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Existence of solutions for a class of second-order sublinear and linear Hamiltonian systems with impulsive effects *

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Abstract: By using the saddle point theorem, some new existence theorems are obtained for second-order Hamiltonian systems with impulsive effects in the cases when the gradient of the nonlinearity grows sublinearly and grows linearly respectively. Our results generalize some existing results and our conditions on the potential are rather relaxed.

Key Words: Hamiltonian systems; Impulse; Critical point theory; Grow sublinearly; Grow linearly.

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1. Introduction

Consider the second-order Hamiltonian systems with impulsive effects

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \\ \Delta \dot{u}^i(t_j) = \dot{u}^i(t_j^+) - \dot{u}^i(t_j^-) = I_{ij}(u^i(t_j)), i = 1, 2, ..., N; j = 1, 2, ..., m. \end{cases} \tag{1.1}$$

where $T > 0, t_0 = 0 < t_1 < t_2 < ... < t_m < t_{m+1} = T, u(t) = (u^1(t), u^2(t), ..., u^N(t)), I_{ij} : \mathbb{R} \to \mathbb{R}(i = 1, 2, ..., N; j = 1, 2, ..., m.)$ are continuous and and $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A) *$F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1([0, T], \mathbb{R}^+)$ such that*

$$|F(t, x)| \le a(|x|)b(t), \quad |\nabla F(t, x)| \le a(|x|)b(t)$$

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for all $x \in \mathbb{R}^N$ *and a.e.* $t \in [0,T]$.

For the sake of convenience, in the sequel, we define $A = \{1, 2, ..., N\}, B = \{1, 2, ..., m\}$.

When $I_{ij} \equiv 0$, (1.1) reduces to the second order Hamiltonian system, it has been proved that problem (1.1) has at least one solution by the least action principle and the minimax methods (see [2, 7-9, 11, 12, 15-18, 20-22, 25, 26]). Many solvability conditions are given, such as the coercive condition (see [2]), the periodicity condition (see [20]); the convexity condition (see [7]); the subadditive condition (see [15]); the bounded condition (see [8]).

When the nonlinearity $\nabla F(t,x)$ is bounded sublinearly, that is, there exist $f, g \in L^1([0,T], \mathbb{R}^+)$ and $\alpha \in [0,1)$ such that

$$|\nabla F(t,x)| \leq f(t)|x|^\alpha + g(t) \tag{1.2}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, Tang [17] also proved the existence of solutions for problem (1.1) when $I_{ij} \equiv 0$ under the condition

$$\lim_{|x|\to+\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt \to +\infty, \tag{1.3}$$

or

$$\lim_{|x|\to+\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt \to -\infty, \tag{1.4}$$

which generalizes Mawhin-Willem's results under bounded condition (see [8]).

When $\alpha = 1$, condition (1.2) reduces to the linearly bounded gradient condition, in this case, Zhao and Wu [21, 22] also proved the existence of solutions for problem (1.1) under the condition

$$\int_0^T f(t)dt < \frac{12}{T} \tag{1.5}$$

and (1.3) or (1.4) with $\alpha = 1$.

For $I_{ij} \not\equiv 0, i \in A, j \in B$, problem (1.1) is an impulsive differential problem. Impulsive differential equations arising from the real world describe the dynamics of processes in which sudden, discontinuous jumps occur. For the background, theory and applications of impulsive differential equations, we refer the readers to the monographs and some recent contributions as [1, 3, 4, 13, 20]. Some classical tools such as fixed point theorems in cones [1, 5, 19], the method of lower and upper solutions [3, 23] have been widely used to study impulsive differential equations.

Recently, the Dirichlet and periodic boundary conditions problems with impulses in the derivative are studied by variational method. For some general and recent works on the theory of critical point theory and variational methods, we refer the readers to [10, 14, 19, 27, 28]. It is a novel approach to apply variational methods to the impulsive boundary value problem (IBVP for short).

In the recent paper [28], based upon the conditions (1.3) and (1.4), Zhou and Li studied the existence of solutions for (1.1). However, there exists F neither satisfies (1.3) nor (1.4) in [28].

Let

$$F(t,x) = \sin\left(\frac{2\pi t}{T}\right)|x|^{7/4} + (0.6T - t)|x|^{3/2}.$$

It is easy to see that

$$|\nabla F(t,x)| \leq \frac{7}{4}\left|\sin\left(\frac{2\pi t}{T}\right)\right||x|^{3/4} + \frac{3}{2}|0.6T - t||x|^{1/2} \leq \frac{7}{4}\left(\left|\sin\left(\frac{2\pi t}{T}\right)\right| + \varepsilon\right)|x|^{3/4} + \frac{T^3}{\varepsilon^2}$$

for all $x \in \mathbb{R}^N$ and $t \in [0,T]$, where $\varepsilon > 0$. The above shows (1.2) holds with $\alpha = 3/4$ and

$$f(t) = \frac{7}{4}\left(\left|\sin\left(\frac{2\pi t}{T}\right)\right| + \varepsilon\right), \quad g(t) = \frac{T^3}{\varepsilon^2}.$$

However, $F(t,x)$ neither satisfies (1.3) nor (1.4). In fact,

$$|x|^{-2\alpha}\int_0^T F(t,x)dt = |x|^{-3/2}\int_0^T \left[\sin\left(\frac{2\pi t}{T}\right)|x|^{7/4} + (0.6T-t)|x|^{3/2}\right]dt = 0.1T^2.$$

The above example shows that it is valuable to improve (1.3) and (1.4) for the problem (1.1).

In the present paper, motivated by the above papers [15, 21, 22, 28], we study the existence of solutions for problem (1.1) under the condition (1.2). We will use the saddle point theorem in critical theory to generalize some results in [28]. In fact, we will establish some new existence criteria to guarantee that system (1.1) has at least one solutions under more relaxed assumptions on $F(t,x)$, which are independent from (1.3) and more general than (1.4) in [17] and [28], to our best knowledge, it seems not to have been considered in the literature.

2. Preliminaries

In this section, we recall some basic facts which will be used in the proofs of our main results. In order to apply the critical point theory, we construct a variational structure. With this variational structure, we can reduce the problem of finding solutions of (1.1) to that of seeking the critical points of a corresponding functional.

Let H_T^1 be the Sobolev space

$$H_T^1 = \left\{u : [0,T] \to \mathbb{R}^N \mid u \text{ is absolutely continuous, } u(0)=u(T),\ \dot{u}\in L^2([0,T],\mathbb{R}^N)\right\},$$

it is a Hilbert space with the inner product

$$< u, v >= \int_0^T (u(t),v(t))dt + \int_0^T (\dot{u}(t),\dot{v}(t))dt, \quad \forall\ u,v \in H_T^1,$$

the corresponding norm is defined by

$$\|u\|_{H_T^1} = \left(\int_0^T \left[|\dot{u}(t)|^2 + |u(t)|^2\right]dt\right)^{\frac{1}{2}}$$

for $u \in H_T^1$.

Let us recall that

$$\|u\|_{L^2} = \left(\int_0^T |u(t)|^2 dt\right)^{\frac{1}{2}} \quad \text{and} \quad \|u\|_\infty = \max_{t\in[0,T]}|u(t)|.$$

Definition 2.1.[28] We say that a function $u \in H_T^1$ is a weak solution of problem (1.1) if the identity

$$\int_0^T (\dot{u}(t),\dot{v}(t))dt + \sum_{j=1}^m\sum_{i=1}^N I_{ij}(u^i(t_j))v^i(t_j)) = -\int_0^T (\nabla F(t,u(t)),v(t))dt$$

holds for any $v \in H_T^1$.

The corresponding functional φ on H_T^1 given by

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t,u(t))dt + \sum_{j=1}^m\sum_{i=1}^N \int_0^{u^i(t_j)} I_{ij}(t)dt \\ &= \psi(u) + \phi(u) \qquad (2.1)\end{aligned}$$

where

$$\psi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, u(t))dt \quad \text{and} \quad \phi(u) = \sum_{j=1}^{m}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)dt.$$

It follows from assumption (A) that $\psi \in C^1(H_T^1, \mathbb{R})$. By the continuity of $I_{ij}, i \in A, j \in B$, one has that $\phi \in C^1(H_T^1, \mathbb{R})$. Thus, $\varphi \in C^1(H_T^1, \mathbb{R})$. For any $v \in H_T^1$, we have

$$< \varphi'(u), v >= \int_0^T (\dot{u}(t), \dot{v}(t))dt + \sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}(u^i(t_j)v^i(t_j)) + \int_0^T (\nabla F(t, u(t)), v(t))dt. \tag{2.2}$$

By Definition 2.1, the weak solutions of problem (1.1) correspond to the critical points of φ.

To prove our main results, we need the following definition and lemma.

Definition 2.2.[8] Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$. I is said to satisfy the (PS) condition on X if any sequence $\{x_n\} \subseteq X$ for which $I(x_n)$ is bounded and $I'(x_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence in X.

Lemma 2.1.[8] *For $u \in H_T^1$, let $\bar{u} = \frac{1}{T}\int_0^T u(t)dt$ and $\tilde{u}(t) = u(t) - \bar{u}$. Then one has*

$$\|\tilde{u}\|_\infty^2 \le \frac{T}{12}\int_0^T |\dot{u}(t)|^2 dt \quad (Sobolev's\ inequality), \tag{2.3}$$

and

$$\|\tilde{u}\|_{L^2}^2 \le \frac{T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2 dt \quad (Writinger's\ inequality). \tag{2.4}$$

3. Main results and Proofs

Theorem 3.1. *Suppose that (A) and (1.2) hold, and the following conditions are satisfied:*

(I1) *There exist $a_{ij}, b_{ij} > 0$ and $\beta_{ij} \in (0, 1), \gamma \in [0, \alpha)$ such that*

$$|I_{ij}(t)| \le a_{ij} + b_{ij}|t|^{\gamma\beta_{ij}}, \quad for\ every\ t \in \mathbb{R}, i \in A, j \in B; \tag{3.1}$$

(I2) *For any $i \in A, j \in B$,*

$$I_{ij}(t)t \le 0, \quad \forall\ t \in \mathbb{R}; \tag{3.2}$$

(F1)

$$\limsup_{|x|\to+\infty} |x|^{-2\alpha}\int_0^T F(t, x)dt < -\frac{T}{8}\left(\int_0^T f(t)dt\right)^2. \tag{3.3}$$

Then problem (1.1) has at least one weak solution in H_T^1.

Theorem 3.2. *Suppose that (A), (1.5), (I2) hold, and the following conditions are satisfied:*

(F2) *There exist $f, g \in L^1([0, T], \mathbb{R}^+)$ such that*

$$|\nabla F(t, x)| \le f(t)|x| + g(t) \tag{3.4}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(I3) *There exist $a_{ij}, b_{ij} > 0$ and $\beta_{ij} \in (0,1), \gamma \in (0,1)$ such that*

$$|I_{ij}(t)| \leq a_{ij} + b_{ij}|t|^{\gamma\beta_{ij}}, \quad \text{for every } t \in \mathbb{R}, i \in A, j \in B; \tag{3.5}$$

(F3)

$$\limsup_{|x|\to+\infty} |x|^{-2} \int_0^T F(t,x)dt \quad < \quad -\frac{3T}{2\left(12 - T\int_0^T f(t)dt\right)} \left(\int_0^T f(t)dt\right)^2. \tag{3.6}$$

Then problem (1.1) has at least one weak solution in H_T^1.

Throughout this paper, for the sake of convenience, we denote

$$M_1 = \int_0^T f(t)dt, \quad M_2 = \int_0^T g(t)dt. \tag{3.7}$$

$$a = \max_{i\in A, j\in B} a_{ij}, \quad b = \max_{i\in A, j\in B} b_{ij}. \tag{3.8}$$

Let $\delta_1, \delta_2, \delta_3, \delta_1', \delta_2', \delta_3'$ denote the positive number and fix

$$\delta_1 + \delta_2 + \delta_3 < \frac{1}{2}, \quad \delta_1' + \delta_2' + \delta_3' < \frac{1}{2} + \frac{TM_1}{24}. \tag{3.9}$$

Let

$$G(\delta_1, \delta_2, \delta_3) = \frac{(1+\delta_1)}{(\frac{1}{2} - \delta_1 - \delta_2 - \delta_3)},$$

when $\delta_1, \delta_2, \delta_3$ are small enough, it is easy to see that $G(\delta_1, \delta_2, \delta_3)$ is monotone increasing for every variable. Furthermore, we have

$$\lim_{(\delta_1,\delta_2,\delta_3)\to(0^+,0^+,0^+)} G(\delta_1, \delta_2, \delta_3) = 2. \tag{3.10}$$

Let

$$H(\delta_1', \delta_2', \delta_3') = \frac{(1 + \frac{TM_1}{24} + \delta_1')}{(\frac{1}{2} - \frac{TM_1}{24} - \delta_1' - \delta_2' - \delta_3')},$$

when $\delta_1', \delta_2', \delta_3'$ are small enough, noting that $M_1 < \frac{T}{12}$ (see (1.5) and (3.7)), it is easy to see that $H(\delta_1', \delta_2', \delta_3')$ is monotone increasing for every variable. Furthermore, we have

$$\lim_{(\delta_1',\ \delta_2',\ \delta_3')\to(0^+,\ 0^+,\ 0^+)} H(\delta_1', \delta_2', \delta_3') = \frac{24 + TM_1}{12 - TM_1}. \tag{3.11}$$

Now, we can prove our results.

Proof of Theorem 3.1. First, we prove that φ satisfies the (PS) condition. Suppose that $\{u_n\} \subset H_T^1$ is a (PS) sequence of φ, that is $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$. By (F1), we can choose an $a_1 > T/12$ such that

$$\limsup_{|x|\to+\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt < -\frac{3}{2}a_1 M_1^2. \tag{3.12}$$

It follows from (1.2) and Lemma 2.1 that

$$
\begin{aligned}
&\left|\int_0^T (F(t,u_n(t))-F(t,\bar{u}_n))dt\right| \\
= &\left|\int_0^T\int_0^1 (\nabla F(t,\bar{u}_n+s\tilde{u}_n(t)),\tilde{u}_n(t))dsdt\right| \\
\le &\int_0^T\int_0^1 f(t)\left|\bar{u}_n+s\tilde{u}_n(t)\right|^\alpha\left|\tilde{u}_n(t)\right|dsdt+\int_0^T\int_0^1 g(t)\left|\tilde{u}_n(t)\right|dsdt \\
\le &\int_0^T f(t)\left(|\bar{u}_n|^\alpha+|\tilde{u}_n(t)|^\alpha\right)\left|\tilde{u}_n(t)\right|dt+\int_0^T g(t)\left|\tilde{u}_n(t)\right|dt \\
\le &\left(|\bar{u}_n|^\alpha\|\tilde{u}_n\|_\infty+\|\tilde{u}_n\|_\infty^{\alpha+1}\right)\int_0^T f(t)dt+\|\tilde{u}_n\|_\infty\int_0^T g(t)dt \\
= &M_1|\bar{u}_n|^\alpha\|\tilde{u}_n\|_\infty+M_1\|\tilde{u}_n\|_\infty^{\alpha+1}+M_2\|\tilde{u}_n\|_\infty \\
\le &\frac{1}{2a_1}\|\tilde{u}_n\|_\infty^2+\frac{a_1}{2}M_1^2|\bar{u}_n|^{2\alpha}+M_1\|\tilde{u}_n\|_\infty^{\alpha+1}+M_2\|\tilde{u}_n\|_\infty \\
\le &\frac{T}{24a_1}\|\dot{u}_n\|_{L^2}^2+\frac{a_1}{2}M_1^2|\bar{u}_n|^{2\alpha}+\left(\frac{T}{12}\right)^{(\alpha+1)/2}M_1\|\dot{u}_n\|_{L^2}^{\alpha+1}+\left(\frac{T}{12}\right)^{1/2}M_2\|\dot{u}_n\|_{L^2} \\
\le &\frac{1}{2}\|\dot{u}_n\|_{L^2}^2+\frac{a_1}{2}M_1^2|\bar{u}_n|^{2\alpha}+\left(\frac{T}{12}\right)^{(\alpha+1)/2}M_1\|\dot{u}_n\|_{L^2}^{\alpha+1}+\left(\frac{T}{12}\right)^{1/2}M_2\|\dot{u}_n\|_{L^2},
\end{aligned}
$$

which means that

$$
\left|\int_0^T (\nabla F(t,u_n(t)),\tilde{u}_n(t))dt\right|\le\left(\frac{1}{2}+\delta_1\right)\|\dot{u}_n\|_{L^2}^2+\frac{a_1}{2}M_1^2|\bar{u}_n|^{2\alpha}+M_3, \tag{3.13}
$$

where M_3 is a positive constant dependent of the arbitrary positive number δ_1 which satisfies (3.9).

By (I1) and Lemma 2.1, we have

$$
\begin{aligned}
&\left|\sum_{j=1}^m\sum_{i=1}^N I_{ij}(u_n^i(t))\tilde{u}_n^i(t)\right| \\
\le &\sum_{j=1}^m\sum_{i=1}^N (a_{ij}+b_{ij}|u_n^i(t)|^{\gamma\beta_{ij}})|\tilde{u}_n^i(t)| \\
\le &\sum_{j=1}^m\sum_{i=1}^N (a_{ij}+b_{ij}|\bar{u}_n^i(t)+\tilde{u}_n^i(t)|^{\gamma\beta_{ij}})|\tilde{u}_n^i(t)| \\
\le &amN\|\tilde{u}_n\|_\infty+b\sum_{j=1}^m\sum_{i=1}^N 2(|\bar{u}_n|^{\gamma\beta_{ij}}+\|\tilde{u}_n\|_\infty^{\gamma\beta_{ij}})\|\tilde{u}_n\|_\infty \\
\le &amN\left(\frac{T}{12}\right)^{1/2}\|\dot{u}_n\|_{L^2}+b\sum_{j=1}^m\sum_{i=1}^N\beta_{ij}|\bar{u}_n|^{2\gamma} \\
&+2b\sum_{j=1}^m\sum_{i=1}^N\frac{2-\beta_{ij}}{2}\|\tilde{u}_n\|_\infty^{\frac{2}{2-\beta_{ij}}}+2b\sum_{j=1}^m\sum_{i=1}^N\|\tilde{u}_n\|_\infty^{\gamma\beta_{ij}+1} \\
\le &amN\left(\frac{T}{12}\right)^{1/2}\|\dot{u}_n\|_{L^2}+b\sum_{j=1}^m\sum_{i=1}^N\beta_{ij}|\bar{u}_n|^{2\gamma} \\
&+b\sum_{j=1}^m\sum_{i=1}^N(2-\beta_{ij})\left(\frac{T}{12}\int_0^T|\dot{u}_n|^2dt\right)^{\frac{1}{2-\beta_{ij}}}+2b\sum_{j=1}^m\sum_{i=1}^N\left(\frac{T}{12}\int_0^T|\dot{u}_n|^2dt\right)^{\frac{\gamma\beta_{ij}+1}{2}},
\end{aligned}
$$

which means that

$$\left|\sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}(u_n^i(t))\tilde{u}_n^i(t)\right| \leq \delta_2\|\dot{u}_n\|_{L^2}^2 + bmN|\bar{u}_n|^{2\gamma} + M_4 \tag{3.14}$$

for all u_n, where M_4 is a positive constant dependent of the arbitrary positive number δ_2 which satisfies (3.9).

Since $\lim_{n\to\infty}\varphi'(x_n) = 0$, we have by (3.13) and (3.14)

$$\begin{aligned}
\|\tilde{u}_n\| &\geq |\langle\varphi'(u_n), \tilde{u}_n\rangle| \\
&= \left|\|\dot{u}_n\|_{L^2}^2 + \int_0^T (\nabla F(t, u_n(t)), \tilde{u}_n(t))dt + \sum_{j=1}^{m}\sum_{i=1}^{N} I_{ij}(u_n^i(t))\tilde{u}_n^i(t)\right| \\
&\geq \left(\frac{1}{2} - \delta_1\right)\|\dot{u}_n\|_{L^2}^2 - \frac{a_1}{2}M_1^2|\bar{u}_n|^{2\alpha} - \delta_2\|\dot{u}_n\|_{L^2}^2 - bmN|\bar{u}_n|^{2\gamma} - M_4.
\end{aligned} \tag{3.15}$$

On the other hand, by (2.4), we have

$$\|\tilde{u}_n\| \leq \frac{(4\pi^2 + T^2)^{1/2}}{2\pi}\|\dot{u}_n\|_{L^2} \leq \delta_3\|\dot{u}_n\|_{L^2}^2 + M_5, \tag{3.16}$$

where M_5 is a positive constant dependent of the arbitrary positive number δ_3 which satisfies (3.9).

It follows from (3.9), (3.15) and (3.16) that there exists $M_6 > 0$ dependent of $\delta_1, \delta_2, \delta_3$ such that

$$\|\dot{u}_n\|_{L^2}^2 \leq \frac{a_1 M_1^2}{2\left(\frac{1}{2} - \delta_1 - \delta_2 - \delta_3\right)}|\bar{u}_n|^{2\alpha} + \frac{bmN}{\left(\frac{1}{2} - \delta_1 - \delta_2 - \delta_3\right)}|\bar{u}_n|^{2\gamma} + M_6. \tag{3.17}$$

Combining with (I2), (3.13) and (3.17), we have

$$\begin{aligned}
\varphi(u_n) &= \frac{1}{2}\|\dot{u}_n\|_{L^2}^2 + \int_0^T [F(t, u_n(t)) - F(t, \bar{u}_n)]\,dt + \int_0^T F(t, \bar{u}_n)dt + \phi(u_n) \\
&\leq (1 + \delta_1)\|\dot{u}_n\|_{L^2}^2 + \frac{a_1}{2}M_1^2|\bar{u}_n|^{2\alpha} + \int_0^T F(t, \bar{u}_n)dt + M_3 \\
&\leq \left[\frac{(1 + \delta_1)a_1 M_1^2}{2(\frac{1}{2} - \delta_1 - \delta_2 - \delta_3)} + \frac{a_1}{2}M_1^2 + |\bar{u}_n|^{-2\alpha}\int_0^T F(t, \bar{u}_n)dt\right]|\bar{u}_n|^{2\alpha} \\
&\quad + \frac{bmN}{\left(\frac{1}{2} - \delta_1 - \delta_2 - \delta_3\right)}|\bar{u}_n|^{2\gamma} + M_7
\end{aligned} \tag{3.18}$$

for some positive constant M_7 dependent of δ_1, δ_2 and δ_3.

We claim that $\{|\bar{u}_n|\}$ is bounded. In fact, if $\{|\bar{u}_n|\}$ is unbounded, we may assume that, going to a subsequence if necessary, $|\bar{u}_n| \to +\infty, \quad n \to +\infty$. It follows from (F1), (3.9), (3.10), (3.12) and (3.18) that

$$\varphi(u_n) \to -\infty, \quad n \to \infty.$$

which contradicts the boundedness of $\{\varphi(u_n)\}$ (see (PS) condition). Hence $\{|\bar{u}_n|\}$ is bounded. Then, it follows from (3.16), (3.17) and the boundedness of $\{|\bar{u}_n|\}$ that $\{u_n\}$ is bounded in H_T^1, going if necessary to a subsequence, we can assume that

$$u_n \rightharpoonup u_0 \quad \text{in} \quad H_T^1, \tag{3.19}$$

by Proposition 1.2 in [8], we have

$$u_n \to u_0 \quad \text{in} \quad C([0, T], \mathbb{R}^N). \tag{3.20}$$

It follows from (2.2) that

$$\begin{aligned}
&\langle \varphi'(u_n) - \varphi'(u_0), u_n - u_0 \rangle \\
= &\int_0^T |\dot{u}_n(t) - \dot{u}(t)|^2 dt \\
&+ \int_0^T (\nabla F(t, u_n(t)) - \nabla F(t, u_0(t)), u_n(t) - u_0(t)) dt \\
&+ \sum_{j=1}^{m} \sum_{i=1}^{N} (I_{ij}(u_n^i(t_j)) - I_{ij}(u^i(t_j)))(u_n^i(t_j)) - u^i(t_j))).
\end{aligned} \tag{3.21}$$

From (3.19)-(3.21), (A) and the continuity of I_{ij}, it follows that $u_n \to u$ in H_T^1. Thus, φ satisfies the (PS) condition.

Let $\tilde{H}_T^1 = \{u \in H_T^1 \mid \bar{u} = 0\}$. Then $H_T^1 = \tilde{H}_T^1 \oplus \mathbb{R}^N$.

In order to use the saddle point theorem ([12], Theorem 4.6), we only need to verify the following conditions:

(A_1) $\varphi(x) \to -\infty$ as $|x| \to \infty$ in $\mathbb{R}^N$.
(A_2) $\varphi(u) \to +\infty$ as $\|u\| \to \infty$ in $\tilde{H}_T^1$.

In fact, by (F1), we get

$$\int_0^T F(t, x) dt \to -\infty \quad \text{as } |x| \to \infty \text{ in } \mathbb{R}^N. \tag{3.22}$$

From (I2) and (3.22), we have

$$\varphi(x) = \int_0^T F(t, x) dt + \phi(x) \to -\infty \quad \text{as } |x| \to \infty \text{ in } \mathbb{R}^N.$$

Thus, (A_1) is verified.

Next, for all $u \in \tilde{H}_T^1$, by (1.2) and Sobolev's inequality, we have

$$\begin{aligned}
&\left| \int_0^T [F(t, u(t)) - F(t, 0)] dt \right| \\
= &\left| \int_0^T \int_0^1 (\nabla F(t, su(t)), u(t)) ds dt \right| \\
\leq &\int_0^T f(t) |u(t)|^{\alpha+1} dt + \int_0^T g(t) |u(t)| dt \\
\leq &M_1 \|u\|_\infty^{\alpha+1} + M_2 \|u\|_\infty \\
\leq &\left(\frac{T}{12} \right)^{(\alpha+1)/2} M_1 \|\dot{u}\|_{L^2}^{\alpha+1} + \left(\frac{T}{12} \right)^{1/2} M_2 \|\dot{u}\|_{L^2}.
\end{aligned} \tag{3.23}$$

It derives from (I1) that

$$
\begin{aligned}
|\phi(u)| &= |\sum_{j=1}^{m}\sum_{i=1}^{N}\int_0^{u^i(t_j)} I_{ij}(t)dt| \\
&\leq \sum_{j=1}^{m}\sum_{i=1}^{N}\int_0^{u^i(t_j)} (a_{ij}+b_{ij}|t|^{\gamma\beta_{ij}})dt \\
&\leq amN\|u\|_\infty + b\sum_{j=1}^{m}\sum_{i=1}^{N}\|u\|_\infty^{\gamma\beta_{ij}+1} \\
&\leq amN\left(\frac{T}{12}\right)^{1/2}\|\dot{u}\|_{L^2} + b\sum_{j=1}^{m}\sum_{i=1}^{N}\left(\frac{T}{12}\right)^{\frac{\gamma\beta_{ij}+1}{2}}\|\dot{u}\|_{L^2}^{\frac{\gamma\beta_{ij}+1}{2}}. \qquad (3.24)
\end{aligned}
$$

It follows from (2.3), (3.23) and (3.24) that

$$
\begin{aligned}
\varphi(u) &= \frac{1}{2}\|\dot{u}\|_{L^2}^2 + \int_0^T [F(t,u(t))-F(t,0)]dt + \int_0^T F(t,0)dt + \phi(u) \\
&\geq \frac{1}{2}\|\dot{u}\|_{L^2}^2 - \left(\frac{T}{12}\right)^{(\alpha+1)/2} M_1\|\dot{u}\|_{L^2}^{\alpha+1} - \left(\frac{T}{12}\right)^{1/2} M_2\|\dot{u}\|_{L^2} + \int_0^T F(t,0)dt \\
&\quad - amN\left(\frac{T}{12}\right)^{1/2}\|\dot{u}\|_{L^2} - b\sum_{j=1}^{m}\sum_{i=1}^{N}\left(\frac{T}{12}\right)^{\frac{\gamma\beta_{ij}+1}{2}}\|\dot{u}\|_{L^2}^{\frac{\gamma\beta_{ij}+1}{2}} \qquad (3.25)
\end{aligned}
$$

for all $u \in \tilde{H}_T^1$. By (2.4), $\|u\| \to \infty$ in $\tilde{H}_T^1$ if and only if $\|\dot{u}\|_{L^2} \to \infty$. So we obtain $\varphi(u) \to +\infty$ as $\|u\| \to \infty$ in $\tilde{H}_T^1$ from (3.25), i.e. (A_2) is verified. The proof of Theorem 3.1 is complete.

Proof of Theorem 3.2. Firstly, we prove that φ satisfies the (PS) condition. Suppose that $\{u_n\} \subset H_T^1$ is a (PS) sequence of φ, that is $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$. By (F3) and (1.5), we can choose an $a_2 > \frac{T}{12}$ such that

$$
\limsup_{|x|\to+\infty} |x|^{-2\alpha}\int_0^T F(t,x)dt < -\frac{18a_2}{12-TM_1}M_1^2. \qquad (3.26)
$$

It follows from (F2) and Lemma 2.1 that

$$
\begin{aligned}
& \left|\int_0^T (F(t,u_n(t)) - F(t,\bar{u}_n))dt\right| \\
= & \left|\int_0^T \int_0^1 (\nabla F(t,\bar{u}_n + s\tilde{u}(t)), \tilde{u}_n(t))dsdt\right| \\
\leq & \int_0^T \int_0^1 f(t)\left(|\bar{u}_n| + s|\tilde{u}_n(t)|\right)|\tilde{u}_n(t)|\,dsdt + \int_0^T \int_0^1 g(t)\,|\tilde{u}_n(t)|\,dsdt \\
= & \int_0^T f(t)\left(|\bar{u}_n| + \frac{1}{2}\,|\tilde{u}_n(t)|\right)|\tilde{u}_n(t)|\,dt + \int_0^T g(t)\,|\tilde{u}_n(t)|\,dt \\
\leq & \left(|\bar{u}_n|\,\|\tilde{u}_n\|_\infty + \frac{1}{2}\,\|\tilde{u}_n\|_\infty^2\right)\int_0^T f(t)dt + \|\tilde{u}_n\|_\infty \int_0^T g(t)dt \\
= & M_1|\bar{u}_n|\|\tilde{u}\|_\infty + \frac{M_1}{2}\|\tilde{u}_n\|_\infty^2 + M_2\|\tilde{u}_n\|_\infty \\
\leq & \frac{1}{2a_2}\|\tilde{u}_n\|_\infty^2 + \frac{a_2}{2}M_1^2|\bar{u}_n|^2 + \frac{M_1}{2}\|\tilde{u}_n\|_\infty^2 + M_2\|\tilde{u}_n\|_\infty \\
\leq & \left(\frac{T}{24a_2} + \frac{TM_1}{24}\right)\|\dot{u}_n\|_{L^2}^2 + \frac{a_2}{2}M_1^2|\bar{u}_n|^2 + \left(\frac{T}{12}\right)^{1/2} M_2\|\dot{u}_n\|_{L^2} \\
\leq & \left(\frac{1}{2} + \frac{TM_1}{24}\right)\|\dot{u}_n\|_{L^2}^2 + \frac{a_2}{2}M_1^2|\bar{u}_n|^2 + \left(\frac{T}{12}\right)^{1/2} M_2\|\dot{u}_n\|_{L^2},
\end{aligned}
$$

which means that

$$
\left|\int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))dt\right| \leq \left(\frac{1}{2} + \frac{TM_1}{24} + \delta_1'\right)\|\dot{u}\|_{L^2}^2 + \frac{a_2}{2}M_1^2|\bar{u}_n|^2 + M_3', \tag{3.27}
$$

where M_3' is a positive constant dependent of the arbitrary positive number δ_1' which satisfies (3.9).

By (1.5), (2.2) and (3.27), we have

$$
\begin{aligned}
\|\tilde{u}_n\| \geq & \ |\langle \varphi'(u_n), \tilde{u}_n\rangle| \\
= & \left|\|\dot{u}_n\|_{L^2}^2 + \int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))dt + \sum_{j=1}^m \sum_{i=1}^N I_{ij}(u_n^i(t))\tilde{u}_n^i(t)\right| \\
\geq & \left(\frac{1}{2} - \frac{TM_1}{24} - \delta_1'\right)\|\dot{u}_n\|_{L^2}^2 - \frac{a_2}{2}M_1^2|\bar{u}_n|^2 \\
& -\delta_2'\|\dot{u}_n\|_{L^2}^2 - bmN|\bar{u}_n|^{2\gamma} - M_4',
\end{aligned} \tag{3.28}
$$

where M_4' is a positive constant dependent of the arbitrary positive number δ_2' which satisfies (3.9).

On the other hand, by (2.4), we have

$$
\|\tilde{u}_n\| \leq \frac{(4\pi^2 + T^2)^{1/2}}{2\pi}\|\dot{u}_n\|_{L^2} \leq \delta_3'\|\dot{u}_n\|_{L^2}^2 + M_5', \tag{3.29}
$$

where M_5' is a positive constant dependent of the arbitrary positive number δ_3' which satisfies (3.9).

It follows from (3.28) and (3.29) that there exists $M_6' > 0$ dependent of δ_1', δ_2' and δ_3' such that

$$
\|\dot{u}_n\|_{L^2}^2 \leq \frac{a_2 M_1^2}{2\left(\frac{1}{2} - \frac{TM_1}{24} - \delta_1' - \delta_2' - \delta_3'\right)}|\bar{u}_n|^2 + \frac{bmN}{\left(\frac{1}{2} - \frac{TM_1}{24} - \delta_1' - \delta_2' - \delta_3'\right)}|\bar{u}_n|^{2\gamma} + M_6'. \tag{3.30}
$$

In a way similar to the proof of Theorem 3.1, we have

$$\begin{aligned}
& \left| \int_0^T (F(t, u_n(t)) - F(t, \bar{u}_n)) dt \right| \\
= & \left| \int_0^T \int_0^1 (\nabla F(t, \bar{u}_n + s\tilde{u}_n(t)), \tilde{u}_n(t)) ds dt \right| \\
\leq & \left(\frac{1}{2} + \delta_1' + \frac{TM_1}{24} \right) \|\dot{u}_n\|_{L^2}^2 + \frac{a_2}{2} M_1^2 |\bar{u}_n|^2 + M_3'. \qquad (3.31)
\end{aligned}$$

By (I2) and (3.31), we have

$$\begin{aligned}
\varphi(u_n) = & \frac{1}{2}\|\dot{u}_n\|_{L^2}^2 + \int_0^T [F(t, u_n(t)) - F(t, \bar{u}_n)]\, dt + \int_0^T F(t, \bar{u}_n) dt + \phi(u) \\
\leq & \left(1 + \frac{TM_1}{24} + \delta_1' \right) \|\dot{u}_n\|_{L^2}^2 + \frac{a_2}{2} M_1^2 |\bar{u}_n|^2 + \int_0^T F(t, \bar{u}_n) dt + M_3' \\
\leq & \left[\frac{(1 + \frac{TM_1}{24} + \delta_1') a_2 M_1^2}{2(\frac{1}{2} - \frac{TM_1}{24} - \delta_1' - \delta_2' - \delta_3')} + \frac{a_2 M_1^2}{2} + \int_0^T F(t, \bar{u}_n) dt \right] |\bar{u}_n|^2 \\
& + \frac{bmN}{(\frac{1}{2} - \frac{TM_1}{24} - \delta_1' - \delta_2' - \delta_3')} |\bar{u}_n|^{2\gamma} + M_7' \qquad (3.32)
\end{aligned}$$

for some positive constant M_7' dependent of δ_1', δ_2' and δ_3'.

We claim that $\{|\bar{u}_n|\}$ is bounded. In fact, if $\{|\bar{u}_n|\}$ is unbounded, we may assume that, going to a subsequence if necessary, $|\bar{u}_n| \to +\infty, \quad n \to +\infty$.

It follows from (F3), (1.5), (3.26) and (3.32) that

$$\varphi(u_n) \to -\infty, \quad n \to \infty,$$

which contradicts the boundedness of $\{\varphi(u_n)\}$ (see (PS) condition). Hence $\{|\bar{u}_n|\}$ is bounded. Arguing then as in the proof in Theorem 3.1, we conclude that the (PS) condition is satisfied.

Similar to the proof of Theorem 3.1, we only need to verify (A_1) and (A_2). It is easy to verify (A_1) by (3.6). In what follows, we verify that (A_2) also holds . For all $u \in \tilde{H}_T^1$, by (3.4) and Sobolev's inequality, we have

$$\begin{aligned}
& \left| \int_0^T [F(t, u(t)) - F(t, 0)] dt \right| \\
= & \left| \int_0^T \int_0^1 (\nabla F(t, su(t)), u(t)) ds dt \right| \\
\leq & \frac{1}{2} \int_0^T f(t) |u(t)|^2 dt + \int_0^T g(t) |u(t)| dt \\
\leq & \frac{M_1}{2} \|u\|_\infty^2 + M_2 \|u\|_\infty \\
\leq & \frac{TM_1}{24} \|\dot{u}\|_{L^2}^2 + \left(\frac{T}{12} \right)^{1/2} M_2 \|\dot{u}\|_{L^2}. \qquad (3.33)
\end{aligned}$$

Similar to the proof in (3.24), by (I3), we have

$$|\phi(u)| \leq amN \left(\frac{T}{12} \right)^{1/2} \|\dot{u}\|_{L^2} + b \sum_{j=1}^{m} \sum_{i=1}^{N} \left(\frac{T}{12} \right)^{\frac{\gamma\beta_{ij}+1}{2}} \|\dot{u}\|_{L^2}^{\frac{\gamma\beta_{ij}+1}{2}}. \qquad (3.34)$$

It follows from (2.1), (3.5) and (3.34) that

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\|\dot{u}\|_{L^2}^2+\int_0^T[F(t,u(t))-F(t,0)]dt+\int_0^T F(t,0)dt+\phi(u)\\ &\geq \frac{12-TM_1}{24}\|\dot{u}\|_{L^2}^2-\left(\frac{T}{12}\right)^{1/2}M_2\|\dot{u}\|_{L^2}+\int_0^T F(t,0)dt\\ &\quad -amN\left(\frac{T}{12}\right)^{1/2}\|\dot{u}\|_{L^2}-b\sum_{j=1}^{m}\sum_{i=1}^{N}\left(\frac{T}{12}\right)^{\frac{\gamma\beta_{ij}+1}{2}}\|\dot{u}\|_{L^2}^{\frac{\gamma\beta_{ij}+1}{2}}\end{aligned} \tag{3.35}$$

for all $u\in\tilde{H}_T^1$. By Wirtinger's inequality (see (2.4)), $\|u\|\to\infty$ in $\tilde{H}_T^1$ if and only if $\|\dot{u}\|_{L^2}\to\infty$. So we obtain $\varphi(u)\to+\infty$ as $\|u\|\to\infty$ in $\tilde{H}_T^1$ from (1.5) and (3.35), i.e. (A_2) is verified. The proof of Theorem 3.2 is complete.

4. Examples

In this section, we give some examples to illustrate our results.

Example 4.1. Let $T=0.0003,\ m=5,\ t_1=0.0002$, consider the second-order Hamiltonian systems with impulsive effects

$$\begin{cases}\ddot{u}(t)=\nabla F(t,u(t)), \quad \text{a.e. } t\in[0,T],\\ u(0)-u(0.0003)=\dot{u}(0)-\dot{u}(0.0003)=0,\\ \Delta\dot{u}^i(0.0002)=\dot{u}^i(0.0002^+)-\dot{u}^i(0.0002^-)=I_{i1}(u^i(0.0002)), i=1,2,...,N; j=1,2,3,4,5.\end{cases} \tag{4.1}$$

Let

$$F(t,x)=\sin\left(\frac{2\pi t}{T}\right)|x|^{7/4}+(0.4T-t)|x|^{3/2}+(h(t),x), I_{i1}=-t^{\frac{1}{7}}, \tag{4.2}$$

where $h\in L^1([0,T],\mathbb{R}^N),\gamma=\beta_{ij}=\frac{1}{\sqrt{7}}$. It is easy to see that

$$\begin{aligned}|\nabla F(t,x)| &\leq \frac{7}{4}\left|\sin\left(\frac{2\pi t}{T}\right)\right||x|^{3/4}+\frac{3}{2}|0.4T-t||x|^{1/2}+|h(t)|\\ &\leq \frac{7}{4}\left(\left|\sin\left(\frac{2\pi t}{T}\right)\right|+\varepsilon\right)|x|^{3/4}+\frac{T^3}{\varepsilon^2}+|h(t)|\end{aligned}$$

for all $x\in\mathbb{R}^N$ and a.e. $t\in[0,T]$, where $\varepsilon>0$. The above shows (1.2) holds with $\alpha=3/4$ and

$$f(t)=\frac{7}{4}\left(\left|\sin\left(\frac{2\pi t}{T}\right)\right|+\varepsilon\right), \qquad g(t)=\frac{T^3}{\varepsilon^2}+|h(t)|. \tag{4.3}$$

However, $F(t,x)$ neither satisfies (1.3) nor (1.4). In fact,

$$\begin{aligned}|x|^{-2\alpha}\int_0^T F(t,x)dt &= |x|^{-3/2}\int_0^T\left[\sin\left(\frac{2\pi t}{T}\right)|x|^{7/4}+(0.4T-t)|x|^{3/2}+(h(t),x)\right]dt\\ &= -0.1T^2+\left(\int_0^T h(t)dt,|x|^{-3/2}x\right).\end{aligned}$$

On the other hand, we have

$$\frac{T}{8}\left(\int_0^T f(t)dt\right)^2=\frac{T}{8}\left[\frac{7}{4}\int_0^T\left(\left|\sin\left(\frac{2\pi t}{T}\right)\right|+\varepsilon\right)dt\right]^2=\frac{49T^3}{128}\left(\frac{2}{\pi}+\varepsilon\right)^2.$$

We can choose ε sufficient small such that

$$\limsup_{|x|\to+\infty} |x|^{-2\alpha} \int_0^T F(t,x)dt = -0.1T^2 < -\frac{49T^3}{128}\left(\frac{2}{\pi}+\varepsilon\right)^2 = -\frac{T}{8}\left(\int_0^T f(t)dt\right)^2.$$

This shows that (F1) holds. By Theorem 3.1, problem (1.1) has at least one solution.

Example 4.2. Let $T=1, m=3, t_1=0.5$, consider the second-order Hamiltonian systems with impulsive effects

$$\begin{cases} \ddot{u}(t) = \nabla F(t,u(t)), \quad \text{a.e. } t\in[0,T], \\ u(0)-u(1) = \dot{u}(0)-\dot{u}(1) = 0, \\ \Delta \dot{u}^i(0.5) = \dot{u}^i(0.5^+) - \dot{u}^i(0.5^-) = I_{ij}(u^i(0.5)), i=1,2,...,N; j=1,2,3. \end{cases} \tag{4.4}$$

Let

$$F(t,x) = (0.4T-t)|x|^2 + t|x|^{3/2} + (h(t),x), \quad I_{i1} = -t^{\frac{1}{9}}, \tag{4.5}$$

where $h\in L^1([0,T],\mathbb{R}^N), \gamma=\beta_{ij}=\frac{1}{3}$. It is easy to see that

$$\begin{aligned} |\nabla F(t,x)| &\le 2|0.4T-t||x| + \frac{3t}{2}|x|^{1/2} + |h(t)| \\ &\le 2\left(|0.4T-t|+\varepsilon\right)|x| + \frac{T^2}{2\varepsilon} + |h(t)| \end{aligned}$$

for all $x\in\mathbb{R}^N$ and a.e. $t\in[0,T]$, where $\varepsilon>0$. The above shows (3.4) holds with

$$f(t) = 2\left(|0.4T-t|+\varepsilon\right), \qquad g(t) = \frac{T^2}{2\varepsilon} + |h(t)|. \tag{4.6}$$

Observe that

$$\begin{aligned} |x|^{-2}\int_0^T F(t,x)dt &= |x|^{-2}\int_0^T \left[(0.4T-t)|x|^2 + t|x|^{3/2} + (h(t),x)\right] dt \\ &= -0.1T^2 + 0.5T^2|x|^{-1/2} + \left(\int_0^T h(t)dt, |x|^{-2}x\right). \end{aligned}$$

On the other hand, we have

$$\int_0^T f(t)dt = 2\int_0^T \left(|0.4T-t|+\varepsilon\right)dt = 0.52T^2 + 2\varepsilon T,$$

$$\left(\int_0^T f(t)dt\right)^2 = (0.52T^2+2\varepsilon T)^2 = 0.2704T^4 + 2.08\varepsilon T^3 + 4\varepsilon^2T^2.$$

We choose $\varepsilon>0$ sufficient small such that

$$\int_0^T f(t)dt = 0.52T^2 + 2\varepsilon T < \frac{12}{T}$$

and

$$\begin{aligned} \limsup_{|x|\to+\infty} |x|^{-2}\int_0^T F(t,x)dt &= -0.1T^2 \\ &< -\frac{3T}{2\left(12 - T\int_0^T f(t)dt\right)}\left(\int_0^T f(t)dt\right)^2 \end{aligned}$$

These show that (F3) holds. By Theorem 3.2, problem (1.1) has at least one solution.

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