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EXISTENCE AND APPROXIMATION OF SOLUTIONS TO THREE-POINT BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study existence and approximation of solutions to some three-point boundary value problems for fractional differential equations of the type

$$
{}^c\mathcal{D}^q_{0+}u(t) + f(t, u(t)) = 0, \quad t \in (0,1), 1 < q < 2
$$
$$
u'(0) = 0, \qquad u(1) = \xi u(\eta),
$$

where $0 < \xi$, $\eta \in (0,1)$ and ${}^c\mathcal{D}^q_{0+}$ is the fractional derivative in the sense of Caputo. For the existence of solution, we develop the method of upper and lower solutions and for the approximation of solutions, we develop the generalized quasilinearization technique (GQT). The GQT generates a monotone sequence of solutions of linear problems that converges monotonically and quadratically to solution of the original nonlinear problem.

1. Introduction

The study of fractional differential equations is of fundamental concern due to its important applications to real world problems. Many problems in applied sciences such as engineering and physics can be modeled by differential equations of fractional order [1, 2, 3]. It has been observed that the models with fractional differential equations provide more realistic and accurate results compared to the analogous models with integer order derivatives, see, [4, 5]. Existence theory for solutions to boundary value problems for fractional differential equations have attracted the attention of many researcher quite recently, see for example [6, 7, 8, 9, 10, 11, 12] and the references therein. However, the method of upper and lower solutions for the existence of solution is less developed and hardly few results can be found in the literature dealing with the upper and lower solutions method to boundary value problems for fractional differential equations [13, 14, 15, 16, 17]. The method of quasilinearization is somewhat developed for initial value problems for fractional differential equations [18, 19, 20, 21] but results dealing with quasilinearization to boundary value problems for fractional differential equations can hardly be seen in the literature. The paper seem to be an attempt

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to develop the generalized quasilinearization to three-point boundary value problems for fractional differential equations.

2. Preliminaries

We recall some basic definitions and lemmas from fractional calculus [4].

Definition 2.1. The fractional integral of order $q > 0$ of a function $g : (0, \infty) \to \mathbb{R}$ is defined by

$$\mathcal{I}^q_{0+} g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds,$$

provided the integral converges.

Definition 2.2. The Caputo fractional derivative of order $q > 0$ of a function $g \in AC^m[0,1]$ is defined by

$${}^c\mathcal{D}^q_{0+} g(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{g^{(n)}(s)}{(t-s)^{q-n+1}} ds, \text{ where } n = \lceil q \rceil,$$

provided that the right side is pointwise defined on $(0, \infty)$.

Remark 2.3. Under the natural conditions on $g(t)$ the Caputo fractional derivative becomes conventional integer order derivative of a function $g(t)$ as $q \to n$.

Lemma 2.4. *For $q > 0$, $g \in C(0,1) \cap L(0,1)$, the homogenous fractional differential equation ${}^c\mathcal{D}^q_{0+} g(t) = 0$ has a solution $g(t) = c_1 + c_2 t + c_3 t^2 + ... + c_n t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0, 1, ..., n$ and $n = \lceil q \rceil + 1$.*

Now, we consider the following nonlinear boundary value problem for fractional differential equation

$$\begin{aligned} &-{}^c\mathcal{D}_{0+} u(t) = f(t, u(t)), \quad t \in (0,1), 1 < q < 2 \\ &u'(0) = 0, \qquad u(1) = \xi u(\eta), \end{aligned} \tag{2.1}$$

where ξ, $\eta \in (0,1)$ and the nonlinearity $f : [0;1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Definition 2.5. A function α is called a lower solution of the BVP (2.1), if $\alpha \in C[0,1]$ and satisfies

$$-{}^c\mathcal{D}^q_{0+} \alpha(t) \leq f(t, \alpha(t)),\ t \in (0,1),\ \alpha'(0) \geq 0,\ \alpha(1) \leq \xi \alpha(\eta).$$

An upper solution $\beta \in C[0,1]$ of the BVP (2.1) is defined similarly by reversing the inequality.

We know that for $h \in C[0,1]$, the linear problem

$$-{}^{c}\mathcal{D}_{0+}^{q}u(t) = h(u(t)), \quad t \in (0,1), 1 < q < 2,\ u'(0) = a, \qquad u(1) - \xi u(\eta) = b,$$

has a unique solution given by

$$u(t) = \frac{1}{1-\xi}[b + a\{\xi(\eta - t) - (1-t)\}] + \int_0^1 G(t,s)h(s)ds,\ t \in [0,1], \tag{2.2}$$

where

$$G(t,s) = \begin{cases} \frac{(1-s)^{q-1} - (1-\xi)(t-s)^{q-1} - \xi(\eta-s)^{q-1}}{(1-\xi)\Gamma(q)}, & 0 \le t \le 1,\ \eta \ge s, \\ \frac{(1-s)^{q-1} - (1-\xi)(t-s)^{q-1}}{(1-\xi)\Gamma(q)}, & 0 < \eta <\le s \le t \le 1, \\ \frac{(1-s)^{q-1} - \xi(\eta-s)^{q-1}}{(1-\xi)\Gamma(q)}, & 0 \le t \le s \le \eta < 1, \\ \frac{(1-s)^{q-1}}{(1-\xi)\Gamma(q)}, & 0 \le t \le s \le 1,\ \eta \le s \end{cases}$$

is the Green's function. The Green's function $G(t,s) > 0$ for all $t,\ s \in (0,1)$. Hence, if $a \le 0$, $b \ge 0$ and $h(t) \ge 0$ on $[0,1]$, then any solution u of the linear BVP is positive on $[0,1]$. Thus, we have the following comparison results.

Comparison results: If $u'(0) \le 0$, $u(1) \ge \xi u(\eta)$ and ${}^{c}\mathcal{D}_{0+}^{q}u(t) \le 0$ on $(0,1)$, then $u \ge 0$ on $(0,1)$.

If $u'(0) \ge 0$, $u(1) \le \xi u(\eta)$ and ${}^{c}\mathcal{D}_{0+}^{q}u(t) \ge 0$ on $(0,1)$, then $u \le 0$ on $(0,1)$.

3. Main Results

Theorem 3.1. *Assume that there exist lower and upper solutions $\alpha, \beta \in C[0,1]$ of the BVP (2.1) such that $\alpha \le \beta$ on $[0,1]$. Assume that $f : [0,1] \times \mathbb{R} \to (0,\infty)$ is continuous and non-decreasing with respect to u on $[0,1]$. Then the BVP (2.1) has $C[0,1]$ positive solution u such that $\alpha(t) \le u(t) \le \beta(t)$, $t \in [0,1]$.*

Proof. Define the modification of f,

$$F(t,u) = \begin{cases} f(t,\beta(t)), & \text{if } u \ge \beta(t), \\ f(t,u(t)), & \text{if } \alpha(t) \le u \le \beta(t), \\ f(t,\alpha(t)), & \text{if } \ u \le \alpha(t). \end{cases} \tag{3.1}$$

Clearly, F is continuous, bounded on $[0,1] \times \mathbb{R}$ and is non-decreasing with respect to u for each fixed $t \in [0,1]$. Hence, the modified BVP

$$-{}^{c}\mathcal{D}_{0+}^{q}u(t) = F(t,u(t)),\ 1 < q < 2, \quad t \in (0,1),\ u'(0) = 0,\ u(1) = \xi u(\eta), \tag{3.2}$$

has a solution u. Moreover, we note that a solution u of the problem (3.2) such that $\alpha(t) \leq u \leq \beta(t)$, $t \in [0,1]$, is a solution of the BVP (2.1). We only need to show that $\alpha(t) \leq u \leq \beta(t)$, $t \in [0,1]$, where u is solution of the BVP (3.2). In view of the non-decreasing property of f, we obtain

$$f(t,\alpha(t)) \leq F(t,u) \leq f(t,\beta(t)),\ (t,u) \in [0,1] \times \mathbb{R}. \tag{3.3}$$

Set $m(t) = \alpha(t) - u(t)$, $t \in [0,1]$, then, $m'(0) \geq 0$, $m(1) \leq \xi m(\eta)$. Using the definition of lower solution and (3.3), we obtain

$$-{}^c\mathcal{D}^q_{0+}m(t) = -{}^c\mathcal{D}^q_{0+}\alpha(t) + {}^c\mathcal{D}^q_{0+}u(t) \leq f(t,\alpha(t)) - F(t,u(t)) \leq 0,\ t \in [0,1].$$

By comparison result $m(t) \leq 0$, $t \in [0,1]$. Similarly, we can show that $u(t) \leq \beta(t)$, $t \in [0,1]$. □

Now, to develop the the iterative scheme, the generalized quasilinearization, choose a function $\phi(t,u)$ with ϕ, ϕ_u, $\phi_{uu} \in C([0,1]\times\mathbb{R})$ such that $\frac{\partial^2}{\partial u^2}\phi(t,u) \geq 0$ for every $t \in [0,1]$ and $u \in [\bar{\alpha},\bar{\beta}]$ and

$$\frac{\partial^2}{\partial u^2}[f(t,u) + \phi(t,u)] \geq 0 \text{ on } [0,1] \times [\bar{\alpha},\bar{\beta}], \tag{3.4}$$

where $\bar{\alpha} = \min\{\alpha(t) : t \in [0,1]\}$ and $\bar{\beta} = \max\{\beta(t) : t \in [0,1]\}$.

Define $F : [0,1]\times\mathbb{R} \to \mathbb{R}$ by $F(t,u) = f(t,u) + \phi(t,u)$. Note that $F \in C([0,1]\times\mathbb{R})$ and

$$\frac{\partial^2}{\partial u^2}F(t,u) \geq 0 \text{ on } [0,1] \times [\bar{\alpha},\bar{\beta}], \tag{3.5}$$

which implies that

$$f(t,u) \geq f(t,y) + F_u(t,y)(u-y) - [\phi(t,u) - \phi(t,y)],\ t \in [0,1], \tag{3.6}$$

where u, $y \in [\bar{\alpha},\bar{\beta}]$. Using the non decreasing property of ϕ_u with respect to u on $[\bar{\alpha},\bar{\beta}]$ for each $t \in [0,1]$, we obtain

$$\phi(t,u) - \phi(t,y) = \phi_u(t,c)(u-y) \leq \phi_u(t,\bar{\beta})(u-y) \text{ for } u \geq y, \tag{3.7}$$

where u, $y \in [\bar{\alpha},\bar{\beta}]$ such that $y \leq c \leq u$. Substituting in (3.6), we have

$$f(t,u) \geq f(t,y) + [F_u(t,y) - \phi_u(t,\bar{\beta})](u-y) \geq f(t,y) + \lambda(u-y), \text{ for } u \geq y, \tag{3.8}$$

where $\lambda = \min\{0, \min\{F_u(t,\bar{\alpha}) - \phi_u(t,\bar{\beta}) :\ t \in [0,1]\}\}$. We note that $\lambda \leq F_u(t,z) - \phi_u(t,\bar{\beta}) \leq f_u(t,\bar{\beta}) : t \in [0,1]$.

Define $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$g(t,u,y) = f(t,y) + \lambda(u-y). \tag{3.9}$$

We note that $g(t,u,y)$ is continuous on $[0,1] \times \mathbb{R} \times \mathbb{R}$ and for $u, y \in [\bar{\alpha}, \bar{\beta}]$, using (3.8) and (3.9), we have

$$\begin{cases} f(t,u) \geq g(t,u,y), \text{for } u \geq y, \\ f(t,u) = g(t,u,u). \end{cases} \tag{3.10}$$

Now, we develop the iterative scheme to approximate the solution. As an initial approximation, we choose $w_0 = \alpha$ and consider the linear problem

$$-{}^c\mathcal{D}_{0+}u(t) = g(t,u(t),w_0(t)),\ t \in [0,1],\ 1 < q < 2,\ u'(0) = 0,\ u(1) = \xi u(\eta). \tag{3.11}$$

The definition of lower and upper solutions and (3.10) imply that

$$g(t,w_0(t),w_0(t)) = f(t,w_0(t)) \geq -{}^c\mathcal{D}^q_{0+}w_0(t),\ t \in [0,1],\ w_0'(0) \geq 0,\ w_0(1) \leq \xi w_0(\eta),$$
$$g(t,\beta(t),w_0(t)) \leq f(t,\beta(t)) \leq -{}^c\mathcal{D}^q_{0+}\beta(t),\ t \in [0,1],\ \beta'(0) \leq 0,\ \beta(1) \geq \xi\beta(\eta),$$

which imply that w_0 and β are lower and upper solutions of (3.11). Hence by Theorem 3.1, there exists a solution $w_1 \in C[0,1]$ of (3.11) such that $w_0 \leq w_1 \leq \beta$ on $[0,1]$. Again, from (3.10) and the fact that w_1 is a solution of (3.11), we obtain

$$-{}^c\mathcal{D}^q_{0+}w_1(t) = g(t,w_1(t),w_0(t)) \leq f(t,w_1(t)),\ t \in [0,1],\ w_1'(0) = 0,\ w_1(1) = \xi w_1(\eta) \tag{3.12}$$

which implies that w_1 is a lower solution of (2.1).

Similarly, we can show that w_1 and β are lower and upper solutions of the linear problem

$$-{}^c\mathcal{D}^q_{0+}u(t) = g(t,u(t),w_1(t)),\ t \in [0,1],\ 1 < q < 2,\ u'(0) = 0,\ u(1) = \xi u(\eta). \tag{3.13}$$

Hence by Theorem 3.1, there exists a solution $w_2 \in C[0,1]$ of (3.13) such that $w_1 \leq w_2 \leq \beta$ on $[0,1]$. Continuing in the above fashion, we obtain a bounded monotone sequence $\{w_n\}$ of solutions of linear problems satisfying

$$w_0 \leq w_1 \leq w_2 \leq w_3 \leq ... \leq w_n \leq \beta \text{ on } [0,1], \tag{3.14}$$

where the element w_n of the sequence is a solution of the linear problem

$$-{}^c\mathcal{D}^q_{0+}u(t) = g(t,u(t),w_{n-1}(t)),\ t \in [0,1],\ 1 < q < 2,\ u'(0) = 0,\ u(1) = \xi u(\eta)$$

and is given by

$$w_n(t) = \int_0^1 G(t,s)g(s, w_n(s), w_{n-1}(s))ds,\ t \in [0,1]. \tag{3.15}$$

The monotonicity and uniform boundedness of the sequence $\{w_n\}$ implies the existence of a pointwise limit w on $[0,1]$ such that $w_n \to w$ uniformly. The dominated convergence theorem implies that for each $t \in [0,1]$,

$$\int_0^1 G(t,s)g(s, w_n(s), w_{n-1}(s))ds \to \int_0^1 G(t,s)f(s, w(s))ds.$$

Passing to the limit as $n \to \infty$, (3.15) yields $w(t) = \int_0^1 G(t,s)f(s,w(s))ds$, $t \in [0,1]$, which implies that w is a solution of (2.1).

To show that the convergence is quadratic, set $e_n(t) = w(t) - w_n(t)$, $t \in [0,1]$, where w is a solution of (2.1). Then, $e_n(t) \geq 0$ on $[0,1]$ and from the boundary conditions, we have $e_n'(0) = 0$, $e_n(1) = \xi e_n(\eta)$. Moreover, for every $t \in [0,1]$, we have

$$-{}^c\mathcal{D}_{0+}^q e_n(t) = F(t, w(t)) - \phi(t, w(t)) - f(t, w_{n-1}(t)) - \lambda(w_n(t) - w_{n-1}(t)). \tag{3.16}$$

Using the mean value theorem and the fact that $\phi_{uu} \geq 0$ on $[0,1] \times [\bar{\alpha}, \bar{\beta}]$, we obtain,

$$\begin{aligned}\phi(t, w(t)) &\geq \phi(t, w_{n-1}(t)) + \phi_u(t, w_{n-1}(t))(w(t) - w_{n-1}(t)) \\ &\geq \phi(t, w_{n-1}(t)) + \phi_u(t, \bar{\alpha})(w(t) - w_{n-1}(t)),\end{aligned}$$

$$\begin{aligned}F(t, w(t)) &= F(t, w_{n-1}(t)) + F_u(t, w_{n-1}(t))(w(t) - w_{n-1}(t)) + \frac{F_{uu}(t,\delta)}{2}(w(t) - w_{n-1}(t))^2 \\ &\leq F(t, w_{n-1}(t)) + F_u(t, \bar{\beta})(w(t) - w_{n-1}(t)) + \frac{F_{uu}(t,\delta)}{2}(w(t) - w_{n-1}(t))^2,\end{aligned}$$

where $w_{n-1} \leq \delta \leq w$. Consequently,

$$\begin{aligned}F(t, w(t)) - \phi(t, w(t)) \leq f(t, w_{n-1}(t)) &+ [F_u(t, \bar{\beta}) - \phi_u(t, \bar{\alpha})](w(t) - w_{n-1}(t)) \\ &+ \frac{F_{uu}(x,\delta)}{2}(w(t) - w_{n-1}(t))^2.\end{aligned}$$

Hence the equation (3.16) can be rewritten as

(3.17)
$$\begin{aligned}
-\mathcal{D}^q e_n(t) &\le [F_u(t,\bar\beta) - \phi_u(t,\bar\alpha)]e_{n-1}(t) + \frac{F_{uu}(t,\delta)}{2}(e_{n-1}(t))^2 - \lambda(e_{n-1}(t) - e_n(t)) \\
&\le [F_u(t,\bar\beta) - \phi_u(t,\bar\alpha) - \lambda]e_{n-1}(t) + \lambda e_n(t) + \frac{F_{uu}(t,\delta)}{2}(e_{n-1}(t))^2 \\
&\le [F_u(t,\bar\beta) - \phi_u(t,\bar\alpha) - \lambda]e_{n-1}(t) + \frac{F_{uu}(t,\delta)}{2}(e_{n-1}(t))^2 \le \rho e_n(t) + d\|e_{n-1}\|^2,\ t \in [0,1],
\end{aligned}$$

where $\rho = \max\{F_u(t,\bar\beta) - \phi_u(t,\bar\alpha) - \lambda :\ t \in [0,1]\} \ge 0$ and $d = \max\{\frac{F_{uu}(t,y)}{2} :\ y \in [\bar\alpha,\bar\beta]\}$. By comparison result, $e_n(t) \le z(t)$, $t \in [0,1]$, where $z(t)$ is a unique solution of the linear BVP

$$-{}^c\mathcal{D}^q_{0+}z(t) - \rho z(t) = d\|e_{n-1}\|^2,\ z'(0) = 0,\ z(1) = \xi z(\eta), \tag{3.18}$$

and is given by

$$e_n(t) \le z(t) = \int_0^1 k(t,s) d\|e_{n-1}\|^2 \le A\|e_{n-1}\|^2, \tag{3.19}$$

where $A = \max\{d\int_0^1 k(t,s)\}$, $k(t,s)$ is the Green's function corresponding to the homogenous problem $-{}^c\mathcal{D}^q_{0+}u(t) - \rho u(t) = 0$, $u'(0) = 0$, $u(1) = \xi u(\eta)$. Hence the convergence is quadratic.

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