every solution of Eq. (1.1)

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is quasi-periodic with period ω . More specifically, we show that under proper conditions, every solution $\{x(n)\}$ of Eq. (1.1) satisfies

$$\lim_{n \rightarrow \infty} (x(n) - \tilde{y}(n)) = 0$$

where $\{\tilde{y}(n)\}$ is a periodic solution with period ω of the following associated difference equation of Eq. (1.1) without forcing term

$$y(n+1) = f(n, y(n)) + g(n, y(n-k)), \quad n = 0, 1, \dots \quad (1.2)$$

Existence and global attractivity of periodic solutions of Eq. (1.2) and some other forms have been studied by numerous authors, see for example [1, 3, 13, 15–17, 19, 20, 22, 23, 31] and the references cited therein. While there has been much progress made in the study of the existence and global attractivity of periodic solutions of Eq. (1.2), the quasi-periodicity of solutions of Eq. (1.1) is relatively scarce. In order to study this phenomenon, we note the following recent result from [15] for the existence of a periodic solution $\tilde{y}(t)$ of Eq. (1.2) (some new results related to those in [15] have been recently presented in [26]).

Theorem A. Assume that there is a nonnegative periodic sequence $\{a(n)\}$ with period ω such that

$$\hat{a} = \prod_{j=0}^{\omega-1} a(j) < 1 \quad \text{and} \quad f(n, y) \leq a(n)y \quad \text{for } n = 0, 1, \dots, \omega - 1 \text{ and } y \geq 0$$

and that $f(n, y) - a(n)y$ is nonincreasing in y . Suppose also that $g(n, y)$ is nonincreasing in y and that there is a positive constant B such that

$$\sum_{i=n}^{n+\omega-1} \left(\prod_{j=i+1}^{n+\omega-1} a(j) \right) [f(i, B) - a(i)B + g(i, B)] \geq 0, \quad n = 0, 1, \dots, \omega - 1 \quad (1.3)$$

and

$$\frac{1}{1 - \hat{a}} \sum_{i=n}^{n+\omega-1} \left(\prod_{j=i+1}^{n+\omega-1} a(j) \right) g(i, 0) \leq B, \quad n = 0, 1, \dots, \omega - 1. \quad (1.4)$$

Then Eq. (1.2) has a nonnegative periodic solution $\{\tilde{y}(n)\}$ with period ω .

We will make use of this theorem in the next section to guarantee a periodic solution of Eq. (1.2), a prerequisite for the existence of quasi-periodic solutions of Eq. (1.1). In Section 3, we show that our main results may be applied to some difference equations derived from applications.

2 Main results

For the sake of convenience, we adopt the notation $\prod_{i=m}^n \rho(i) = 1$ and $\sum_{i=m}^n \rho(i) = 0$ whenever $\{\rho(n)\}$ is a real sequence and $m > n$ in the following discussion.

The following lemma – which is needed in the proof of our main result – is folklore, and all the ingredients for its proof can be found in some papers dealing with the linear first-order difference equation (see, for example, [18] and [23] and the related references therein). Nevertheless, we will give a proof for the sake of completeness.

Lemma 2.1. Assume that $\{a(n)\}$ is a nonnegative periodic sequence with period ω and $\{b(n)\}$ is a real sequence. If

$$\prod_{i=0}^{\omega-1} a(i) < 1 \quad \text{and} \quad b(n) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (2.1)$$

then

$$\sum_{i=0}^n \left(\prod_{j=i+1}^n a(j) \right) b(i) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.2)$$

Proof. First we show that there is a positive constant A such that

$$\sum_{i=0}^n \left(\prod_{j=i+1}^n a(j) \right) \leq A, \quad n = 0, 1, \dots \quad (2.3)$$

Observe that for any $n \geq 0$, there are nonnegative integers m and l such that

$$n = m\omega + l, \quad 0 \leq l \leq \omega - 1.$$

Then

$$\begin{aligned} \sum_{i=0}^n \left(\prod_{j=i+1}^n a(j) \right) &= \sum_{i=0}^{m\omega+l} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right) \\ &= \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right) + \sum_{i=\omega}^{2\omega-1} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right) + \dots + \sum_{i=(m-1)\omega}^{m\omega-1} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right) \\ &\quad + \sum_{i=m\omega}^{m\omega+l} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right) \\ &= \prod_{j=\omega}^{m\omega+l} a(j) \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) + \prod_{j=2\omega}^{m\omega+l} a(j) \sum_{i=\omega}^{2\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) \\ &\quad + \dots + \prod_{j=m\omega}^{m\omega+l} a(j) \sum_{i=(m-1)\omega}^{m\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) + \sum_{i=0}^l \left(\prod_{j=i+1}^l a(j) \right) \\ &= \prod_{j=\omega}^{m\omega-1} a(j) \prod_{j=m\omega}^{m\omega+l} a(j) \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) + \prod_{j=2\omega}^{m\omega-1} a(j) \prod_{j=m\omega}^{m\omega+l} a(j) \sum_{i=\omega}^{2\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) \\ &\quad + \dots + \prod_{j=m\omega}^{m\omega-1} a(j) \prod_{j=m\omega}^{m\omega+l} a(j) \sum_{i=(m-1)\omega}^{m\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) + \sum_{i=0}^l \left(\prod_{j=i+1}^l a(j) \right) \\ &= \left(\prod_{i=0}^{\omega-1} a(j) \right)^{m-1} \prod_{i=0}^l a(j) \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) \\ &\quad + \left(\prod_{i=0}^{\omega-1} a(j) \right)^{m-2} \prod_{i=0}^l a(j) \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) \\ &\quad + \dots + \prod_{i=0}^l a(j) \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) + \sum_{i=0}^l \left(\prod_{j=i+1}^l a(j) \right) \end{aligned}$$

$$\begin{aligned}
&= \left[\left(\prod_{i=0}^{\omega-1} a(i) \right)^{m-1} + \left(\prod_{i=0}^{\omega-1} a(j) \right)^{m-2} + \cdots + 1 \right] \prod_{i=0}^l a(j) \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) \\
&\quad + \sum_{i=0}^l \left(\prod_{j=i+1}^l a(j) \right) \\
&= \frac{1 - \left(\prod_{j=0}^{\omega-1} a(j) \right)^m}{1 - \prod_{j=0}^{\omega-1} a(j)} \prod_{j=0}^l a(j) \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) + \sum_{i=0}^l \left(\prod_{j=i+1}^l a(j) \right).
\end{aligned}$$

Thus

$$\sum_{i=0}^n \left(\prod_{j=i+1}^n a(j) \right) \leq \frac{\prod_{j=0}^l a(j)}{1 - \prod_{j=0}^{\omega-1} a(j)} \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) + \sum_{i=0}^l \left(\prod_{j=i+1}^l a(j) \right), \quad l = 0, 1, \dots, \omega - 1. \quad (2.4)$$

Let

$$A_1 = \max_{0 \leq l \leq \omega-1} \prod_{j=0}^l a(j), \quad A_2 = \max_{0 \leq l \leq \omega-1} \sum_{i=0}^l \left(\prod_{j=i+1}^l a(j) \right)$$

and

$$A = \frac{A_1}{1 - \prod_{j=0}^{\omega-1} a(j)} \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{\omega-1} a(j) \right) + A_2.$$

Then from (2.4) we see that (2.3) holds. Next, we show that (2.2) holds. Since $b(n) \rightarrow 0$ as $n \rightarrow \infty$, there is a positive constant $C (\geq A)$ such that

$$|b(n)| \leq C, \quad n \geq 0$$

and for each $\epsilon > 0$, there is a positive integer N_1 such that

$$|b(n)| < \frac{\epsilon}{2C}, \quad n > N_1.$$

Hence, by noting (2.3), we see that

$$\sum_{i=N_1+1}^n \left(\prod_{j=i+1}^n a(j) \right) |b(i)| \leq \sum_{i=N_1+1}^n \left(\prod_{j=i+1}^n a(j) \right) \frac{\epsilon}{2C} \leq A \frac{\epsilon}{2C} \leq \epsilon/2, \quad n > N_1.$$

Since for each $t = 1, 2, \dots, N_1 + 1$, $\prod_{j=t}^n a(j) \rightarrow 0$ as $n \rightarrow \infty$, there is a positive integer $N_2 (> N_1)$ such that

$$\prod_{j=t}^n a(j) < \frac{\epsilon}{2(N_1 + 1)C}, \quad n > N_2, \quad t = 1, 2, \dots, N_1 + 1.$$

Hence,

$$\sum_{i=0}^{N_1} \left(\prod_{j=i+1}^n a(j) \right) |b(i)| \leq \sum_{i=0}^{N_1} \left(\prod_{j=i+1}^n a(j) \right) C \leq (N_1 + 1) \frac{\epsilon}{2(N_1 + 1)C} C = \epsilon/2, \quad n > N_2.$$

Then it follows that

$$\begin{aligned} \left| \sum_{i=0}^n \left(\prod_{j=i+1}^n a(j) \right) b(i) \right| &= \left| \sum_{i=0}^{N_1} \left(\prod_{j=i+1}^n a(j) \right) b(i) + \sum_{i=N_1+1}^n \left(\prod_{j=i+1}^n a(j) \right) b(i) \right| \\ &\leq \sum_{i=0}^{N_1} \left(\prod_{j=i+1}^n a(j) \right) |b(i)| + \sum_{i=N_1+1}^n \left(\prod_{j=i+1}^n a(j) \right) |b(i)| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad n > N_2 \end{aligned}$$

which yields (2.2). The proof is complete. \square

Now, consider the linear difference equation

$$u(n+1) = a(n)u(n) + b(n), \quad n = 0, 1, \dots, \quad (2.5)$$

where $\{a(n)\}$ and $\{b(n)\}$ satisfy the hypotheses in Lemma 2.1. Assume that $\{u(n)\}$ is a solution of Eq. (2.5). It is known that the general solution to the equation is

$$u(n+1) = \left(\prod_{j=0}^n a(j) \right) u(0) + \sum_{i=0}^n \left(\prod_{j=i+1}^n a(j) \right) b(i), \quad n = 0, 1, \dots,$$

which is frequently used in the literature (see, e.g., recent papers [21, 23–25], as well as many related references therein, where some applications to ordinary and partial difference equations, as well as many historical facts on the equation and related solvable ones can be found). Clearly, by noting the periodicity of $\{a(n)\}$ and (2.1), we see that

$$\left(\prod_{j=0}^n a(j) \right) u(0) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, the following conclusion comes from Lemma 2.1 immediately.

Corollary 2.2. *Assume that $\{a(n)\}$ and $\{b(n)\}$ satisfy the hypotheses in Lemma 2.1. Then every solution $\{u(n)\}$ of Eq. (2.5) converges to zero as $n \rightarrow \infty$.*

The following corollary is about the difference inequality

$$v(n+1) \leq a(n)v(n) + b(n), \quad n = 0, 1, \dots \quad (2.6)$$

Assume that $\{v(n)\}$ is a nonnegative solution of (2.6). Clearly, $\{v(n)\}$ satisfies

$$0 \leq v(n) \leq u(n), \quad n = 0, 1, \dots$$

where $\{u(n)\}$ is the solution of Eq. (2.5) with $u(0) = v(0)$. Hence, the following conclusion is a direct consequence of Corollary 2.2.

Corollary 2.3. *Assume that $\{a(n)\}$ and $\{b(n)\}$ satisfy the hypotheses in Lemma 2.1. Then every nonnegative solution $\{v(n)\}$ of (2.6) converges to zero as $n \rightarrow \infty$.*

The following lemma is straightforward but will be referenced multiple times in the main result.

Lemma 2.4. Suppose $f(n, x), g(n, x)$ are real functions and that $\{a(n)\}$ is a real sequence, and assume $f(n, x) - a(n)x$ and $g(n, x)$ are nonincreasing. Then for any $y \geq 0$,

$$f(n, x + y) - f(n, x) \leq a(n)y$$

and

$$f(n, x + y) - f(n, x) + g(n, x + y) - g(n, x) \leq a(n)y.$$

Proof. Let $y \geq 0$. As $f(n, x) - a(n)x$ is nonincreasing we have

$$f(n, x + y) - a(n)(x + y) \leq f(n, x) - a(n)x.$$

Thus, $f(n, x + y) - f(n, x) \leq a(n)y$. As $g(n, x)$ is nonincreasing, we see that $g(n, x + y) - g(n, x) \leq 0$. Combining the above inequalities completes the proof. \square

The following theorem is our main result.

Theorem 2.5. Consider Eq. (1.1) and assume that $f(n, x)$ is nondecreasing in x . Suppose that $\{a(n)\}$ is a nonnegative periodic sequence with period ω , and $\{b(n)\}$ is a real sequence such that $\{a(n)\}$ and $\{b(n)\}$ satisfy (2.1), $f(n, x) \leq a(n)x$ and $f(n, x) - a(n)x$ is nonincreasing in x . Suppose also that $g(n, x)$ is nonincreasing in x and there is a positive constant B such that (1.3) and (1.4) are satisfied. Suppose there is a nonnegative sequence $\{L(n)\}$ with period ω such that

$$|g(n, x) - g(n, y)| \leq L(n)|x - y|, \quad n = 0, 1, \dots, \omega - 1 \quad (2.7)$$

and that either

$$a(n) \leq 1 \text{ and } \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) L(i) < 1, \quad n = 0, 1, \dots, \omega - 1 \quad (2.8)$$

or

$$\sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) L(i) < 1, \quad n = 0, 1, \dots, \omega - 1. \quad (2.9)$$

Then every solution $\{x(n)\}$ of Eq. (1.1) satisfies

$$\lim_{n \rightarrow \infty} (x(n) - \tilde{y}(n)) = 0 \quad (2.10)$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution of Eq. (1.2) with period ω .

Proof. In view of Theorem A, we know that Eq. (1.2) has a unique periodic solution $\{\tilde{y}(n)\}$. Let $z(n) = x(n) - \tilde{y}(n)$. Then $\{z(n)\}$ satisfies

$$z(n + 1) + \tilde{y}(n + 1) = f(n, z(n) + \tilde{y}(n)) + g(n, z(n - k) + \tilde{y}(n - k)) + b(n), \quad n = 0, 1, \dots$$

Since $\{\tilde{y}(n)\}$ is a solution of Eq. (1.2), $\tilde{y}(n + 1) = f(n, \tilde{y}(n)) + g(n, \tilde{y}(n - k))$. Hence, it follows that

$$\begin{aligned} z(n + 1) &= f(n, z(n) + \tilde{y}(n)) - f(n, \tilde{y}(n)) \\ &\quad + g(n, z(n - k) + \tilde{y}(n - k)) - g(n, \tilde{y}(n - k)) + b(n), \quad n = 0, 1, \dots \end{aligned} \quad (2.11)$$

Clearly, to complete the proof of the theorem and show that (2.10) holds, it suffices to show that every solution $\{z(n)\}$ of Eq. (2.11) tends to zero as $n \rightarrow \infty$. First assume that $\{z(n)\}$ is

a nonoscillatory solution of Eq. (2.11). Then $\{z(n)\}$ is either eventually positive or eventually negative. We assume that $\{z(n)\}$ is eventually positive. The proof for the case that $\{z(n)\}$ is eventually negative is similar and will be omitted. Hence, there is a positive integer n_0 such that $z(n) > 0$ for $n \geq n_0$. Then by noting $f(n, x) - a(n)x$ and $g(n, x)$ are nonincreasing in x , it follows from Lemma 2.4 and (2.11) that

$$z(n+1) \leq a(n)z(n) + b(n), \quad n \geq n_0 + k$$

and so by Corollary 2.3, $z(n) \rightarrow 0$ as $n \rightarrow \infty$.

Next, assume that $\{z(n)\}$ is an oscillatory solution of Eq. (2.11). Then there is an increasing sequence $\{n_t\}$ of positive integers such that $y(n_1) \leq 0$ and for $\tau = 1, 2, \dots$,

$$\begin{cases} y(n) > 0 & \text{when } n_{2\tau-1} < n \leq n_{2\tau} \text{ and} \\ y(n) \leq 0 & \text{when } n_{2\tau} < n \leq n_{2\tau+1}. \end{cases} \quad (2.12)$$

Case 1. Assume that (2.8) holds. Then there is a positive number μ such that

$$\mu < 1 \quad \text{and} \quad \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) L(i) \leq \mu, \quad n = 0, 1, \dots$$

We show that

$$z(n) \leq \mu \max_{n_1-k \leq l \leq n_1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|, \quad n_1 < n \leq n_2. \quad (2.13)$$

In fact, from (2.12) we see that $z(n_1) \leq 0$ and $z(n) > 0$, $n_1 < n \leq n_2$. As $f(n, x) - a(n)x$ is nonincreasing in x , from Lemma 2.4 we see that $f(n, z(n) + \tilde{y}(n)) - f(n, \tilde{y}(n)) \leq a(n)z(n)$ and (2.11) becomes

$$z(n+1) \leq a(n)z(n) + g(n, z(n-k) + \tilde{y}(n-k)) - g(n, \tilde{y}(n-k)) + b(n).$$

Then by using (2.7), it follows that when $n_1 < n \leq n_2$,

$$\begin{aligned} z(n) &= \left(\prod_{j=n_1}^{n-1} a(j) \right) z(n_1) + \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) [g(i, z(i-k) + \tilde{y}(i-k)) - g(i, \tilde{y}(i-k)) + b(i)] \\ &\leq \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |g(i, z(i-k) + \tilde{y}(i-k)) - g(i, \tilde{y}(i-k))| + \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \quad (2.14) \\ &\leq \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) L(i) |z(i-k)| + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \end{aligned}$$

Now, consider two cases $n_2 \leq n_1 + k + 1$ and $n_2 > n_1 + k + 1$, respectively. When $n_2 \leq n_1 + k + 1$, for any $n_1 < n \leq n_2$, $n - k - 1 \leq n_1$ and so (2.14) yields

$$\begin{aligned} z(n) &\leq \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) L(i) \max_{n_1-k \leq l \leq n_1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \\ &\leq \sum_{i=n-k-1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) L(i) \max_{n_1-k \leq l \leq n_1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \\ &\leq \mu \max_{n_1-k \leq l \leq n_1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|. \end{aligned}$$

Hence, (2.13) holds in this case. Next, consider the case that $n_2 > n_1 + k + 1$. When $n_1 < n \leq n_1 + k + 1$, as we have shown above, (2.13) holds. In particular,

$$z(n_1 + k + 1) \leq \mu \max_{n_1 - k \leq l \leq n_1} \{|z(l)|\} + \sum_{i=0}^{n_1+k} \left(\prod_{j=i+1}^{n_1+k} a(j) \right) |b(i)|. \quad (2.15)$$

When $n_1 + k + 1 < n \leq n_2$, by noting $z(n - k - 1) > 0$, (2.15) holds and Lemma 2.4, (2.11) yields

$$\begin{aligned} z(n) &\leq a(n-1)z(n-1) + b(n-1) \\ &= \left(\prod_{j=n_1+k+1}^{n-1} a(j) \right) z(n_1+k+1) + \sum_{i=n_1+k+1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) b(i) \\ &\leq \left(\prod_{j=n_1+k+1}^{n-1} a(j) \right) \left(\mu \max_{n_1-k \leq l \leq n_1} \{|z(l)|\} + \sum_{i=0}^{n_1+k} \left(\prod_{j=i+1}^{n_1+k} a(j) \right) |b(i)| \right) \\ &\quad + \sum_{i=n_1+k+1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) b(i) \\ &\leq \mu \max_{n_1-k \leq l \leq n_1} \{|z(l)|\} + \sum_{i=0}^{n_1+k} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| + \sum_{i=n_1+k+1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) b(i) \\ &= \mu \max_{n_1-k \leq l \leq n_1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \end{aligned}$$

and so $z(n)$ satisfies (2.13). Hence for any case, (2.13) holds. Then by a similar argument, we may show that

$$z(n) \geq - \left[\mu \max_{n_2-k \leq l \leq n_2} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \right], \quad n_2 < n \leq n_3,$$

and in general,

$$|z(n)| \leq \mu B(t) + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|, \quad n_t < n \leq n_{t+1}. \quad (2.16)$$

where

$$B(t) = \max_{n_t-k \leq l \leq n_t} \{|z(l)|\}, \quad t = 1, 2, \dots$$

Since $b(n) \rightarrow 0$ as $n \rightarrow \infty$, $|b(n)| \rightarrow 0$ as $n \rightarrow \infty$. Then it follows from Lemma 2.1,

$$\sum_{i=0}^n \left(\prod_{j=i+1}^n a(j) \right) |b(i)| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.17)$$

Hence, from (2.16) we see that if $B(t) \rightarrow 0$ as $t \rightarrow \infty$, then $z(n) \rightarrow 0$ as $n \rightarrow \infty$. In the following, we assume that $B(t) \not\rightarrow 0$ as $t \rightarrow \infty$. Then there is a subsequence $\{B(t_s)\}$ of $\{B(t)\}$ such that

$$B(t_s) \geq \eta, \quad s = 1, 2, \dots$$

where η is a positive constant.

By noting (2.17) again, we may choose a positive number δ such that

$$\mu + \delta < 1$$

and a subsequence $\{n_{t_{s_r}}\}$ of $\{n_{t_s}\}$ such that for each $r = 1, 2, \dots$,

$$n_{t_{s_{r+1}}} - n_{t_{s_r}} \geq 1 + 2k$$

and

$$\sum_{i=0}^n \left(\prod_{j=i+1}^n a(j) \right) |b(i)| < \eta \delta^r, \quad n \geq n_{t_{s_r}} - 1. \quad (2.18)$$

We claim that

$$B(t) \leq B(t_{s_r}) \quad \text{for } t \geq t_{s_r}, \quad r = 1, 2, \dots \quad (2.19)$$

In fact, if $n_{t_{s_r+1}} - k > n_{t_{s_r}}$, we see that when $n_{t_{s_r+1}} - k \leq n \leq n_{t_{s_r+1}}$, it follows from (2.16) and (2.18) that

$$|z(n)| \leq \mu B(t_{s_r}) + \eta \delta^r \leq (\mu + \delta^r) B(t_{s_r}) \leq B(t_{s_r}). \quad (2.20)$$

If $n_{t_{s_r+1}} - k \leq n_{t_{s_r}}$ we see that (2.20) holds when $n_{t_{s_r}} < n \leq n_{t_{s_r+1}}$; while when $n_{t_{s_r+1}} - k \leq n \leq n_{t_{s_r}}$, by noting $n_{t_{s_r}} - k < n_{t_{s_r+1}} - k$, we see that

$$|z(n)| \leq \max_{n_{t_{s_r}} - k \leq l \leq n_{t_{s_r}}} \{|z(l)|\} = B(t_{s_r}).$$

Hence, from the above discussion we see that for any case when $n_{t_{s_r+1}} - k \leq n \leq n_{t_{s_r+1}}$,

$$|z(n)| \leq B(t_{s_r})$$

and so

$$B(t_{s_r} + 1) = \max_{n_{t_{s_r+1}} - k \leq l \leq n_{t_{s_r+1}}} \{|z(l)|\} \leq B(t_{s_r}).$$

Then by a similar argument and induction, we may show that for any $l \geq 1$,

$$B(t_{s_r} + l) \leq B(t_{s_r})$$

that is, (2.19) holds. Then it follows from (2.16) and (2.19) that

$$|z(n)| \leq \mu B(t_{s_r}) + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|, \quad n > n_{s_r}. \quad (2.21)$$

Next, we show that

$$|z(n)| \leq (\mu + \delta)^r B(t_{s_1}), \quad n > n_{t_{s_1}}, \quad r = 1, 2, \dots \quad (2.22)$$

When $r = 1$, from (2.18) and (2.21) we see that

$$|z(n)| \leq \mu B(t_{s_1}) + \eta \delta \leq (\mu + \delta) B(t_{s_1}), \quad n > n_{t_{s_1}}$$

which satisfies (2.22) with $r = 1$. Assume that when $r = m$, (2.22) holds, that is,

$$|z(n)| \leq (\mu + \delta)^m B(t_{s_1}), \quad n > n_{t_{s_m}}. \quad (2.23)$$

Then from (2.21) and (2.23) we see that when $n > n_{t_{s_{m+1}}}$,

$$\begin{aligned} |z(n)| &\leq \mu B(t_{s_{m+1}}) + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \\ &\leq \mu(\mu + \delta)^m B(t_{s_1}) + \eta \delta^{m+1} \\ &\leq (\mu(\mu + \delta)^m + \delta^{m+1}) B(t_{s_1}) \\ &\leq (\mu + \delta)^{m+1} B(t_{s_1}), \end{aligned}$$

which satisfies (2.22) with $r = m + 1$. Hence, by induction, (2.22) holds. Clearly, (2.22) implies that $z(n) \rightarrow 0$ as $n \rightarrow \infty$.

Case 2. Assume that (2.9) holds. Then there is a positive number ν such that

$$\nu < 1 \quad \text{and} \quad \sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) L(i) \leq \nu, \quad n = 0, 1, \dots$$

We claim that

$$z(n) \leq \nu \max_{n_1-k \leq l \leq n_1+\omega-1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|, \quad n_1 < n \leq n_2. \quad (2.24)$$

First, from the proof of Case 1, we see that when $n_1 < n \leq n_2$, (2.14) holds. Next, consider two cases $n_2 \leq n_1 + k + \omega$ and $n_2 > n_1 + k + \omega$, respectively. When $n_2 \leq n_1 + k + \omega$, for any $n_1 < n \leq n_2$, $n - k - \omega \leq n_1$ and so (2.14) yields

$$\begin{aligned} z(n) &\leq \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) L(i) \max_{n_1-k \leq l \leq n_1+\omega-1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \\ &\leq \sum_{i=n-k-\omega}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) L(i) \max_{n_1-k \leq l \leq n_1+\omega-1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \\ &\leq \nu \max_{n_1-k \leq l \leq n_1+\omega-1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|. \end{aligned} \quad (2.25)$$

Hence, (2.24) holds in this case. Next, consider the case that $n_2 > n_1 + k + \omega$. When $n_1 < n \leq n_1 + k + \omega$, as we have shown above, (2.24) holds. Hence, we only need to show that (2.24) holds also when $n_1 + k + \omega < n \leq n_2$. In fact, by noting that when $n_1 + k + 1 < n \leq n_2$, $z(n - k - 1) > 0$, and the result of Lemma 2.4, (2.11) yields

$$z(n) \leq a(n-1)z(n-1) + b(n-1), \quad n_1 + k + 1 < n \leq n_2. \quad (2.26)$$

Hence, it follows from (2.25) and (2.26) that

$$\begin{aligned} z(n_1 + k + \omega + 1) &\leq \left(\prod_{j=n_1+k+1}^{n_1+k+\omega} a(j) \right) z(n_1 + k + 1) + \sum_{i=n_1+k+1}^{n_1+k+\omega} \left(\prod_{j=i+1}^{n_1+k+\omega} a(j) \right) |b(i)| \\ &\leq \left(\prod_{j=n_1+k+1}^{n_1+k+\omega} a(j) \right) \left(\nu \max_{n_1-k \leq l \leq n_1+\omega-1} \{|z(l)|\} + \sum_{i=0}^{n_1+k} \left(\prod_{j=i+1}^{n_1+k} a(j) \right) |b(i)| \right) \\ &\quad + \sum_{i=n_1+k+1}^{n_1+k+\omega} \left(\prod_{j=i+1}^{n_1+k+\omega} a(j) \right) |b(i)| \end{aligned}$$

$$\begin{aligned}
 &\leq \nu \max_{n_1-k \leq l \leq n_1+\omega-1} \{|z(l)|\} + \sum_{i=0}^{n_1+k} \left(\prod_{j=i+1}^{n_1+k+\omega} a(j) \right) |b(i)| \\
 &\quad + \sum_{i=n_1+k+1}^{n_1+k+\omega} \left(\prod_{j=i+1}^{n_1+k+\omega} a(j) \right) |b(i)| \\
 &= \nu \max_{n_1-k \leq l \leq n_1+\omega-1} \{|z(l)|\} + \sum_{i=0}^{n_1+k+\omega} \left(\prod_{j=i+1}^{n_1+k+\omega} a(j) \right) |b(i)|
 \end{aligned}$$

and similarly,

$$\begin{aligned}
 z(n_1 + k + \omega + 2) &\leq \nu \max_{n_1-k \leq l \leq n_1+\omega-1} \{|z(l)|\} + \sum_{i=0}^{n_1+k+\omega+1} \left(\prod_{j=i+1}^{n_1+k+\omega+1} a(j) \right) |b(i)| \\
 &\quad \vdots \\
 z(n_2) &\leq \nu \max_{n_1-k \leq l \leq n_1+\omega-1} \{|z(l)|\} + \sum_{i=0}^{n_2-1} \left(\prod_{j=i+1}^{n_2-1} a(j) \right) |b(i)|.
 \end{aligned}$$

Hence for any case, (2.24) holds. Then by a similar argument, we may show that

$$z(n) \geq - \left[\nu \max_{n_2-k \leq l \leq n_2+\omega-1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \right], \quad n_2 < n \leq n_3,$$

and in general,

$$|z(n)| \leq \mu C(t) + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|, \quad n_t < n \leq n_{t+1}.$$

where

$$C(t) = \max_{n_t-k \leq l \leq n_t+\omega-1} \{|z(l)|\}, \quad t = 1, 2, \dots$$

Then by an argument similar to that for Case 1, we may show the following.

If $C(t) \rightarrow 0$ as $t \rightarrow \infty$, then $z(n) \rightarrow 0$ as $n \rightarrow \infty$; If $C(t) \not\rightarrow 0$ as $t \rightarrow \infty$, then there is a subsequence $\{C(t_s)\}$ of $\{C(t)\}$ such that

$$C(t_s) \geq \eta, \quad s = 1, 2, \dots$$

where η is a positive constant. A positive number δ such that

$$\nu + \delta < 1$$

and a subsequence $\{n_{t_{sr}}\}$ of $\{n_{t_s}\}$ such that for each $r = 1, 2, \dots$,

$$n_{t_{sr+1}} - n_{t_{sr}} \geq 1 + 2k$$

could be chosen such that

$$\sum_{i=0}^n \left(\prod_{j=i+1}^n a(i) \right) |b(i)| < \eta \delta^r, \quad n \geq n_{t_{sr}} - 1$$

and

$$|z(n)| \leq (\mu + \delta)^r C(t_{s_1}), \quad n > n_{t_{s_1}}, \quad r = 1, 2, \dots$$

Clearly, the above inequalities imply that $z(n) \rightarrow 0$ as $n \rightarrow \infty$. The proof is complete. \square

When $g(n, x) = p(n)h(x)$, where $\{p(n)\}$ is a nonnegative periodic sequence with period ω and h is a nonnegative continuous function, Eq. (1.1) becomes

$$x(n+1) = f(n, x(n)) + p(n)h(x(n-k)) + b(n), \quad n = 0, 1, \dots \quad (2.27)$$

and the following result is a direct consequence of Theorem 2.5.

Corollary 2.6. Consider Eq. (2.27) and assume that $f(n, x)$ is nondecreasing in x . Assume also that $\{a(n)\}$ is a nonnegative periodic sequence with period ω and $\{b(n)\}$ is a real sequence such that $\{a(n)\}$ and $\{b(n)\}$ satisfy (2.1), $f(n, x) \leq a(n)x$ and that $f(n, x) - a(n)x$ is nonincreasing in x . Suppose that h is nonincreasing and L -Lipschitz and that there is a positive constant B such that

$$\sum_{i=n}^{n+\omega-1} \left(\prod_{j=i+1}^{n+\omega-1} a(j) \right) [f(i, B) - a(i)B + p(i)h(B)] \geq 0, \quad n = 0, 1, \dots, \omega - 1 \quad (2.28)$$

and

$$\frac{1}{1 - \prod_{j=0}^{\omega-1} a(j)} \sum_{i=n}^{n+\omega-1} \left(\prod_{j=i+1}^{n+\omega-1} a(j) \right) p(i)h(0) \leq B, \quad n = 0, 1, \dots, \omega - 1. \quad (2.29)$$

Suppose also that either

$$a(n) \leq 1 \quad \text{and} \quad L \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) p(i) < 1, \quad n = 0, 1, \dots, \omega - 1$$

or

$$L \sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) p(i) < 1, \quad n = 0, 1, \dots, \omega - 1.$$

Then every solution $\{x(n)\}$ of Eq. (2.27) satisfies

$$\lim_{n \rightarrow \infty} (x(n) - \tilde{y}(n)) = 0$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution with period ω of the equation

$$y(n+1) = f(n, y(n)) + p(n)h(y(n-k)), \quad n = 0, 1, \dots$$

When $f(n, x) = a(n)x(n)$, Eq. (2.27) becomes

$$x(n+1) = a(n)x(n) + p(n)h(x(n-k)) + b(n), \quad n = 0, 1, \dots \quad (2.30)$$

(2.28) is satisfied for any $B > 0$ and (2.29) holds for B large enough. Thus the following result is a direct consequence of Corollary 2.6.

Corollary 2.7. Consider Eq. (2.30) and assume that $\{a(n)\}$ is a nonnegative periodic sequence with period ω and $\{b(n)\}$ is a real sequence such that $\{a(n)\}$ and $\{b(n)\}$ satisfy (2.1). Suppose also that $h(x)$ is nonincreasing and L -Lipschitz, and that either

$$a(n) \leq 1 \quad \text{and} \quad L \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) p(i) < 1, \quad n = 0, 1, \dots, \omega - 1 \quad (2.31)$$

or

$$L \sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) p(i) < 1, \quad n = 0, 1, \dots, \omega - 1. \quad (2.32)$$

Then every solutions $\{x(n)\}$ of Eq. (2.30) satisfies

$$\lim_{n \rightarrow \infty} (x(n) - \tilde{y}(n)) = 0$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution with period ω of the equation

$$y(n+1) = a(n)y(n) + p(n)h(y(n-k)), \quad n = 0, 1, \dots$$

In particular, when $h(x) \equiv 1$, Eq. (2.27) reduces to the first order linear equation

$$x(n+1) = a(n)x(n) + p(n) + b(n), \quad n = 0, 1, \dots \quad (2.33)$$

Since we may choose $L = 0$, (2.31) and (2.32) hold. Hence, from Corollary 2.7, we have the following result immediately.

Corollary 2.8. Consider Eq. (2.33) and assume that $\{a(n)\}$ is a nonnegative periodic sequence with period ω and $\{b(n)\}$ is a real sequence such that $\{a(n)\}$ and $\{b(n)\}$ satisfy (2.1). Then every solution $\{x(n)\}$ of Eq. (2.33) satisfies

$$\lim_{n \rightarrow \infty} (x(n) - \tilde{y}(n)) = 0$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution with period ω of the equation

$$y(n+1) = a(n)y(n) + p(n), \quad n = 0, 1, \dots \quad (2.34)$$

Remark 2.9. When $a(n) \equiv a$ and $p(n) \equiv p$ are nonnegative constants, Eqs. (2.33) and (2.34) become

$$x(n+1) = ax(n) + p + b(n), \quad n = 0, 1, \dots \quad (2.35)$$

and

$$y(n+1) = ay(n) + p, \quad n = 0, 1, \dots \quad (2.36)$$

respectively. The nonnegative periodic solution $\{\tilde{y}(n)\}$ of Eq. (2.36) becomes the nonnegative equilibrium point $\bar{y} = \frac{p}{1-a}$. Then by Corollary 2.8, when $a < 1$, every nonnegative solution $\{x(n)\}$ of Eq. (2.35) converges to \bar{y} as $n \rightarrow \infty$. In fact, in this case, the solution of Eq. (2.35) is

$$x(n) = a^n x(0) + p \frac{1-a^n}{1-a} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) b(i), \quad n = 1, 2, \dots$$

By noting (2.1) and Lemma 2.1, we know that $\sum_{i=0}^n \left(\prod_{j=i+1}^n a(j) \right) b(i) \rightarrow 0$ as $n \rightarrow \infty$ and so

$$x(n) \rightarrow \frac{p}{1-a} \quad \text{as } n \rightarrow \infty.$$

Remark 2.10. Clearly, Corollary 2.8 implies that for the equation

$$x(n+1) = a(n)x(n) + q(n), \quad n = 0, 1, \dots$$

where $\{a(n)\}$ is nonnegative and periodic with period ω , and $\{q(n)\}$ is nonnegative and quasi-periodic with period ω , if $\sum_{i=0}^{\omega-1} a(j) < 1$, then every nonnegative solution of the equation is quasi-periodic with period ω .

3 Applications

In this section, we apply our results obtained in Section 2 to some equations derived from mathematical biology. In applications, there are often external factors – known or unknown – that affect the mathematical model. Two such factors that have been studied in related models are migration and subsets of populations which become isolated and unchanged by density-dependent effects, see [11, 27] and references cited therein.

Consider the difference equations

$$x(n+1) = \frac{a(n)x^2(n)}{x(n) + \delta(n)} + \frac{v(n)\rho(n)\sigma(n)}{1 + e^{\beta(n)x(n-k) - \alpha(n)}} + b(n), \quad n = 0, 1, \dots, \quad (3.1)$$

$$x(n+1) = a(n)x(n) + \beta(n)e^{-\sigma(n)x(n-k)} + b(n), \quad n = 0, 1, \dots \quad (3.2)$$

and

$$x(n+1) = a(n)x(n) + \frac{\beta(n)}{1 + x^\gamma(n-k)} + b(n), \quad n = 0, 1, \dots \quad (3.3)$$

where $\{a(n)\}$, $\{\alpha(n)\}$, $\{\beta(n)\}$, $\{v(n)\}$, $\{\delta(n)\}$, $\{\rho(n)\}$, $\{\sigma(n)\}$ are nonnegative periodic sequences with period ω , $\{b(n)\}$ is a real sequence, γ is a positive constant and k is a nonnegative integer. When $a(n) \equiv a$, $\alpha(n) \equiv \alpha$, $\beta(n) \equiv \beta$, $v(n) \equiv v$, $\delta(n) \equiv \delta$, $\rho(n) \equiv \rho$ and $\sigma(n) \equiv \sigma$ are nonnegative constants and $b(n) \equiv 0$, Eqs. (3.1), (3.2) and (3.3) reduce to

$$x(n+1) = \frac{ax^2(n)}{x(n) + \delta} + \frac{v\rho\sigma}{1 + e^{\beta x(n-k) - \alpha}}, \quad n = 0, 1, \dots, \quad (3.4)$$

$$x(n+1) = ax(n) + \beta e^{-\sigma x(n-k)}, \quad n = 0, 1, \dots \quad (3.5)$$

and

$$x(n+1) = ax(n) + \frac{\beta}{1 + x^\gamma(n-k)}, \quad n = 0, 1, \dots \quad (3.6)$$

respectively. Eq. (3.4) is derived from a model of the energy cost for new leaf growth in citrus crops, see [30]. When $b(n) \neq 0$, $\{b(n)\}$ may represent defoliation that does not occur naturally or is not considered natural defoliation by the model parameters. A similar equation is given for the litter mass in perennial grasses, and the results that follow will apply directly to this model, see [28]. Eq. (3.5) is a discrete version of a model of the survival of red blood cells in an animal [29], and Eq. (3.6) is a discrete analog of a model that has been used to study blood cell production [10]. The global attractivity of positive solutions of Eqs. (3.5), (3.6) and some extensions of them has been studied by numerous authors, see for example [4–7, 9, 12, 14] and references cited therein. When $b(n) \neq 0$, $\{b(n)\}$ may represent the medical replacement of blood cells or administration of antibodies, see [2, 8] and references cited therein.

Suppose $\{b(n)\}$ is quasi-periodic, that is, there exist real sequences $\{q(n)\}$ and $\{r(n)\}$ such that $\{q(n)\}$ is periodic with period ω , $\{r(n)\}$ is such that $r(n) \rightarrow 0$ as $n \rightarrow \infty$, and $b(n) = q(n) + r(n)$. Then Eqs. (3.1), (3.2) and (3.3) become

$$x(n+1) = \frac{a(n)x^2(n)}{x(n) + \delta(n)} + \frac{\gamma(n)\rho(n)\sigma(n)}{1 + e^{\beta(n)x(n-k) - \alpha(n)}} + q(n) + r(n), \quad n = 0, 1, \dots, \quad (3.7)$$

$$x(n+1) = a(n)x(n) + \beta(n)e^{-\sigma(n)x(n-k)} + q(n) + r(n), \quad n = 0, 1, \dots \quad (3.8)$$

and

$$x(n+1) = a(n)x(n) + \frac{\beta(n)}{1 + x^\gamma(n-k)} + q(n) + r(n), \quad n = 0, 1, \dots \quad (3.9)$$

respectively.

First, consider Eq. (3.7). It is of the form of Eq. (1.1) with

$$f(n, x) = \frac{a(n)x^2}{x + \delta(n)} \quad \text{and} \quad g(n, x) = \frac{v(n)\rho(n)\sigma(n)}{1 + e^{\beta(n)x - \alpha(n)}} + q(n).$$

As

$$\frac{df}{dx} = \frac{a(n)x(x + 2\delta(n))}{(x + \delta(n))^2}, \quad x \geq 0,$$

we see that $f(n, x)$ is nondecreasing in x . We next note that

$$f(n, x) - a(n)x = \frac{-a(n)\delta(n)x}{x + \delta(n)}, \quad x \geq 0$$

and

$$\frac{d}{dx}(f(n, x) - a(n)x) = \frac{-a(n)\delta^2(n)}{(x + \delta(n))^2}, \quad x \geq 0,$$

thus $f(n, x) \leq a(n)x$ and $f(n, x) - a(n)x$ is nonincreasing in x . As

$$\frac{dg}{dx} = -\beta(n)v(n)\rho(n)\sigma(n) \frac{e^{\beta(n)x - \alpha(n)}}{(1 + e^{\beta(n)x - \alpha(n)})^2}, \quad x \geq 0$$

and

$$\frac{d^2g}{dx^2} = -\beta^2(n)v(n)\rho(n)\sigma(n) \frac{e^{\beta(n)x - \alpha(n)}(1 - e^{\beta(n)x - \alpha(n)})}{(1 + e^{\beta(n)x - \alpha(n)})^3}, \quad x \geq 0,$$

we see that $g(n, x)$ is nonincreasing in x , and for each n , $\left|\frac{dg(n, x)}{dx}\right|$ achieves a maximum when $x = \frac{\alpha(n)}{\beta(n)}$, and

$$\left|\frac{dg(n, x)}{dx}\right|_{x=\frac{\alpha(n)}{\beta(n)}} = \frac{\beta(n)v(n)\rho(n)\sigma(n)}{4}.$$

Thus $g(n, x)$ is L -Lipschitz with $L(n) = \frac{\beta(n)v(n)\rho(n)\sigma(n)}{4}$. Hence, we have the following conclusion from Theorem 2.5.

Corollary 3.1. *Assume that*

$$\hat{a} = \prod_{j=0}^{\omega-1} a(j) < 1.$$

Suppose there exists a positive constant B such that

$$\sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) \left[q(i) + \frac{v(i)\rho(i)\sigma(i)}{1 + e^{B\cdot\beta(i)-\alpha(i)}} - \frac{B^2a(i)\delta(i)}{B + \delta(i)} \right] \geq 0, \quad n = 0, 1, \dots, \omega - 1$$

and

$$\frac{1}{1 - \hat{a}} \sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) \left(\frac{v(i)\rho(i)\sigma(i)}{1 + e^{-\alpha(i)}} + q(i) \right) \leq B, \quad n = 0, 1, \dots, \omega - 1.$$

Suppose also that either

$$a(n) \leq 1 \quad \text{and} \quad \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) \beta(i)v(i)\rho(i)\sigma(i) < 4, \quad n = 0, 1, \dots, \omega - 1$$

or

$$\sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) \beta(i) \nu(i) \rho(i) \sigma(i) < 4, \quad n = 0, 1, \dots, \omega - 1.$$

Then every solution $\{x(n)\}$ of Eq. (3.7) satisfies

$$\lim_{n \rightarrow \infty} (x(n) - \tilde{y}(n)) = 0$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution with period ω of the following equation

$$y(n+1) = \frac{a(n)y^2(n)}{y(n) + \delta(n)} + \frac{\gamma(n)\rho(n)\sigma(n)}{1 + e^{\beta(n)y(n-k) - \alpha(n)}} + q(n), \quad n = 0, 1, \dots$$

Next consider Eq. (3.8). It is in the form of Eq. (1.1) with

$$f(n, x) = a(n)x \quad \text{and} \quad g(n, x) = \beta(n)e^{-\sigma(n)x} + q(n).$$

(1.3) is satisfied for any $B > 0$ and (1.4) holds for B large enough. Observing

$$\frac{dg}{dx} = -\beta(n)\sigma(n)e^{-\sigma(n)x}, \quad x \geq 0,$$

we see that $g(n, x)$ is nonincreasing in x and

$$\left| \frac{dg}{dx} \right| \leq \beta(n)\sigma(n) \quad \text{for } x \geq 0,$$

which implies that for each n , $g(n, x)$ is L -Lipschitz with $L(n) = \beta(n)\sigma(n)$. Hence, we have the following conclusion from Theorem 2.5.

Corollary 3.2. *Assume that*

$$\prod_{j=0}^{\omega-1} a(j) < 1$$

and that either

$$a(n) \leq 1 \quad \text{and} \quad \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) \beta(i)\sigma(i) < 1, \quad n = 0, 1, \dots, \omega - 1$$

or

$$\sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) \beta(i)\sigma(i) < 1, \quad n = 0, 1, \dots, \omega - 1.$$

Then every solution $\{x(n)\}$ of Eq. (3.8) satisfies

$$\lim_{n \rightarrow \infty} (x(n) - \tilde{y}(n)) = 0$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution with period ω of the following equation

$$y(n+1) = a(n)y(n) + \beta(n)e^{-\sigma(n)y(n-k)} + q(n), \quad n = 0, 1, \dots$$

Finally, consider Eq. (3.9). It is in the form of (1.1) with

$$f(n, x) = a(n)x \quad \text{and} \quad g(n, x) = \frac{\beta(n)}{1+x^\gamma} + q(n).$$

gain, (1.3) is satisfied for any $B > 0$ and (1.4) hold for B large enough. Observing that

$$\frac{dg}{dx} = -\beta(n) \frac{\gamma x^{\gamma-1}}{(1+x^\gamma)^2} \quad \text{and} \quad \frac{d^2g}{dx^2} = \beta(n) \frac{\gamma x^{\gamma-2}((\gamma+1)x^\gamma - (\gamma-1))}{(1+x^\gamma)^3}$$

we see that for each n , when $\gamma = 1$,

$$\left| \frac{dg}{dx} \right| \leq \left| \frac{dg}{dx} \right|_{x=0} = \beta(n) \quad \text{for } x \geq 0$$

and when $\gamma > 1$, $\left| \frac{dg}{dx} \right|$ attains its maximum at $x^* = \left(\frac{\gamma-1}{\gamma+1} \right)^{1/\gamma}$ and

$$\left| \frac{dg}{dx} \right|_{x=x^*} = \frac{(\gamma-1)^{\frac{\gamma-1}{\gamma}} (\gamma+1)^{\frac{\gamma+1}{\gamma}}}{4\gamma} \beta(n), \quad n = 0, 1, \dots, \omega-1.$$

Hence, $g(n, x)$ is L -Lipschitz with

$$L(n) = \begin{cases} \beta(n), & \gamma = 1, \\ \frac{(\gamma-1)^{\frac{\gamma-1}{\gamma}} (\gamma+1)^{\frac{\gamma+1}{\gamma}}}{4\gamma} \beta(n), & \gamma > 1. \end{cases}$$

It follows from Theorem 2.5 that the following conclusion holds.

Corollary 3.3. *Assume that*

$$\prod_{j=0}^{\omega-1} a(j) < 1.$$

Suppose also that when $\gamma = 1$, either

$$a(n) \leq 1 \quad \text{and} \quad \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) \beta(i) < 1, \quad n = 0, 1, \dots, \omega-1$$

or

$$\sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) \beta(i) < 1, \quad n = 0, 1, \dots, \omega-1;$$

when $\gamma > 1$, either

$$a(n) \leq 1 \quad \text{and} \quad \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) \beta(i) < \frac{4\gamma}{(\gamma-1)^{\frac{\gamma-1}{\gamma}} (\gamma+1)^{\frac{\gamma+1}{\gamma}}}, \quad n = 0, 1, \dots, \omega-1$$

or

$$\sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) \beta(i) < \frac{4\gamma}{(\gamma-1)^{\frac{\gamma-1}{\gamma}} (\gamma+1)^{\frac{\gamma+1}{\gamma}}}, \quad n = 0, 1, \dots, \omega-1.$$

Then every solution $\{x(n)\}$ of Eq. (3.9) satisfies

$$\lim_{n \rightarrow \infty} (x(n) - \tilde{y}(n)) = 0$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution of with period ω of the following equation

$$y(n+1) = a(n)y(n) + \frac{\beta(n)}{1+y^\gamma(n-k)} + q(n), \quad n = 0, 1, \dots$$

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