Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **72**, 1–15; https://doi.org/10.14232/ejqtde.2019.1.72 www.math.u-szeged.hu/ejqtde/

Resolution methods for mathematical models based on differential equations with Stieltjes derivatives

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Received 15 March 2019, appeared 2 October 2019
Communicated by Gennaro Infante

Abstract. Stieltjes differential equations, i.e. differential equations with usual derivatives replaced by derivatives with respect to given functions (derivators), are useful to model processes which exhibit dead times and/or sudden changes. These advantages of Stieltjes equations are exploited in this paper in the analysis of two real life models: first, the frictionless motion of a vehicle equipped with an electric engine and, second, the evolution of populations of cyanobacteria *Spirullina plantensis* in semicontinuous cultivation processes. Furthermore, this is not only a paper on applications of known results. For the adequate analysis of our mathematical models we first deduce the solution formula for Stieltjes equations with separate variables. Finally, we show that differential equations with Stieltjes derivatives reduce to ODEs when the derivator is continuous, thus obtaining another resolution method for more general cases.

Keywords: Stieltjes differential equations, dynamic equations, separation of variables, biological models.

2010 Mathematics Subject Classification: 34A36, 34K05, 34K05.

1 Introduction and preliminary results

In this paper we will obtain resolution methods for differential equations with Stieltjes derivatives to be applied in the exact computation of solutions of two mathematical models. This will be done in two different ways. We will first study Stieltjes differential equations with separate variables and later we show that, in general, Stieltjes differential equations are equivalent to ODEs under certain hypotheses.

Consider a Stieltjes differential system

$$x_g'(t) = f(t, x(t)), \quad t \in I = [t_0, t_0 + T], \quad x(t_0) = x_0 \tag{1.1}$$

where $T > 0$ and $x_0 \in \mathbb{R}$ are fixed, $f : I \times \mathbb{R} \to \mathbb{R}$ is a given function, and $x_g'(t)$ stands for the g-derivative of the unknown with respect to a nondecreasing and left-continuous derivator $g : \mathbb{R} \to \mathbb{R}$. The precise definition and background on g-derivatives are collected in Section 2.

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It is shown in [3, Section 3], see also [1, Section 8], that this kind of equation contains as particular cases Δ-differential equations on time scales or differential equations with countably many impulses.

In Section 3, we look at (1.1) in the particular case of separable variables, namely

$$x'_g(t) = c(t)f(x(t)), \quad t \geq t_0, \quad x(t_0) = x_0,$$

for which, under certain hypotheses, we obtain an explicit solution using the chain rule for g-derivatives. Then, in Section 4 we introduce and study our mathematical models in the form of these problems. We present a model for the motion of a vehicle and another one for a bacteria population. These examples are meant to show the interest of Stieltjes differential equations for modelling different processes which could hardly be studied by means of ordinary differential equations. It was precisely in the analysis of those models where we found the main idea for Section 5: Stieltjes differential equations of the form of (1.1) reduce to ODEs when the derivator is continuous, a most useful result in the exact computation of solutions to Stieltjes equations.

2 Preliminares

Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left-continuous function. Let us recall the definition of the g-derivative introduced in [3]. To that end, we first introduce the following two sets: the set of points around which g is constant,

$$C_g = \{s \in \mathbb{R} : g \text{ is constant on } (s-\varepsilon, s+\varepsilon) \text{ for some } \varepsilon > 0\},$$

and the set of discontinuity points of g that can be written as

$$D_g = \{s \in \mathbb{R} : g(s^+) - g(s) > 0\},$$

where $g(s^+)$ denotes the limit of g at s from the right. Now the g-derivative of a function $x : \mathbb{R} \to \mathbb{R}$ at a point $t \in \mathbb{R} \setminus C_g$ is

$$x'_g(t) = \begin{cases} \lim\limits_{s \to t} \dfrac{x(s) - x(t)}{g(s) - g(t)}, & \text{if } t \notin D_g, \\ \dfrac{x(t^+) - x(t)}{g(t^+) - g(t)}, & \text{if } t \in D_g \text{ and } t < t_0 + T, \end{cases}$$

provided that the corresponding limit exists.

Notice that we do not define g-derivatives at points $t \in C_g$, nor it is necessary because C_g is a null-measure set for μ_g (the Lebesgue–Stieltjes measure induced by g), see [3, Proposition 2.5]. Therefore, the differential equation in (1.1) is not really defined for $t \in I \cap C_g$. Roughly speaking, connected components of C_g correspond to negligible times, i.e. lapses when our system does not evolve at all. In turn, discontinuities of g correspond with times when sudden changes occur and which are usually introduced in models in the form of impulses. For the remaining set of times $I \setminus (C_g \cup D_g)$ we note that different slopes of the derivator g correspond to different influences of the corresponding times, namely, the bigger the slope of g the more important the corresponding times are for the process.

Finally, we recover an interesting set introduced in [3]. By definition, the set C_g is open in the usual topology, so it can be uniquely expressed as the countable union of open disjoint intervals, say

$$C_g = \bigcup_{n \in \mathbb{N}} (a_n, b_n). \tag{2.1}$$

Without loss of generality, we can assume that $a_n < a_{n+1}$ for all $n \in \mathbb{N}$. The set N_g is defined then as the endpoints of such intervals that are continuity point of g, that is

$$N_g = \{a_n, b_n : n \in \mathbb{N}\} \setminus D_g.$$

We also define $N_g^- = \{a_n : n \in \mathbb{N}\} \setminus D_g$ and $N_g^+ = \{b_n : n \in \mathbb{N}\} \setminus D_g$. Clearly, $N_g = N_g^- \cup N_g^+$.

Remark 2.1. Note that if $u \notin C_g \cup N_g \cup D_g$, then $g(v) \neq g(u)$ for $v = u$. Hence, if g is continuous, $u \notin C_g \cup N_g$ implies that $g(v) \neq g(u)$ for $v = u$.

By a solution of (1.1), we mean a function $x : [t_0, t_0 + T] \to \mathbb{R}$ such that $x(t_0) = x_0$ and x is g-absolutely continuous function in the sense of the definition included in the following Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral [3, Theorem 5.4].

Theorem 2.2 (Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral)**.** *Let $a, b \in \mathbb{R}$, $a < b$, and $F : [a, b] \to \mathbb{R}$. The following conditions are equivalent.*

(1) *The function F is* absolutely continuous with respect to g on $[a, b]$ *(or* g-absolutely continuous*) according to the following definition: to each $\varepsilon > 0$ there is some $\delta > 0$ such that, for any family $\{(a_n, b_n)\}_{n=1}^m$ of pairwise disjoint open subintervals of $[a, b]$, the inequality*

$$\sum_{n=1}^{m} (g(b_n) - g(a_n)) < \delta$$

implies

$$\sum_{n=1}^{m} |F(b_n) - F(a_n)| < \varepsilon.$$

(2) *The function F fulfills the following properties:*

(a) *There exists $F'_g(t)$ for g-almost all $t \in [a, b)$ (i.e., for all t except on a set of μ_g measure zero);*

(b) *$F'_g \in \mathcal{L}^1_g([a, b))$, the set of Lebesgue–Stieltjes integrable functions with respect to μ_g; and*

(c) *For each $t \in [a, b]$, we have*

$$F(t) = F(a) + \int_{[a,t)} F'_g(s)\, d\mu_g. \tag{2.2}$$

In this paper we consider integration in the Lebesgue–Stieltjes sense mainly, and we shall call "g-measurable" any function (or set) which is measurable with respect to the Lebesgue–Stieltjes σ-algebra generated by g. Moreover, integrals such as that in (2.2) shall be denoted also as

$$\int_{[a,t)} F'_g(s)\, dg(s).$$

For the particular case of $g(t) = t$, we have that $\mu_g = m$, the usual Lebesgue measure, for which we use the notation

$$\int_a^t F'(s)\, ds.$$

For properties of g-absolutely continuous functions we refer readers to [1, 3]. For convenience of readers, we include the following results.

Proposition 2.3 ([3, Proposition 5.3]). *If F is g-absolutely continuous on $[a,b]$, then it has bounded variation and it is continuous from the left at every $t \in [a,b)$.*

Moreover, F is continuous in $[a,b] \setminus D_g$, where D_g is the set of discontinuity points of g, and if g is constant on some interval $(\alpha, \beta) \subset [a,b]$, then F is constant on (α,β) as well.

Proposition 2.4 ([1, Proposition 5.3]). *Let $F_1 : [a,b] \to \mathbb{R}$ be g-absolutely continuous. Assume that $F_1([a,b]) \subset [c,d]$ for some $c,d \in \mathbb{R}$, $c < d$, and let $F_2 : [c,d] \to \mathbb{R}$ satisfy a Lipschitz condition on $[c,d]$. Then the composition $F_2 \circ F_1$ is g-absolutely continuous on $[a,b]$.*

Finally, we recall the chain rule for g-derivatives and g-differentiation of indefinite integrals, which we shall use in order to obtain the formula of the solution of a problem with separate variables.

Theorem 2.5 (Chain rule for g-derivatives [3, Theorem 2.3]). *Let f be a real-valued real function defined on a neighborhood of $t \in \mathbb{R} \setminus D_g$, and let h be another function defined in a neighborhood of $f(t)$. The following results hold for the g-derivative of the composition $h \circ f$ at t:*

1. *If there exist $h'(f(t))$ and $f'_g(t)$, then there exists*
$$(h \circ f)'_g(t) = h'(f(t)) f'_g(t).$$
2. *If there exist $h'_g(f(t))$, $g'(f(t))$, and $f'_g(t)$, then there exists*
$$(h \circ f)'_g(t) = h'_g(f(t)) g'(f(t)) f'_g(t).$$

Theorem 2.6 ([3, Theorem 2.4, Proposition 5.2]). *Assume that $c : [a,b) \to \overline{\mathbb{R}}$ is integrable on $[a,b)$ with respect to μ_g and consider its indefinite Lebesgue–Stieltjes integral*

$$C(t) = \int_{[a,t)} c \, d\mu_g \quad \textit{for all } t \in [a,b].$$

Then C is g-absolutely continuous on $[a,b]$ and there is a g-measurable set $N \subset [a,b]$ such that $\mu_g(N) = 0$ and

$$C'_g(t) = c(t) \quad \textit{for all } t \in [a,b] \setminus N.$$

3 Separation of variables

This section is devoted to the explicit resolution of the separable initial value problem

$$x'_g(t) = c(t) f(x(t)), \quad t \geq t_0, \quad x(t_0) = x_0. \tag{3.1}$$

Note that for the particular case of $f(x) = x$ the problem has been solved in [1]. As in the ODE case, problem (3.1) can be solved with an exponential map, which we recall here for the convenience of the reader.

Definition 3.1. Let $c \in \mathcal{L}^1_g([a,b))$ be such that

$$c(t)\big(g(t^+) - g(t)\big) > -1 \quad \text{for every } t \in [a,b) \cap D_g, \tag{3.2}$$

and

$$\sum_{t \in [a,b) \cap D_g} \left| \log\left(1 + c(t)(g(t^+) - g(t))\right) \right| < \infty. \tag{3.3}$$

We define $e_c(\cdot, a) : [a,b] \to (0,\infty)$ by

$$e_c(t,a) = e^{\int_{[a,t)} \widetilde{c}(s)\, d\mu_g}, \tag{3.4}$$

where

$$\widetilde{c}(t) = \begin{cases} c(t) & \text{if } t \in [a,b] \setminus D_g, \\ \frac{\log\left(1+c(t)(g(t^+)-g(t))\right)}{g(t^+)-g(t)} & \text{if } t \in [a,b) \cap D_g. \end{cases} \tag{3.5}$$

Proposition 3.2 ([1, Lemma 6.3])**.** *Let $c \in \mathcal{L}^1_g([a,b))$ satisfy* (3.2) *and* (3.3)*. Then for every $x_a \in \mathbb{R}$ the mapping $t \mapsto x_a\, e_c(t,a)$ is g-absolutely continuous and solves the initial value problem*

$$x'_g(t) = c(t)x(t) \quad \textit{for } g\textit{-almost all } t \in [a,b),\ x(a) = x_a. \tag{3.6}$$

It is important to note that c has to be redefined at discontinuity points of g. Bearing this idea in mind, we consider the particular case of (3.1) corresponding to a continuous derivator g.

Theorem 3.3. *Let $c \in \mathcal{L}^1_{g,\mathrm{loc}}([t_0,+\infty))$ (i.e., c is g-integrable on compact subsets of $[t_0,\infty)$) and assume that there is some $R > 0$ such that f is continuous and positive on $J = (x_0 - R, x_0 + R)$. Define*

$$F(x) = \int_{x_0}^{x} \frac{dr}{f(r)} \quad \textit{for every } x \in J. \tag{3.7}$$

If there exists $r > 0$ such that

$$\int_{[t_0,t)} c(s)\, dg(s) \in F(J) \quad \textit{for all } t \in [t_0, t_0 + r), \textit{ and } g \textit{ is continuous on } [t_0, t_0 + r), \tag{3.8}$$

then a solution of (3.1) *is given by the following formula:*

$$x(t) = F^{-1}\left(\int_{[t_0,t)} c(s)\, dg(s)\right) \quad \textit{for all } t \in [t_0, t_0 + r). \tag{3.9}$$

Proof. Clearly, $x(t_0) = x_0$. Since F^{-1} is locally Lipschitzian, we can deduce from Proposition 2.4 and Theorem 2.6 that x is g-absolutely continuous on any interval $[t_0, t_0 + s]$, $s \in (0,r)$. In particular, there exists $x'_g(t)$ for g-almost all $t \in [t_0, t_0 + r)$. Using the chain rule (Theorem 2.5) and Theorem 2.6, we compute for g-almost all $t \in [t_0, t_0 + r)$

$$x'_g(t) = (F^{-1})'\left(\int_{[t_0,t)} c(s)\, dg(s)\right)\left(\int_{[t_0,\cdot)} c(s)\, dg(s)\right)'_g(t) = \frac{1}{F'(x(t))}c(t) = f(x(t))c(t).$$

□

Remark 3.4. Formula (3.9) is equivalent to

$$\int_{x_0}^{x(t)} \frac{dr}{f(r)} = \int_{[t_0,t)} c(s)\, dg(s).$$

Theorem 3.3 is false, in general, when g is discontinuous. Indeed, observe that (3.9) does not give (3.4) when f is the identity and g has at least one discontinuity point in $(t_0, t_0 + r)$. At discontinuity points of g we cannot use Theorem 2.5 to compute derivatives of compositions by means of the chain rule, so we need an alternative approach and an alternative formula for the solutions.

Here and henceforth, we assume that g is discontinuous exactly at the points of a sequence $\{\tau_k\}_{k=1}^{\infty}$, where

$$t_0 < \tau_1 < \tau_2 < \cdots .$$

Assuming that g is continuous at the initial time t_0 is not really a restriction, see [1, Section 5].

Remark 3.5. In general, D_g is just a countable set. Here we assume that D_g is discrete, i.e. all its elements are isolated points. We have no solution formula for the general case.

Solving (3.1) on the interval $[t_0, \tau_1)$ can be done with the aid of Theorem 3.3, because g is continuous on $[t_0, \tau_1)$. Furthermore, since solutions are continuous from the left everywhere, we get the solution on $[t_0, \tau_1]$ with the same formula. Specifically, under suitable conditions, a solution of (3.1) on the interval $[t_0, \tau_1]$ is implicitly given by the expression

$$\int_{x_0}^{x(t)} \frac{dr}{f(r)} = \int_{[t_0,t)} c(s)\, dg(s) \quad \text{for all } t \in [t_0, \tau_1]. \tag{3.10}$$

Obtaining the solution formula on the right of τ_1 is a matter of induction. First, according to the definition of g-derivative at discontinuity points, the differential equation in (3.1) for $t = \tau_1$ reads simply as follows:

$$x(\tau_1^+) = x(\tau_1) + c(\tau_1) f(x(\tau_1))(g(\tau_1^+) - g(\tau_1)) \equiv x_1. \tag{3.11}$$

Therefore, we have to solve another initial value problem

$$x_g'(t) = c(t) f(x(t)), \quad t \in (\tau_1, \tau_2], \quad x(\tau_1^+) = x_1, \tag{3.12}$$

by means of (3.9), with obvious modifications: under suitable conditions (see Remark 3.6), a solution of (3.12) is defined by

$$\int_{x_1}^{x(t)} \frac{dr}{f(r)} = \int_{(\tau_1,t)} c(s)\, dg(s) \quad \text{for all } t \in (\tau_1, \tau_2], \tag{3.13}$$

where x_1 is defined in (3.11).

Remark 3.6. Formula (3.13) gives a solution of (3.12) provided that, for instance, f is continuous and positive on $J = (x_1 - R, x_1 + R)$, for some $R > 0$, and

$$\int_{(\tau_1,t)} c(s)\, dg(s) \in F(J) \quad \text{for all } t \in (\tau_1, \tau_2],$$

where $F(x) = \int_{x_1}^{x} dr/f(r)$, $x \in J$.

Summing up, a solution of (3.1) can be recursively computed as follows: define $x(t)$ on $[t_0, \tau_1]$ by means of (3.10); assume that we have defined $x(t)$ on $[t_0, \tau_k]$, for some $k \in \{1, 2, \dots\}$, then compute the number

$$x_k = x(\tau_k) + c(\tau_k) f(x(\tau_k))(g(\tau_k^+) - g(\tau_k)), \tag{3.14}$$

and define $x(t)$ implicitly on $(\tau_k, \tau_{k+1}]$ by the expression

$$\int_{x_k}^{x(t)} \frac{dr}{f(r)} = \int_{(\tau_k,t)} c(s)\, dg(s) \quad \text{for all } t \in (\tau_k, \tau_{k+1}]. \tag{3.15}$$

4 Real life applications

4.1 An irregularly forced frictionless motion

We want to set up a simple model for the motion of a vehicle impulsed by an electric engine which we can turn on and off as often as we please. We disregard any other force. In particular, the speed increases when the engine is turned on, and the vehicle never slows down, it just keeps its speed when the engine is turned off.

Let $g(t)$ denote the number of seconds that the engine has been on until time t. This function $g(t)$ is continuous, nondecreasing, and constant on the time intervals when the engine is turned off.

Let $s(t)$ denote the vehicle's speed after t seconds. For simplicity, we assume that speed increases on every time interval $[t, t+h]$, $h > 0$, at a rate proportional to the time the engine has been on during that time interval. Moreover, we consider that accelerating the vehicle is harder at very slow or at very high speeds, so we assume a proportionality "constant" which depends on $s(t)$, at least for small values of $h > 0$. This leads to

$$s(t+h) - s(t) = f(s(t))(g(t+h) - g(t)),$$

which, considering the limit as $h \to 0^+$, yields the g-differential model

$$s_g'(t) = f(s(t)), \quad t \geq 0, \quad s(0) = s_0. \tag{4.1}$$

We suggest using logistic-type functions like

$$f(s) = \alpha \max\{0, (s+\beta)(s_{\max} - s)\}, \quad \text{for } s \geq 0,$$

where α, β and $s_{\max}$ are positive constants. Observe that $f(s) > 0$ for $s \in [0, s_{\max})$ and $f(s) = 0$ for $s \geq s_{\max}$, which means that the engine can accelerate the vehicle only when its speed belongs to the interval $[0, s_{\max})$. Observe also that, in case $\beta < s_{\max}$ (which we assume from now on), f attains a maximum at $(s_{\max} - \beta)/2$, which means that the engine is more efficient when the vehicle is moving at that specific value of speed.

As an instance, we consider

$$f(s) = \max\{0, (s+1)(2-s)\} \tag{4.2}$$

and we solve (4.1) for $s_0 = 0$. We note that f is continuous and positive for $s \in [0, 2)$, and a solution is implicitly given by

$$\int_0^{s(t)} \frac{dr}{(r+1)(2-r)} = \int_{[0,t)} dg(s) = g(t).$$

Elementary computations yield the solution

$$s(t) = \frac{2e^{3g(t)} - 2}{e^{3g(t)} + 2}, \quad t \geq 0. \tag{4.3}$$

Keeping the engine on at every time corresponds to the derivator $g(t) = t$ for all $t \geq 0$. The corresponding solution is displayed in Figure 4.1.

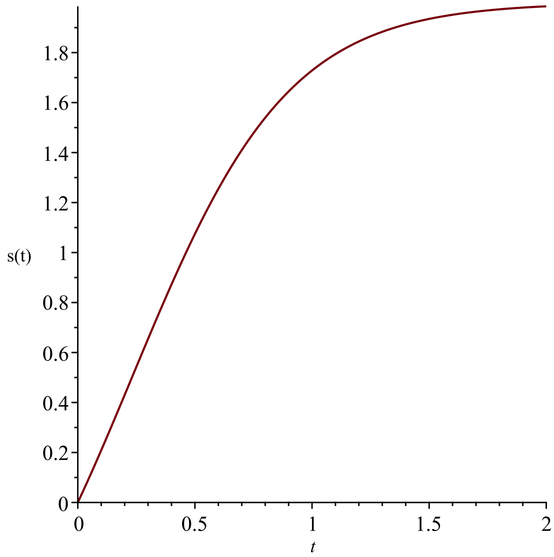

Figure 4.1: Solution (4.3) for $g(t) = t$ (engine constantly on).

If we turn the device off for $t \in [1/2, 1] \cup [3/2, 2]$, then we should take

$$g(t) = \begin{cases} t, & \text{for } t \in [0, 1/2], \\ 1/2, & \text{for } t \in [1/2, 1], \\ t - 1/2, & \text{for } t \in [1, 3/2], \\ 3/2, & \text{for } t \in [3/2, 2]. \end{cases}$$

The graph of g and the solution given by (4.3) can be seen in Figures 4.2 and 4.3, respectively.

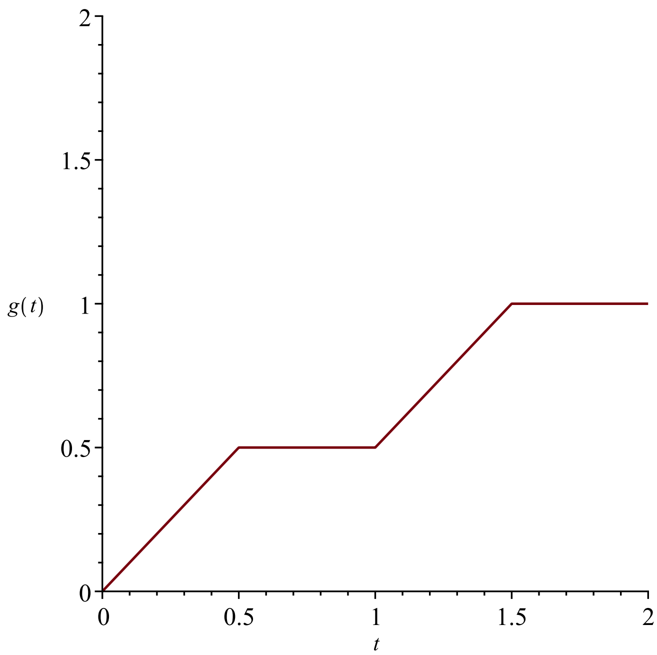

Figure 4.2: Graph of $g(t)$.

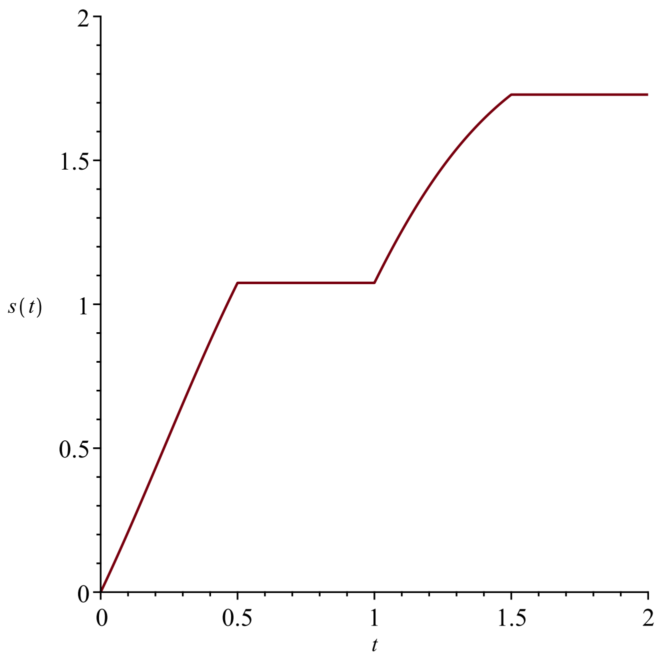

Figure 4.3: Solution (4.3) with dead times (engine off).

4.2 Semicontinuous culture systems

Semicontinuous cultivation is a system to produce bacteria in which a portion of the culture medium is periodically removed and the remaining culture is used as the starting point for continuation of the culture. For the model in this section we take into account [5], where semicontinuous cultivation of the cyanobacteria *Spirulina plantensis* is studied. We highlight the following important feature of the system in [5]: illumination was controlled to have a 12 hours light/dark photoperiod, which resulted in two different reproduction phases every day.

Bearing the above considerations in mind, we shall set up a mathematical model for the production of *Spirulina plantensis* in a semicontinuous culture system. First, we consider days as our time units. We assign light periods to be the time intervals $[k, k+1/2)$, $k = 0, 1, 2, \ldots,$ while dark periods are $[k+1/2, k+1)$, $k = 0, 1, 2, \ldots.$ Second, we assume that half of the culture is removed every 10 days and immediately refilled with new nutrients so that the remaining bacteria start reproducing again. Finally, we take as derivator a nondecreasing function $g : [0, \infty) \to [0, \infty)$, continuous everywhere with the exception of the positive multiples of 10, and such that

$$g'(t) = \begin{cases} 1, & \text{if } t \in (k, k+1/2), k = 0, 1, 2, \ldots, \\ 1/2, & \text{if } t \in (k+1/2, k+1), k = 0, 1, 2, \ldots, \end{cases}$$

and $g((10k)^+) - g(10k) = 1$ for $k = 1, 2, 3, \ldots$ A concise explicit expression for $g(t)$ can be obtained by defining first its values on the first day, namely

$$h(t) = \begin{cases} t, & \text{if } t \in [0, 1/2), \\ t/2 + 1/4, & \text{if } t \in (1/2, 1], \end{cases}$$

and then we can define the remaining values by "periodicity", and introducing jump discontinuities at relevant places, as

$$g(t) = h(t - [t]) + 3[t]/4 + [t/10],$$

where $[\cdot]$ stands for the floor function. Observe that we should modify the values $g(10k)$ $(k = 1, 2, \dots)$ so that g be left-continuous, but we shall not do it to avoid technicalities. See Figure 4.4 for a plot of this function.

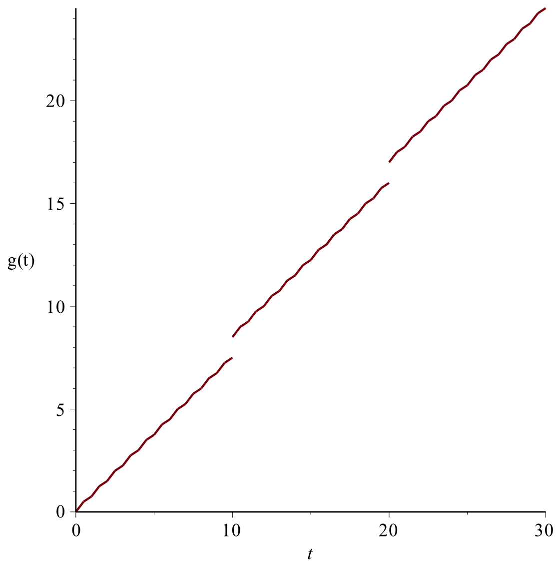

Figure 4.4: Graph of $g(t)$ for a semicontinuous bacteria culture. Observe different slopes for light and dark periods, and discontinuities at the renewal moments.

We are now ready to introduce a g-differential model for the biomass concentration $x(t)$, measured in grams per liter at time t, with a given initial concentration $x(0) = x_0$. Biomass concentration should satisfy

$$x'_g(t) = f(t, x(t)) \quad t \geq 0, \quad x(0) = x_0, \tag{4.4}$$

where $f(t, x)$ is assumed to be logistic except at the renewal moments (positive multiples of 10), when we remove half of the culture and immediately refill the flask with new nutrients. Specifically, we define

$$f(t, x) = \begin{cases} \alpha\, x(N - x), & \text{if } t \neq 10k,\ k = 1, 2, \dots, \\ -x/2, & \text{if } t = 10k,\ k = 1, 2, \dots, \end{cases}$$

where $\alpha > 0$ and $N > 0$ are biological parameters to be adjusted from experimental results.

Using the formulas (3.10) and (3.15), we compute the solution: for $t \in [0, 10]$ the solution is

$$x(t) = \frac{\dfrac{x_0 N}{N - x_0} e^{\alpha N g(t)}}{1 + \dfrac{x_0}{N - x_0} e^{\alpha N g(t)}}.$$

Assume we have computed $x(t)$ for all $t \in [0, 10k]$, for some $k = 1, 2, \dots$, then we define

$$x_k = x(10k^+) = \frac{x(10k)}{2},$$

and the solution for $t \in (10k, 10k + 10]$ is given by

$$x(t) = \frac{\dfrac{x_k N}{N - x_k} e^{\alpha N[g(t) - g(10k^+)]}}{1 + \dfrac{x_k}{N - x_k} e^{\alpha N[g(t) - g(10k^+)]}}.$$

Observe that $g(10k^+) = 8.5k$ for all $k = 1, 2, \dots$

See Figure 4.5 for a plot of the solution corresponding to $x_0 = 0.4$ grams per liter, $\alpha = 0.1$ and $N = 1.5$. These choices yield a good approximation of the experimental results obtained in [5] for the cyanobacteria *Spirulina platensis*, see [5, Figure 2].

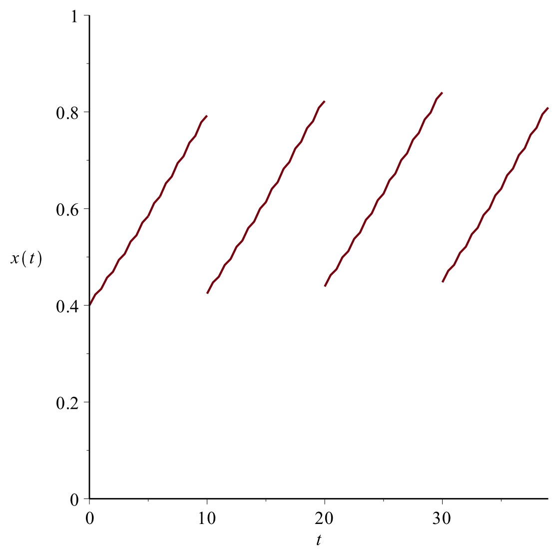

Figure 4.5: Biomass $x(t)$ grams per liter in a semicontinuous culture, with initial density of 0.4 g/L, and parameters $\alpha = 0.1$, $N = 1.5$.

5 Stieltjes equations with continuous derivators are just ODEs

The careful reader might have noticed that the solutions obtained for the previous examples are just the solutions of the corresponding ODEs, composed with the derivator g. In this section we will show that, under the assumption of continuity there is an equivalence between Stieltjes differential equations and ODEs.

It is pretty straightforward that a Stieltjes differential equation reduces to an ODE when the derivator is not only continuous, but also differentiable. Indeed, assume that $g \in \mathcal{C}^1([t_0, t_0 + T])$. Then, for any $t \in [t_0, t_0 + T] \setminus (C_g \cup N_g)$ and any $x : [t_0, t_0 + T] \to \mathbb{R}$ such that $x'_g(t)$ exists, it follows directly from the definition that $x'(t)$ exists and

$$x'(t) = x'_g(t) g'(t).$$

Therefore, if x solves (1.1) on $[t_0, t_0+T] \setminus (C_g \cup N_g)$ then x solves

$$x'(t) = f(t, x(t))g'(t), \quad t \in [t_0, t_0+T] \setminus (C_g \cup N_g). \tag{5.1}$$

More than that is true: since $g'(t) = 0$ for all $t \in C_g \cup N_g$, the ODE (5.1) is satisfied on the whole interval $I = [t_0, t_0+T]$.

Conversely, let $c \in \mathcal{L}^1([t_0, t_0+T])$, $c \geq 0$, and assume that x solves

$$x'(t) = f(t, x(t))c(t), \quad t \in I = [t_0, t_0+T]. \tag{5.2}$$

Then x solves (1.1) on $[t_0, t_0+T] \setminus N$ with

$$g(t) = \int_{t_0}^{t} c(s)\, ds, \quad t \in [t_0, t_0+T],$$

and $N = \{t \in I : c(t) = 0\}$.

In what follows, we will show that we can still transform a Stieltjes differential equation into an ODE when we change the hypothesis of differentiability for just continuity.

Consider equation (1.1) with $g : \mathbb{R} \to \mathbb{R}$ nondecreasing and continuous, i.e., $D_g = \emptyset$. Without loss of generality, we assume that $g(\mathbb{R}) = \mathbb{R}$ (if not, it suffices to redefine g linearly outside the interval $I = [t_0, t_0+T]$, which has no influence on the equation).

We define the pseudo-inverse of g as the function $\gamma : \mathbb{R} \to \mathbb{R}$ such that

$$\gamma(x) = \min\{t \in \mathbb{R} : g(t) = x\} \quad \text{for each } x \in \mathbb{R}. \tag{5.3}$$

This definition is good. To prove it, just notice that g is continuous, nondecreasing and $g(\pm\infty) = \pm\infty$, which implies that

$$g^{-1}(\{x\}) = \{t \in \mathbb{R} : g(t) = x\} \tag{5.4}$$

is a compact interval (even a singleton if $x \notin C_g \cup N_g$).

The most important properties of γ are gathered in the following statement.

Proposition 5.1. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing, continuous and $g(\mathbb{R}) = \mathbb{R}$.*

If $\gamma : \mathbb{R} \to \mathbb{R}$ is defined as in (5.3), then the following properties hold:

1. *for all $x \in \mathbb{R}$, $g(\gamma(x)) = x$;*
2. *for all $t \in \mathbb{R}$, $\gamma(g(t)) \leq t$;*
3. *for all $t \in \mathbb{R}$, $t \notin C_g \cup N_g^+$, $\gamma(g(t)) = t$;*
4. *γ is strictly increasing: $x < y$ implies $\gamma(x) < \gamma(y)$;*
5. *γ is left-continuous everywhere and continuous at every $x \in \mathbb{R}$, $x \notin g(C_g)$.*

Proof. Property 1 is a direct consequence of the definition (5.3): $g(\gamma(x))) = g(\min\{t : g(t) = x\}) = x$. For 2 observe that $\gamma(g(t)) = \min\{s : g(s) = g(t)\} \leq t$. To prove 3 just note that $t \notin C_g \cup N_g$ implies that the set (5.4) for $x = g(t)$ is the singleton $\{t\}$. Now, if $t \in N_g^-$, we have that $t = a_{n_0}$ for some $n_0 \in \mathbb{N}$. In that case, it is clear that $g(t) = g(s)$ for all $s \in (t, b_{n_0})$ and $g(s) < g(t)$ for all $s < t$, as any other case would lead to a contradiction. Thus, we have that

$$\gamma(g(t)) = \min\{s \in \mathbb{R} : g(t) = g(s)\} = \min(a_{n_0}, b_{n_0}) = a_{n_0} = t.$$

For 4 we fix $x < y$ and we note that if $t \in g^{-1}(\{x\})$ and $s \in g^{-1}(\{y\})$, then $t < s$, for otherwise we would have $x = g(t) \geq g(s) = y$, a contradiction. Hence $\gamma(x) = \min g^{-1}(\{x\}) < \min g^{-1}(\{y\}) = \gamma(y)$.

Finally, we prove 5. First, property 4 ensures that

$$\gamma(x^-) \leq \gamma(x) \leq \gamma(x^+) \quad \text{for all } x \in \mathbb{R}. \tag{5.5}$$

Assume, reasoning by contradiction, that $\gamma(x^-) < \gamma(x)$ for some x. Since γ is increasing, we can fix τ such that

$$\gamma(y) < \tau < \gamma(x) \quad \text{for all } y < x. \tag{5.6}$$

Now we deduce from the monotonicity of g and property 1 that

$$y = g(\gamma(y)) \leq g(\tau) \leq g(\gamma(x)) = x \quad \text{for all } y < x,$$

which implies that $g(\tau) = x$. Now property 2 yields $\gamma(x) = \gamma(g(\tau)) \leq \tau$, a contradiction with (5.6). Hence γ is left-continuous everywhere.

We shall prove that γ is right-continuous at every $x \in \mathbb{R} \setminus g(C_g)$ from (5.5) and a similar contradiction argument. Assume that for one of those x we can find τ such that

$$\gamma(x) < \tau < \gamma(y) \quad \text{for all } y > x.$$

Since g is nondecreasing we have

$$x = g(\gamma(x)) \leq g(\tau) \leq g(\gamma(y)) = y \quad \text{for all } y > x,$$

and therefore $x = g(\gamma(x)) = g(\tau)$. Since $\gamma(x) < \tau$, for any $t \in (\gamma(x), \tau)$ we have $g(t) = x$ and $t \in C_g$, a contradiction with the choice of x. □

We now have the necessary tools to reduce a Stieltjes differential equation to an ODE. First, we show how to compute solutions of Stieltjes equations by solving related ODEs.

Theorem 5.2. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing, continuous and $g(\mathbb{R}) = \mathbb{R}$.*

If $y : [g(t_0), g(t_0 + T)] \to \mathbb{R}$ is a solution of

$$y'(s) = f(\gamma(s), y(s)), \quad s \in [g(t_0), g(t_0 + T)] \setminus C, \tag{5.7}$$

for some set C, then $x : [t_0, t_0 + T] \to \mathbb{R}$ given by $x(t) = y(g(t))$ solves (1.1) for all $t \in [t_0, t_0 + T] \setminus (g^{-1}(C) \cup C_g \cup N_g)$.

In particular, $x = y \circ g$ solves (1.1) g-almost everywhere in $[t_0, t_0 + T]$ provided that $g^{-1}(C)$ be a null g-measure subset of $[t_0, t_0 + T]$.

Proof. Fix $t \in [t_0, t_0 + T] \setminus (g^{-1}(C) \cup C_g \cup N_g)$. Then assertion 2 in Proposition 5.1 yields that $\gamma(g(t)) = t$ and, moreover $g(t) \in [g(t_0), g(t_0 + T)] \setminus C$. Hence $y'(g(t))$ exists and

$$y'(g(t)) = f(\gamma(g(t)), y(g(t))) = f(t, x(t)).$$

Thus, it is enough to show that $x'_g(t)$ exists and equals $y'(g(t))$.

Fix $\varepsilon > 0$. Since $y'(g(t))$ exists, there exists $\tilde{\delta} > 0$ such that

$$[z \in [g(t_0), g(t_0 + T)],\ 0 < |z - g(t)| < \tilde{\delta}] \implies \left| \frac{y(z) - y(g(t))}{z - g(t)} - y'(g(t)) \right| < \varepsilon.$$

On the other hand, since g is continuous at t, there exists $\delta > 0$ such that

$$[s \in [t_0, t_0 + T],\ |s - t| < \delta] \implies |g(s) - g(t)| < \tilde{\delta}.$$

Now, Remark 2.1 ensures that if $0 < |s - t| < \delta$ then $0 < |g(s) - g(t)| < \tilde{\delta}$. Hence, it follows that

$$[s \in [t_0, t_0 + T],\ 0 < |s - t| < \delta] \implies \left| \frac{y(g(s)) - y(g(t))}{g(s) - g(t)} - y'(g(t)) \right| < \varepsilon,$$

that is, $x'_g(t)$ exists and $x'_g(t) = y'(g(t))$. □

Example 5.3. We consider problem (4.1) again, for $s_0 = 0$ and

$$f(s) = [(s+1)(2-s)]^+.$$

Assuming a derivator g in the conditions of Theorem 5.2 and $g(0) = 0$, we just have to solve the initial value problem

$$y'(s) = f(y(s)), \quad y(0) = 0,$$

and then we get a solution of the Stieltjes problem (4.1) in the form

$$s(t) = y(g(t)) = \frac{2e^{3g(t)} - 2}{e^{3g(t)} + 2}, \quad t \geq 0.$$

Next we prove the converse result, thus showing that problems (1.1) and (5.7) are equivalent.

Theorem 5.4. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing, continuous and $g(\mathbb{R}) = \mathbb{R}$.*

If $x : [t_0, t_0 + T] \to \mathbb{R}$ is a solution of (1.1) for all $t \in [t_0, t_0 + T] \setminus (C \cup C_g \cup N_g)$ for some set C, then $y : [g(t_0), g(t_0 + T)] \to \mathbb{R}$ given by $y(t) = x(\gamma(t))$ solves (5.7) for all $t \in [g(t_0), g(t_0 + T)] \setminus g(C \cup C_g \cup N_g)$.

In particular, $y = x \circ \gamma$ solves m-almost everywhere in $[g(t_0), g(t_0 + T)]$ provided that $m^{-1}(C)$ be a null m-measure subset of $[g(t_0), g(t_0 + T)]$.

Proof. Fix $s \in [g(t_0), g(t_0 + T)] \setminus g(C \cup C_g \cup N_g)$. Then there exists $u \in [t_0, t_0 + T] \setminus C \cup C_g \cup N_g$ such that $g(u) = s$. Moreover, Remark 2.1 ensures that $g^{-1}(\{s\}) = \{u\}$ so $\gamma(s) = u \in [t_0, t_0 + T] \setminus C \cup C_g \cup N_g$ and $x'_g(\gamma(s))$ exists. Furthermore,

$$x'_g(\gamma(s)) = f(\gamma(s), x(\gamma(s))) = f(\gamma(s), y(s)),$$

so, it is enough to show that $y'(s)$ exists and equals $x'_g(\gamma(s))$.

Fix $\varepsilon > 0$. Since $x'_g(\gamma(s))$ exists, there exists $\tilde{\delta} > 0$ such that

$$[z \in [t_0, t_0 + T],\ 0 < |z - \gamma(s)| < \tilde{\delta}] \implies \left| \frac{x(z) - x(\gamma(s))}{g(z) - g(\gamma(s))} - x'_g(\gamma(s)) \right| < \varepsilon.$$

On the other hand, $s \in [g(t_0), g(t_0 + T)] \setminus g(C \cup C_g \cup N_g)$ so assertion 4 in Proposition 5.1 ensures that γ is continuous at s. Hence, there exists $\delta > 0$ such that

$$[r \in [g(t_0), g(t_0 + T)],\ |r - s| < \delta] \implies |\gamma(r) - g(s)| < \tilde{\delta}.$$

Now, assertion 3 in Proposition 5.1 guarantees that if $0 < |s - t| < \delta$ then $0 < |g(s) - g(t)| < \tilde{\delta}$. Hence, it follows that

$$[r \in [g(t_0), g(t_0 + T)],\ 0 < |r - s| < \delta] \implies \left| \frac{x(\gamma(r)) - x(\gamma(s))}{g(\gamma(r)) - g(\gamma(s))} - x'_g(\gamma(s)) \right| < \varepsilon.$$

The result now follows by property 1 in Proposition 5.1. □

Remark 5.5. It is possible to extend theorems 5.2 and 5.4 to problem (1.1) with $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ with the obvious changes.

As a final comment, note that the results in this section are not valid for discontinuous derivators g. However, if D_g is a discrete set, then we can argue "piece-by-piece" to obtain the general solution of (1.1). That is, we can use the results in this section to solve (5.7) in each of the subintervals generated by D_g.

Acknowledgements

Rodrigo López Pouso was partially supported by Ministerio de Economía y Competitividad, Spain, and FEDER, Project MTM2016-75140-P and Xunta de Galicia under grant ED431C 2019/02. Ignacio Márquez Albés was supported by Xunta de Galicia under grants ED481A-2017/095 and ED431C 2019/02.

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