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Infinitely many homoclinic solutions for perturbed second-order Hamiltonian systems with subquadratic potentials

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Abstract. In this paper, we consider the following perturbed second-order Hamiltonian system

$$-\ddot{u}(t) + L(t)u = \nabla W(t, u(t)) + \nabla G(t, u(t)), \qquad \forall\, t \in \mathbb{R},$$

where $W(t, u)$ is subquadratic near origin with respect to u; the perturbation term $G(t, u)$ is only locally defined near the origin and may not be even in u. By using the variant Rabinowitz's perturbation method, we establish a new criterion for guaranteeing that this perturbed second-order Hamiltonian system has infinitely many homoclinic solutions under broken symmetry situations. Our result improves some related results in the literature.

Keywords: broken symmetry, Hamiltonian system, homoclinic solutions, subquadratic potential, Rabinowitz's perturbation method.

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1 Introduction

Consider the following second-order Hamiltonian system

$$-\ddot{u}(t) + L(t)u(t) = \nabla W\big(t, u(t)\big) + \nabla G\big(t, u(t)\big), \qquad \forall\, t \in \mathbb{R}, \tag{1.1}$$

where $u = (u_1, u_2, \dots, u_N) \in \mathbb{R}^N$ and $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a symmetric matrix-valued function. As usual, a solution u of problem (1.1) is homoclinic (to 0), if $|u(t)| \to 0$ as $|t| \to +\infty$. In addition, if $u \not\equiv 0$ then u is called a nontrivial homoclinic solution.

When $G \equiv 0$, (1.1) reduces to the second-order Hamiltonian system

$$-\ddot{u}(t) + L(t)u(t) = \nabla W\big(t, u(t)\big), \qquad \forall\, t \in \mathbb{R}. \tag{1.2}$$

In the past twenty years, the existence and multiplicity of homoclinic solutions for problem (1.2) have been extensively investigated by variational methods. Next we recall some results in

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this aspect. For problem (1.2), under the assumption that $L(t)$ and $W(t,x)$ are T-periodic in t, Rabinowitz [16] proved the existence of homoclinic orbits as a limit of $2kT$-periodic solutions of problem (1.2). Then this trick has been developed to study the existence and multiplicity of homoclinic solutions for more general Hamiltonian systems (see, e.g., [8,21,28]).

When $L(t)$ and $W(t,x)$ are not periodic in t, the problem of existence of homoclinic solutions for (1.2) is quite different from the one just described, since the Sobolev embedding is no longer compact. To overcome this difficulty, Rabinowitz and Tanaka [17] introduced the following coercive condition:

(L_0) $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$ and there is a continuous function $l : \mathbb{R} \to \mathbb{R}$ such that $l(t) > 0$ for all $t \in \mathbb{R}$ and $(L(t)u, u) \geq l(t)|u|^2$, $\forall\, u \in \mathbb{R}^N$ and $l(t) \to +\infty$ as $|t| \to +\infty$.

The condition (L_0) implies that the self-adjoint operator of $-d^2/dt^2 + L(t)$ in $L^2(\mathbb{R}, \mathbb{R}^N)$ has a sequence of eigenvalues λ_n (counted with multiplicity) and

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty. \tag{1.3}$$

Under this assumption on L, they obtained the existence of a nontrivial homoclinic solution for problem (1.2) by using a variant of the Mountain Pass Theorem without the Palais–Smale condition. Subsequently, Omana and Willem [13] showed that the Palais–Smale condition is satisfied under the coercive condition (L_0), and they used the usual Mountain Pass Theorem to prove the same result as in [17]. Since then, the coercive condition (L_0) and its variants have been used in a number of papers, and we refer the readers to [10,23,25–27] and the references therein.

Assume that $W(t,x)$ is of subquadratic growth as $|x| \to 0$ for all $t \in \mathbb{R}$, Ding [6] considered this case and presented the following condition

(L_0') there is a constant $\alpha < 2$ such that

$$l(t)|t|^{\alpha-2} \to +\infty \quad \text{as } |t| \to +\infty,$$

where $l(t)$ is given in (L_0). The main purpose of (L_0') is to guarantee some better properties of Sobolev embedding in the subquadratic case. If $W(t,x)$ is even in x, Ding proved a sequence of homoclinic solutions for problem (1.2). After the work of Ding [6], there are many papers concerning the existence of infinitely many homoclinic solutions in the subquadratic case (see, e.g., [20,22,34,35]). It is worth pointing out that most of these mentioned papers assumed that $W(t,x)$ is even with respect to x. Actually, the approaches used in these works depend on the notion of genus for symmetric sets. Therefore, the condition that $W(t,x)$ is even with respect to x is crucial in the application of these methods. When $W(t,x)$ is not even in x, the symmetry of the corresponding functional for problem (1.2) is broken. It is natural to ask whether an infinite number of homoclinic solutions can be maintained in broken symmetry case, and such a problem is often called perturbation from symmetry problem.

Since 1980s, many scholars have developed different methods to study the perturbation from symmetry problem for elliptic equations and Hamiltonian systems (see, e.g., [1,3,9,11,18,19,24,31–33]. If $G(t,x)$ is not even in x, problem (1.1) loses its symmetry under the assumption that $W(t,x)$ is even in x, and the authors [30] studied the perturbation from symmetry problem for (1.1). Specifically speaking, when $W(t,x)$ is locally superquadratic as

$|x| \to +\infty$, we obtained an unbounded sequence of homoclinic solutions by means of Bolle's perturbation method introduced in [3].

If $W(t,x)$ is subquadratic near origin with respect to x, i.e., $\lim_{x\to 0} W(t,x)/|x|^2 = +\infty$ for all $t \in \mathbb{R}$, an interesting question is whether the infinite number of homoclinic solutions persists under symmetry breaking situations. To the best of our knowledge, there are very few results on this topic. The main purpose of this paper is to give a positive answer to this question. To be precise, if the non-even perturbation term G is locally defined and satisfies some growth conditions near the origin, the existence of infinitely many homoclinic solutions for (1.1) can be preserved. Our tool is a variant of the perturbation method developed by Rabinowitz in [14]. The main idea of our proof is to introduce a modified functional by subtle truncation of the original functional, then the nonsymmetric part of this modified functional can be estimated. Then we can prove that the modified functional has almost the same small critical values as the original functional. Next we state the main result of this paper.

Theorem 1.1. *Let the condition (L_0) hold. Moreover, assume that the following condition hold:*

(H_1) $W(t,x) = W_1(t,x) + W_2(t,x)$, $W_1, W_2 \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ *and there exist a constant* $1 < p < 2$ *such that*

$$\left|\nabla W_1(t,x)\right| \le a(t)|x|^{p-1}, \qquad \forall\, (t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{1.4}$$

where $a:\ \mathbb{R} \to \mathbb{R}^+$ *is a continuous function such that* $a \in L^{\frac{2}{2-p}}(\mathbb{R})$;

(H_2) $W_1(t,0) \equiv 0$ *and there exist constants* $C_1 > 0$, $1 < \mu < 2$ *and* $\alpha_1 > 2$ *such that*

$$-C_1|x|^{\alpha_1} \le (\nabla W_1(t,x), x) - \mu W_1(t,x) \le 0, \qquad \forall\, (t,x) \in \mathbb{R}\times\mathbb{R}^N; \tag{1.5}$$

(H_3) *there exist constants* $C_2 > 0$, $1 < \alpha_2 < 2$ *and* $\alpha_3 > 2$ *such that*

$$W_1(t,x) \ge b(t)|x|^{\alpha_2} - C_2|x|^{\alpha_3}, \qquad \forall\, (t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{1.6}$$

where $b:\ \mathbb{R} \to \mathbb{R}^+$ *is a continuous function such that* $b \in L^{\frac{2}{2-\alpha_2}}(\mathbb{R})$;

(H_4) $W_2(t,0) \equiv 0$ *and there exist constants* $C_3 > 0$ *and* $\alpha_4 > 2$ *such that*

$$|\nabla W_2(t,x)| \le C_3|x|^{\alpha_4 - 1}, \qquad \forall\, (t,x) \in \mathbb{R}\times\mathbb{R}^N; \tag{1.7}$$

(H_5) $W_i(t,x) = W_i(t,-x)$, $i = 1,2$, $\forall\, (t,x) \in \mathbb{R}\times\mathbb{R}^N$;

(G_1) $G \in C^1(\mathbb{R}\times B_{r_0}(0),\mathbb{R})$, $G(t,0) \equiv 0$ *and there exist constants* $C_4 > 0$ *and* $\sigma > 2$ *such that*

$$\left|\nabla G(t,x)\right| \le C_4|x|^{\sigma-1}, \qquad \forall\, (t,x) \in \mathbb{R}\times B_{r_0}(0), \tag{1.8}$$

where $B_{r_0}(0)$ *denotes the open ball in* $\mathbb{R}^N$ *centred at 0 with radius* r_0;

(G_2) *there exist constants* $C_5 > 0$, $\beta > \frac{2(2-p)}{p(\sigma-2)}$ *and* $n_0 \in \mathbb{N}$ *such that* $\lambda_n \ge C_5 n^\beta$, $n \ge n_0$, *where the eigenvalues* λ_n *are given in* (1.3).

Then problem (1.1) *has a sequence of homoclinic solutions* $\{u_n\}$ *such that* $\max_{t\in\mathbb{R}} |u_n(t)| \to 0$ *as* $n \to \infty$.

Notation. Throughout the paper, we denote by C_n various positive constants which may vary from line to line and are not essential to the proof.

2 Variational setting and preliminaries

Let

$$E = \left\{ u \in H^1(\mathbb{R}, \mathbb{R}^N) : \int_{\mathbb{R}} \left[|\dot{u}(t)|^2 + \big(L(t)u(t), u(t)\big) \right] dt < +\infty \right\}$$

endowed with the inner product

$$(u, v) = \int_{\mathbb{R}} \Big[\big(\dot{u}(t), \dot{v}(t)\big) + \big(L(t)u(t), u(t)\big) \Big] dt.$$

Then E is a Hilbert space with this inner product and we denote by $\|\cdot\|$ the induced norm. As usual, for $1 \le \nu < +\infty$, let

$$\|u\|_\nu = \Big(\int_{\mathbb{R}} |u(t)|^\nu dt \Big)^{1/\nu}, \qquad u \in L^\nu(\mathbb{R}, \mathbb{R}^N).$$

It is evident that E is continuously embedded into $H^1(\mathbb{R}, \mathbb{R}^N)$, so E is continuously embedded into $L^\nu(\mathbb{R}, \mathbb{R}^N)$ for any $\nu \in [2, \infty]$, i.e., there exists $\tau_\nu > 0$ such that

$$\|u\|_\nu \le \tau_\nu \|u\|, \qquad \forall\, u \in E. \tag{2.1}$$

Moreover, E is compactly embedded into $L^\nu_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$ for all $\nu \in [1, \infty]$.

Next we introduce a useful result proved in Lemma 2.3 of [21] by Tang and Xiao.

Lemma 2.1. *For any $u \in E$, the following inequalities hold:*

$$|u(t)| \le \left\{ \int_t^\infty \frac{1}{\sqrt{l(s)}} \Big[|\dot{u}(s)|^2 + \big(L(s)u(s), u(s)\big) \Big] ds \right\}^{1/2}, \qquad t \in \mathbb{R}, \tag{2.2}$$

and

$$|u(t)| \le \left\{ \int_{-\infty}^t \frac{1}{\sqrt{l(s)}} \Big[|\dot{u}(s)|^2 + \big(L(s)u(s), u(s)\big) \Big] ds \right\}^{1/2}, \qquad t \in \mathbb{R}. \tag{2.3}$$

In view of condition (G_1) in Theorem 1.1, the perturbation term G is only locally defined, so we can't apply the variational methods directly. To overcome this difficulty, we use cut-off method to modify $G(t, x)$ for x outside a neighbourhood of the origin. In detail, we have the following lemma.

Lemma 2.2. *Suppose that (G_1) is satisfied. Then there exists a new function $\widetilde{G}$ possessing the following properties:*

(i) $\widetilde{G} \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $\widetilde{G}(t, 0) \equiv 0$ *and*

$$\big|\nabla \widetilde{G}(t, x)\big| \le 16 C_4 |x|^{\sigma - 1}, \qquad \forall\, (t, x) \in \mathbb{R} \times \mathbb{R}^N; \tag{2.4}$$

(ii) *there exists a positive constant $r_1 \le \min\{r_0/2, 1/2\}$ such that*

$$\widetilde{G}(t, x) = G(t, x), \qquad \forall\, (t, x) \in \mathbb{R} \times B_{r_1}(0); \tag{2.5}$$

where $B_{r_1}(0)$ denotes the open ball in $\mathbb{R}^N$ centred at 0 with radius r_1.

Proof. Since $G(t,0) = 0$, by (1.8) and direct computation we have

$$|G(t,x)| \le C_4|x|^\sigma, \qquad \forall\ (t,x) \in \mathbb{R} \times B_{r_0}(0). \tag{2.6}$$

Choose a constant $r_1 = \min\{r_0/2, 1/2\}$ and define a cut-off function $h \in C^1(\mathbb{R},\mathbb{R})$ such that $h(t) = 1$ for $t \le 1$, $h(t) = 0$ for $t \ge 2$ and $-2 \le h'(t) < 0$ for $1 < t < 2$. Set

$$\begin{cases} \widetilde{G}(t,x) = h\big(|x|^2/r_1^2\big)G(t,x), & \forall\ (t,x) \in \mathbb{R} \times B_{\sqrt{2}r_1}(0), \\ \widetilde{G}(t,x) \equiv 0, & \forall\ (t,x) \in \mathbb{R} \times \big(\mathbb{R}^N \backslash B_{\sqrt{2}r_1}(0)\big). \end{cases} \tag{2.7}$$

In view of (2.7), for $i = 1, 2, \dots, N$, we have

$$\frac{\partial \widetilde{G}}{\partial x_i} = \frac{2x_i}{r_1^2} h'\left(\frac{|x|^2}{r_1^2}\right) G(t,x) + h\left(\frac{|x|^2}{r_1^2}\right)\frac{\partial G}{\partial x_i}, \qquad \forall\ (t,x) \in \mathbb{R} \times B_{\sqrt{2}r_1}(0), \tag{2.8}$$

and $\partial\widetilde{G}/\partial x_i = 0$, $\forall (t,x) \in \mathbb{R} \times \big(\mathbb{R}^N \backslash B_{\sqrt{2}r_1}(0)\big)$. By (2.7) and (2.8), $\widetilde{G} \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $\widetilde{G}(t,0) \equiv 0$ and $\widetilde{G}(t,x) = G(t,x)$, $\forall\ (t,x) \in \mathbb{R} \times B_{r_1}(0)$. Moreover, it is easy to verify (2.4) by (1.8), (2.6) and (2.8). □

Next we introduce the following modified Hamiltonian system

$$-\ddot{u}(t) + L(t)u(t) = \nabla W\big(t,u(t)\big) + \nabla\widetilde{G}\big(t,u(t)\big), \qquad \forall\ t \in \mathbb{R}. \tag{2.9}$$

Let $I : E \to \mathbb{R}$ be defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W_1(t,u)dt - \int_{\mathbb{R}} W_2(t,u)dt - \int_{\mathbb{R}} \widetilde{G}(t,u)dt. \tag{2.10}$$

Under assumptions (L_0), (H_1), (H_2), (H_4) and (G_1), $I \in C^1(E,\mathbb{R})$ and

$$\langle I'(u), v\rangle = (u,v) - \int_{\mathbb{R}} \nabla W_1(t,u)v dt - \int_{\mathbb{R}} \nabla W_2(t,u)v dt - \int_{\mathbb{R}} \nabla\widetilde{G}(t,u)v dt \tag{2.11}$$

for all $u, v \in E$. The critical points of I in E are solutions of (2.9). Moreover, by the coercivity of l, (2.2) and (2.3), these solutions are homoclinic to 0.

Next we introduce a cut-off function $\zeta_\mu \in C^\infty(\mathbb{R},\mathbb{R})$ satisfying

$$\begin{cases} \zeta_\mu(t) = 1, & t \in (-\infty, A/2], \\ 0 \le \zeta_\mu(t) \le 1, & t \in (A/2, A/4), \\ \zeta_\mu(t) = 0, & t \in [A/4, \infty), \\ |\zeta'_\mu(t)| \le -8A^{-1}, & t \in \mathbb{R}, \end{cases} \tag{2.12}$$

where $A := (4\mu)^{-1}(\mu - 2) < 0$. Setting $T_0 := \min\{T_1, T_2, T_3, 1/2\}$, where

$$T_1 = \left\{\frac{2-\mu}{8\mu\big(C_1\tau_{\alpha_1}^{\alpha_1} + 10C_3\tau_{\alpha_4}^{\alpha_4} + 16(10 - 32A^{-1})C_4\tau_\sigma^\sigma\big)}\right\}^{\frac{1}{\alpha-2}}, \tag{2.13}$$

$$T_2 = \left\{\frac{1}{12\big(2^{\frac{\alpha_4+4}{2}}C_3\tau_{\alpha_4}^{\alpha_4} - 2^{\frac{\sigma+12}{2}}C_4\tau_\sigma^\sigma A^{-1}\big)}\right\}^{\frac{2}{\alpha-2}} \quad \text{and} \quad T_3 = \left\{\frac{-A}{2^{\frac{\sigma+18}{2}}C_4\tau_\sigma^\sigma}\right\}^{\frac{2}{\sigma-2}}, \tag{2.14}$$

$\alpha := \min\{\alpha_1, \alpha_4, \sigma\}$ and τ_{α_1}, τ_{α_4} and τ_σ are embedding constants given in (2.1). By the definition of T_0, T_0 is a fixed positive constant.

With the help of T_0 and the cut-off function h introduced in Lemma 2.2, define

$$k_{T_0}(u) = h\left(\frac{\|u\|^2}{T_0}\right), \qquad \forall\, u \in E. \tag{2.15}$$

Lemma 2.3. *The functional k_{T_0} defined by (2.15) is of $C^1(E, \mathbb{R})$ and*

$$|\langle k'_{T_0}(u), u\rangle| \le 8, \qquad \forall\, u \in E.$$

Proof. By (2.15) and direct calculation we have

$$\langle k'_{T_0}(u), v\rangle = 2h'\left(\frac{\|u\|^2}{T_0}\right)\frac{(u,v)}{T_0}, \qquad \forall\, u,\, v \in E. \tag{2.16}$$

Assume that $u_n \to u_0$ in E. In view of (2.16), for any $v \in E$, we obtain

$$\begin{aligned}
&|\langle k'_{T_0}(u_n) - k'_{T_0}(u_0), v\rangle| \\
&\quad = 2\left|h'\left(\frac{\|u_n\|^2}{T_0}\right)\frac{(u_n,v)}{T_0} - h'\left(\frac{\|u_0\|^2}{T_0}\right)\frac{(u_0,v)}{T_0}\right| \\
&\quad \le 2{T_0}^{-1}\|v\|\left[\left|h'\left(\frac{\|u_n\|^2}{T_0}\right)\right|\|u_n - u_0\| + \left|h'\left(\frac{\|u_n\|^2}{T_0}\right) - h'\left(\frac{\|u_0\|^2}{T_0}\right)\right|\|u_0\|\right],
\end{aligned}$$

which implies that $\|k'_{T_0}(u_n) - k'_{T_0}(u_0)\|_{E^*} \to 0$, $n \to \infty$. So $k_{T_0} \in C^1(E, \mathbb{R})$. By the definition of h and (2.16), we get $|\langle k'_{T_0}(u), u\rangle| \le 8$, $\forall\, u \in E$. □

With the help of this functional k_{T_0}, we define a new functional $\bar{I}_{T_0}$ on E by

$$\bar{I}_{T_0}(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W_1(t,u)dt - k_{T_0}(u)\left(\int_{\mathbb{R}} W_2(t,u)dt + \int_{\mathbb{R}} \tilde{G}(t,u)dt\right), \qquad \forall\, u \in E. \tag{2.17}$$

By (2.16), $\bar{I}_{T_0} \in C^1(E, \mathbb{R})$ and one can easily check that

$$\begin{aligned}
\langle \bar{I}'_{T_0}(u), v\rangle = (u,v) &- \int_{\mathbb{R}} \nabla W_1(t,u)v dt - k_{T_0}(u)\left(\int_{\mathbb{R}} \nabla W_2(t,u)v dt + \int_{\mathbb{R}} \nabla\tilde{G}(t,u)v dt\right) \\
&- \langle k'_{T_0}(u), v\rangle\left(\int_{\mathbb{R}} W_2(t,u)dt + \int_{\mathbb{R}} \tilde{G}(t,u)dt\right), \qquad \forall\, u,\, v \in E.
\end{aligned} \tag{2.18}$$

We will give some prior bounds for critical points of $\bar{I}_{T_0}$ based on the corresponding critical values in the following lemma, which is useful to introduce a modified functional.

Lemma 2.4. *Assume that (H_2), (H_4) and (G_1) are satisfied, if u is a critical point of $\bar{I}_{T_0}$, then*

$$\bar{I}_{T_0}(u) \le \frac{\mu - 2}{4\mu}\|u\|^2. \tag{2.19}$$

Proof. When u is a critical point of $\bar{I}_{T_0}$ and $\|u\|^2 > 2T_0$, by (2.16) and (2.17), $k_{T_0}(u) = 0$ and $k'_{T_0}(u) = 0$. In view of (2.18) and (2.19), we conclude that

$$\bar{I}_{T_0}(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W_1(t,u)dt, \qquad \langle \bar{I}'_{T_0}(u), u\rangle = \|u\|^2 - \int_{\mathbb{R}} (\nabla W_1(t,u), u)dt. \tag{2.20}$$

By (1.5) and (2.20), we get

$$\begin{aligned}\bar{I}_{T_0}(u) &= \bar{I}_{T_0}(u) - \mu^{-1}\langle \bar{I}'_{T_0}(u), u\rangle \\ &= \frac{\mu-2}{2\mu}\|u\|^2 + \mu^{-1}\int_{\mathbb{R}} \big((\nabla W_1(t,u), u) - \mu W_1(t,u)\big)dt \\ &\leq \frac{\mu-2}{4\mu}\|u\|^2. \end{aligned} \tag{2.21}$$

If u is a critical point of $\bar{I}_{T_0}$ with $\|u\|^2 \leq 2T_0$, by Lemma 2.2, Lemma 2.3, (1.5), (1.7), (2.17) and (2.18) we have

$$\begin{aligned}\bar{I}_{T_0}(u) &= \bar{I}_{T_0}(u) - \mu^{-1}\langle \bar{I}'_{T_0}(u), u\rangle \\ &\leq \frac{\mu-2}{2\mu}\|u\|^2 + C_1\tau_{\alpha_1}^{\alpha_1}\|u\|^{\alpha_1} + 10(C_3\tau_{\alpha_4}^{\alpha_4}\|u\|^{\alpha_4} + 16C_4\tau_\sigma^\sigma\|u\|^\sigma). \end{aligned} \tag{2.22}$$

By the definition of T_0 and (2.13), we get

$$C_1\tau_{\alpha_1}^{\alpha_1}\|u\|^{\alpha_1} + 10C_3\tau_{\alpha_4}^{\alpha_4}\|u\|^{\alpha_4} + 16(10-32A^{-1})C_4\tau_\sigma^\sigma\|u\|^\sigma < \frac{2-\mu}{4\mu}\|u\|^2. \tag{2.23}$$

In both cases, it follows from (2.21)–(2.23) that (2.19) holds. □

By the cut-off function ζ_μ and $\bar{I}_{T_0}$, define a functional as follows

$$l_\mu(u) = \zeta_\mu\big(\|u\|^{-2}\bar{I}_{T_0}(u)\big), \qquad \forall\, u \in E\backslash\{0\}. \tag{2.24}$$

By direct computation, for any $u \in E\backslash\{0\}$ and any $v \in E$,

$$\langle l'_\mu(u), v\rangle = \zeta'_\mu(\theta_{T_0}(u))\|u\|^{-4}\Big(\|u\|^2\langle \bar{I}'_{T_0}(u), v\rangle - 2\bar{I}_{T_0}(u)(u,v)\Big), \tag{2.25}$$

where $\theta_{T_0}(u) := \|u\|^{-2}\bar{I}_{T_0}(u)$, $\forall\, u \in E\backslash\{0\}$. Under assumptions of Theorem 1.1, it is easy to check that l_μ is continuously differentiable at any $u \in E\backslash\{0\}$.

By these functionals k_{T_0} and l_μ, we can introduce a modified functional J_{T_0} as follows:

$$J_{T_0}(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W_1(t,u)dt - k_{T_0}(u)\int_{\mathbb{R}} W_2(t,u)dt - \psi(u), \qquad \forall\, u \in E, \tag{2.26}$$

where

$$\psi(u) := \begin{cases} k_{T_0}(u)\; l_\mu(u)\; Q(u), & u \in E\setminus\{0\}, \\ 0, & u = 0, \end{cases} \tag{2.27}$$

and $Q(u) := \int_{\mathbb{R}} \tilde{G}(t,u)dt$, $\forall\, u \in E$. It follows from (2.1) and (2.4) that

$$\int_{\mathbb{R}} |\tilde{G}(t,u)|dt \leq 16C_4\tau_\sigma^\sigma\|u\|^\sigma, \qquad \forall\, u \in E. \tag{2.28}$$

Moreover, it is easy to check that $Q \in C^1(E,\mathbb{R})$ and

$$\langle Q'(u), v\rangle = \int_{\mathbb{R}} \nabla\tilde{G}(t,u)v dt, \qquad \forall\, u, v \in E. \tag{2.29}$$

Next we give a bound on $|\langle\psi'(u), u\rangle|$, $\forall\, u \in E$, which is used to obtain the estimate of $|J_{T_0}(u) - J_{T_0}(-u)|$, $\forall\, u \in E$. Then we show that J_{T_0} has no critical point with positive critical value on E.

Lemma 2.5. *Assume that (L_0), (H_1), (H_2), (H_4), (H_5) and (G_1) holds. Then*

(i) *the functional ψ defined by (2.27) is of class $C^1(E,\mathbb{R})$ and*

$$|\langle\psi'(u),u\rangle| \le 16(9-32A^{-1})C_4\tau_\sigma^\sigma\|u\|^\sigma, \qquad \forall\, u\in E; \tag{2.30}$$

(ii) *$J_{T_0}\in C^1(E,\mathbb{R})$ and there exists a constant $C_6>0$ independent of u such that*

$$|J_{T_0}(u)-J_{T_0}(-u)| \le C_6|J_{T_0}(u)|^{\frac{\sigma}{2}}, \qquad \forall\, u\in E; \tag{2.31}$$

(iii) *J_{T_0} has no critical point with positive critical value on E and $K_0=\{0\}$, where $K_0:=\{u\in E: J_{T_0}(u)=0,\ J'_{T_0}(u)=0\}$.*

Proof. For $u=0$ and any $v\in E$, by (2.4), (2.15), (2.24) and (2.27) we have

$$|\langle\psi'(0),v\rangle| = \left|\lim_{\lambda\to 0}\frac{\psi(\lambda v)-\psi(0)}{\lambda}\right| \le 16C_4\lim_{\lambda\to 0}|\lambda|^{\sigma-1}\int_{\mathbb{R}}|v(t)|^\sigma dt = 0,$$

so $\psi'(0)=0$. Combining (2.16), (2.25), (2.27) and (2.29), for $u\in E\setminus\{0\}$ and $v\in E$, we obtain

$$\langle\psi'(u),v\rangle = \langle k'_{T_0}(u),v\rangle l_\mu(u)Q(u) + k_{T_0}(u)\langle l'_\mu(u),v\rangle Q(u) + k_{T_0}(u)l_\mu(u)\langle Q'(u),v\rangle. \tag{2.32}$$

Next we prove $\psi'\in C^1(E,\mathbb{R})$. Suppose that $u_n\to u_0$ in E. We consider two possible cases.

Case 1. $u_0\neq 0$. In view of Lemma 2.3, (2.25), (2.29) and (2.32), $\psi'(u_n)\to\psi'(u_0)$, $n\to\infty$.

Case 2. $u_0=0$. Without loss of generality, we can assume $\|u_n\|^2<T_0$. It follows from (2.15) and (2.16) that $k'_{T_0}(u_n)=0$ and $k_{T_0}(u_n)=1$. Then (2.32) reduces to

$$\langle\psi'(u_n),v\rangle = \langle l'_\mu(u_n),v\rangle Q(u_n) + l_\mu(u_n)\langle Q'(u_n),v\rangle, \qquad \forall\, v\in E. \tag{2.33}$$

By (2.25), we can divide $\langle l'_\mu(u_n),v\rangle\ Q(u_n)$ into two parts as follows

$$\langle l'_\mu(u_n),v\rangle\ Q(u_n) = Q_1(u_n,v) - Q_2(u_n,v), \tag{2.34}$$

where

$$Q_1(u_n,v) = \zeta'_\mu(\theta_{T_0}(u_n))\|u_n\|^{-2}\langle\bar I'_{T_0}(u_n),v\rangle Q(u_n) \qquad \forall\, v\in E, \tag{2.35}$$

and

$$\begin{aligned} Q_2(u_n,v) &= 2\zeta'_\mu(\theta_{T_0}(u_n))\|u_n\|^{-4}\bar I_{T_0}(u_n)(u_n,v)Q(u_n)\\ &= 2\zeta'_\mu(\theta_{T_0}(u_n))\theta_{T_0}(u_n)\|u_n\|^{-2}(u_n,v)Q(u_n) \qquad \forall\, v\in E. \end{aligned} \tag{2.36}$$

In view of (2.12), (2.28), (2.35) and (2.36), we deduce that

$$|Q_1(u_n,v)| \le C_7\|\bar I'_{T_0}(u_n)\|_{E^*}\|u_n\|^{\sigma-2}\|v\|, \tag{2.37}$$

and

$$|Q_2(u_n,v)| \le C_8\|u_n\|^{\sigma-1}\|v\|. \tag{2.38}$$

Since $k'_{T_0}(u_n)=0$, $k_{T_0}(u_n)=1$ and $u_n\to 0$, by (1.4), (1.7), (2.4), (2.18) and (2.29),

$$\|\bar I'_{T_0}(u_n)\|_{E^*}\to 0 \quad\text{and}\quad \|Q'(u_n)\|_{E^*}\to 0, \qquad n\to\infty. \tag{2.39}$$

In combination with (2.24)-(2.25), (2.33), (2.34), (2.37)-(2.39), we have

$$\|\psi'(u_n) - \psi'(0)\|_{E^*} = \sup_{\|v\|\le 1} \left|\langle l'_\mu(u_n), v\rangle Q(u_n) + l_\mu(u_n)\langle Q'(u_n), v\rangle\right| \to 0, \qquad n \to \infty,$$

which implies the continuity of ψ' at 0. So we have $\psi \in C^1(E, \mathbb{R})$.

If $\|u\|^2 > 2T_0$ or $u = 0$, by (2.15), (2.16) and (2.26), it is easy to see that $\langle \psi'(u), u\rangle = 0$. Otherwise, $\|u\|^2 \le 2T_0$ and $u \ne 0$. Arguing similarly as in (2.22), we obtain

$$|\bar{I}_{T_0}(u) - \mu^{-1}\langle \bar{I}'_{T_0}(u), u\rangle| \le 2|A|\|u\|^2 + C_1\tau_{\alpha_1}^{\alpha_1}\|u\|^{\alpha_1} + 10(C_3\tau_{\alpha_4}^{\alpha_4}\|u\|^{\alpha_4} + 16C_4\tau_\sigma^\sigma\|u\|^\sigma). \tag{2.40}$$

Since $\|u\|^2 \le 2T_0$, by (2.13), (2.23) and (2.40) we get

$$|\langle \bar{I}'_{T_0}(u), u\rangle| \le \mu\big(3|A|\|u\|^2 + |\bar{I}_{T_0}(u)|\big). \tag{2.41}$$

In combination with (2.12) and (2.25), if $\theta_{T_0}(u) \notin [A/2, A/4]$, we have $l'_\mu(u) = 0$. Otherwise, $A/2 \le \theta_{T_0}(u) \le A/4$, then the definition of θ_{T_0} imply that

$$|\bar{I}_{T_0}(u)| \le |A|\|u\|^2. \tag{2.42}$$

When $\|u\|^2 \le 2T_0$ and $u \ne 0$, it follows from (2.25), (2.28), (2.41)–(2.42) that

$$\begin{aligned} \left|k_{T_0}(u)\langle l'_\mu(u), u\rangle Q(u)\right| &\le -16A^{-1}\|u\|^{-2}\big(|\bar{I}_{T_0}(u)| + |\langle \bar{I}'_{T_0}(u), u\rangle|\big)|Q(u)| \\ &\le -512A^{-1}C_4\tau_\sigma^\sigma\|u\|^\sigma. \end{aligned} \tag{2.43}$$

In view of Lemma 2.3, (2.4), (2.12), (2.15), (2.24), (2.28) and (2.29), we have

$$\left|\langle k'_{T_0}(u), u\rangle l_\mu(u)Q(u) + k_{T_0}(u)l_\mu(u)\langle Q'(u), u\rangle\right| \le 144C_4\tau_\sigma^\sigma\|u\|^\sigma, \qquad \forall\, u \in E\setminus\{0\}. \tag{2.44}$$

It follows from (2.32), (2.43) and (2.44) that (2.30) holds.

Next we prove (ii). By (1.4), (1.7), Lemma 2.3 and (i) in Lemma 2.5, we deduce that $J_{T_0} \in C^1(E, \mathbb{R})$ and

$$\begin{aligned} \langle J'_{T_0}(u), v\rangle = (u, v) &- \int_{\mathbb{R}} \nabla W_1(t, u)v dt - k_{T_0}(u)\int_{\mathbb{R}} \nabla W_2(t, u)v dt \\ &- \langle k'_{T_0}(u), v\rangle \int_{\mathbb{R}} W_2(t, u)dt - \langle \psi'(u), v\rangle, \qquad \forall\, u,\ v \in E. \end{aligned} \tag{2.45}$$

When $\|u\|^2 > 2T_0$ or $\theta_{T_0}(u) > A/4$, by (2.15) or (2.24) and (2.27) we have $\psi_{T_0}(u) = 0$. Then (2.31) holds by (H_5) and (2.26). If $\theta_{T_0}(u) \le A/4$, then the definition of θ_{T_0} imply that

$$|\bar{I}_{T_0}(u)| \ge \frac{|A|}{4}\|u\|^2. \tag{2.46}$$

When $\|u\|^2 \le 2T_0$ and $\theta_{T_0}(u) \le A/4$, by (2.13), (2.17), (2.26), (2.28) and (2.46) we get

$$|J_{T_0}(u)| \ge |\bar{I}_{T_0}(u)| - 2|Q(u)| \ge \frac{|A|}{4}\|u\|^2 - 32C_4\tau_\sigma^\sigma\|u\|^\sigma \ge \frac{|A|}{20}\|u\|^2. \tag{2.47}$$

In view of (H_5), (2.15), (2.24), (2.26)–(2.28), we obtain that

$$|J_{T_0}(u) - J_{T_0}(-u)| \le 32C_4\tau_\sigma^\sigma\|u\|^\sigma, \qquad \forall\, u \in E. \tag{2.48}$$

So (2.31) holds by (2.47) and (2.48).

Next we prove (iii) by contradiction. If u_0 is a critical point of J_{T_0} with $J_{T_0}(u_0) > 0$, by (H_2), (H_4), (2.26) and (2.27) we have $u_0 \neq 0$. Without loss of generality, we assume $\|u_0\|^2 \leq 2T_0$. Otherwise, (2.15)–(2.16) and (2.32) imply that $k_{T_0}(u_0) = 0$, $k'_{T_0}(u_0) = 0$ and $\psi'(u_0) = 0$. By (2.26), (2.27) and (2.45), we get

$$J_{T_0}(u_0) = \frac{1}{2}\|u_0\|^2 - \int_{\mathbb{R}} W_1(t, u_0)dt, \tag{2.49}$$

and

$$\langle J'_{T_0}(u_0), u_0\rangle = \|u_0\|^2 - \int_{\mathbb{R}} (\nabla W_1(t, u_0), u_0)dt. \tag{2.50}$$

In combination with (1.5), (2.49) and (2.50), it is easy to verify that

$$\begin{aligned} 0 < J_{T_0}(u_0) &= J_{T_0}(u_0) - \mu^{-1}\langle J'_{T_0}(u_0), u_0\rangle \\ &= 2A\|u_0\|^2 + \mu^{-1}\int_{\mathbb{R}} \left((\nabla W_1(t, u_0), u_0) - \mu W_1(t, u_0)\right)dt \\ &\leq 2A\|u_0\|^2 < 0, \end{aligned}$$

which is a contradiction, so $\|u_0\|^2 \leq 2T_0$.

It follows from Lemma 2.3, (2.26)–(2.28), (2.30) and (2.45) that

$$J_{T_0}(u_0) \leq \frac{1}{2}\|u_0\|^2 - \int_{\mathbb{R}} W_1(t, u_0)dt + C_3\tau_{\alpha_4}^{\alpha_4}\|u_0\|^{\alpha_4} + 16C_4\tau_\sigma^\sigma\|u_0\|^\sigma,$$

and

$$\langle J'_{T_0}(u_0), u_0\rangle \geq \|u_0\|^2 - \int_{\mathbb{R}} (\nabla W_1(t, u_0), u_0)dt - 9C_3\tau_{\alpha_4}^{\alpha_4}\|u_0\|^{\alpha_4} - 16(9 - 32A^{-1})C_4\tau_\sigma^\sigma\|u_0\|^\sigma.$$

Since $\|u_0\|^2 \leq 2T_0$, by (1.5), (2.13) and two inequalities above, we have

$$\begin{aligned} 0 < J_{T_0}(u_0) &= J_{T_0}(u_0) - \mu^{-1}\langle J'_{T_0}(u_0), u_0\rangle \\ &\leq 2A\|u_0\|^2 + C_1\tau_{\alpha_1}^{\alpha_1}\|u_0\|^{\alpha_1} + 10C_3\tau_{\alpha_4}^{\alpha_4}\|u_0\|^{\alpha_4} + 16(10 - 32A^{-1})C_4\tau_\sigma^\sigma\|u_0\|^\sigma \\ &< A\|u_0\|^2 < 0, \end{aligned}$$

which is also a contradiction. Moreover, by a similar proof, we have $K_0 = \{0\}$. □

3 Proofs of main results

Lemma 3.1. *Suppose that (L_0), (H_1), (H_4) and (G_1) are satisfied. Then the functional J_{T_0} satisfies the Palais–Smale condition.*

Proof. First we prove that J_{T_0} is bounded from below. From Hölder's inequality, (1.4), (2.15), (2.26) and (2.27), if $\|u\|^2 > 2T_0$,

$$J_{T_0}(u) \geq \frac{1}{2}\|u\|^2 - C_9\|u\|^p. \tag{3.1}$$

Since $1 < p < 2$, (3.1) implies that $J_{T_0}(u) \to +\infty$ as $\|u\| \to +\infty$.

Next we show that J_{T_0} satisfies the Palais–Smale condition. Let $\{u_n\}_{n\in\mathbb{N}} \subset E$ be a Palais–Smale sequence, i.e., $\{J_{T_0}(u_n)\}_{n\in\mathbb{N}}$ is bounded and $J'_{T_0}(u_n) \to 0$ as $n \to +\infty$. Since J_{T_0} is coercive, $\{u_n\}$ is bounded in E. Then there is a positive constant A such that $\|u_n\| \leq A$,

$n \in \mathbb{N}$, passing to subsequence, also denoted by $\{u_n\}$, it can be assumed that $u_n \rightharpoonup u_0$, $n \to \infty$ for some $u_0 \in E$.

Since $a \in L^{\frac{2}{2-p}}(\mathbb{R})$, for any given number $\varepsilon > 0$, we can choose $T_\varepsilon > 0$ such that

$$\left(\int_{|t|>T_\varepsilon} |a(t)|^{2/(2-p)} dt\right)^{(2-p)/2} < \varepsilon. \tag{3.2}$$

By (1.4) and the Hölder inequality, we have

$$\int_{-T_\varepsilon}^{T_\varepsilon} |\nabla W_1(t, u_n(t))||u_n(t) - u_0(t)| dt \leq (\tau_2 A)^{p-1} \|a\|_{2/(2-p)} \left(\int_{-T_\varepsilon}^{T_\varepsilon} |u_n - u_0|^2 dt\right)^{1/2}. \tag{3.3}$$

By Sobolev embedding theorem, we also get

$$u_n \to u_0 \quad \text{in } L^2_{loc}(\mathbb{R}, \mathbb{R}^N),\ n \to \infty. \tag{3.4}$$

Consequently, in view of (3.3) and (3.4),

$$\int_{-T_\varepsilon}^{T_\varepsilon} |\nabla W_1(t, u_n(t))||u_n(t) - u_0(t)| dt \to 0, \qquad n \to \infty. \tag{3.5}$$

On the other hand, it follows from (1.4), (3.2) and the Hölder inequality that

$$\begin{aligned}
&\int_{|t|>T_\varepsilon} |\nabla W_1(t, u_n(t))||u_n(t) - u_0(t)| dt \\
&\quad \leq \int_{|t|>T_\varepsilon} |a(t)||u_n(t)|^{p-1}\big(|u_n(t)| + |u_0(t)|\big) dt \\
&\quad \leq 2\int_{|t|>T_\varepsilon} |a(t)|\big(|u_n(t)|^p + |u_0(t)|^p\big) dt \\
&\quad \leq 2\tau_2^p \Big(\int_{|t|>T_\varepsilon} |a(t)|^{2/(2-p)} dt\Big)^{(2-p)/2} (\|u_n\|^p + \|u_0\|^p) \\
&\quad \leq 2\tau_2^p (A^p + \|u_0\|^p)\varepsilon, \qquad n \in \mathbb{N}.
\end{aligned} \tag{3.6}$$

Note that ε is arbitrary, combining (3.5) with (3.6),

$$\int_{\mathbb{R}} |\nabla W_1(t, u_n(t))||u_n(t) - u_0(t)| dt \to 0, \qquad n \to \infty. \tag{3.7}$$

Since l is coercive, for any given number $\varepsilon > 0$, there exists $T'_\varepsilon > 0$ such that

$$\varepsilon l(t) > 1, \qquad |t| > T'_\varepsilon. \tag{3.8}$$

It follows from (1.7), (3.4) and the Hölder inequality that

$$\int_{-T'_\varepsilon}^{T'_\varepsilon} |\nabla W_2(t, u_n(t))||u_n(t) - u_0(t)| dt \to 0, \qquad n \to \infty. \tag{3.9}$$

Since E is continuously embedded into $L^\infty(\mathbb{R}, \mathbb{R}^N)$ and $\|u_n\| \leq A$, we get

$$\|u_n\|_\infty \leq \tau_\infty A, \qquad n \in \mathbb{N}. \tag{3.10}$$

By (L_0), (1.7), (3.8) and (3.10), we have

$$\begin{aligned}
&\int_{|t|>T'_\varepsilon} |\nabla W_2(t,u_n(t))||u_n(t)-u_0(t)|dt \\
&\quad\le C_3(\tau_\infty A)^{\alpha_4-2}\int_{|t|>T'_\varepsilon} |u_n(t)|\big(|u_n(t)|+|u_0(t)|\big)dt \\
&\quad\le 2C_3(\tau_\infty A)^{\alpha_4-2}\varepsilon\int_{|t|>T'_\varepsilon} l(t)\big(|u_n(t)|^2+|u_0(t)|^2\big)dt \\
&\quad\le 2C_3(\tau_\infty A)^{\alpha_4-2}\varepsilon\int_{|t|>T'_\varepsilon} \Big[\big(L(t)u_n(t),u_n(t)\big)+\big(L(t)u_0(t),u_0(t)\big)\Big]dt \\
&\quad\le 2C_3(\tau_\infty A)^{\alpha_4-2}(A^2+\|u_0\|^2)\varepsilon, \qquad n\in\mathbb{N}. \qquad (3.11)
\end{aligned}$$

Since ε is arbitrary, it follows from (3.9) and (3.11) that

$$\int_{\mathbb{R}} |\nabla W_2(t,u_n(t))||u_n(t)-u_0(t)|dt \to 0, \qquad n\to\infty. \qquad (3.12)$$

By a similar proof as (3.9) and (3.11), we also have

$$\int_{\mathbb{R}} |\nabla \tilde{G}(t,u_n(t))||u_n(t)-u_0(t)|dt \to 0, \qquad n\to\infty. \qquad (3.13)$$

Next we consider the following two possible cases.

Case 1. $\|u_n\|^2 > 2T_0$ or $u_n = 0$. From (2.15), (2.16) and (2.32), $k_{T_0}(u_n)=0$, $k'_{T_0}(u_n)=0$ and $\psi'(u_n)=0$. Therefore, by (2.45), we have

$$|\langle J'_{T_0}(u_n),u_n-u_0\rangle| \ge \|u_n-u_0\|^2+(u_0,u_n-u_0)-\int_{\mathbb{R}}|\nabla W_1(t,u_n)||u_n-u_0|dt. \qquad (3.14)$$

Case 2. $\|u_n\|^2 \le 2T_0$ and $u_n \ne 0$. In combination with (2.16) and (2.28), we get

$$\begin{aligned}
\big|\langle k'_{T_0}(u_n),u_n-u_0\rangle Q(u_n)\big| &\le 32C_4\tau_\sigma^\sigma h'\left(\frac{\|u_n\|^2}{T_0}\right)\frac{(u_n,u_n-u_0)}{T_0}\|u_n\|^\sigma \\
&\le 2^{\frac{\sigma+12}{2}}C_4\tau_\sigma^\sigma T_0^{\frac{\sigma-2}{2}}\big(\|u_n-u_0\|^2+(u_0,u_n-u_0)\big). \qquad (3.15)
\end{aligned}$$

In view of (2.12) and (2.24), $|l(u_n)|\le 1$. Arguing as in (3.15), we also have

$$\Big|\langle k'_{T_0}(u_n),u_n-u_0\rangle l(u_n)Q(u_n)\Big| \le 2^{\frac{\sigma+12}{2}}C_4\tau_\sigma^\sigma T_0^{\frac{\sigma-2}{2}}\big(\|u_n-u_0\|^2+(u_0,u_n-u_0)\big). \qquad (3.16)$$

It follows from (1.7) and (2.10) that

$$\begin{aligned}
\left|\langle k'_{T_0}(u_n),u_n-u_0\rangle\int_{\mathbb{R}}W_2(t,u_n)dt\right| &\le 2C_3\tau_{\alpha_4}^{\alpha_4}h'\left(\frac{\|u_n\|^2}{T_0}\right)\frac{(u_n,u_n-u_0)}{T_0}\|u_n\|^{\alpha_4} \\
&\le 2^{\frac{\alpha_4+4}{2}}C_3\tau_{\alpha_4}^{\alpha_4}T_0^{\frac{\alpha_4-2}{2}}\big(\|u_n-u_0\|^2+(u_0,u_n-u_0)\big). \qquad (3.17)
\end{aligned}$$

By (2.16) and (2.34), we have

$$\big|k'_{T_0}(u_n)\langle l'(u_n),u_n-u_0\rangle Q(u_n)\big| \le \big|Q_1(u_n,u_n-u_0)\big|+\big|Q_2(u_n,u_n-u_0)\big|. \qquad (3.18)$$

In view of (2.12), (2.28) and (2.35), we obtain

$$\begin{aligned}
\big|Q_1(u_n,u_n-u_0)\big| &= |\zeta'_\mu(\theta_{T_0}(u_n))|\|u_n\|^{-2}|\langle \bar{I}'_{T_0}(u_n),u_n-u_0\rangle||Q(u_n)| \\
&\le -2^{\frac{\sigma+12}{2}}A^{-1}C_4\tau_\sigma^\sigma T_0^{\frac{\sigma-2}{2}}\big|\langle \bar{I}'_{T_0}(u_n),u_n-u_0\rangle\big|. \qquad (3.19)
\end{aligned}$$

It follows from (2.18), (3.7), (3.12), (3.13), (3.15) and (3.17) that

$$\begin{aligned} |\langle \bar{I}'_{T_0}(u_n), u_n - u_0\rangle| &\le \|u_n - u_0\|^2 + \left|\langle k'_{T_0}(u_n), u_n - u_0\rangle \int_{\mathbb{R}} W_2(t, u_n)dt\right| \\ &\quad + \left|\langle k'_{T_0}(u_n), u_n - u_0\rangle Q(u_n)\right| + o_n(1) \\ &\le (C_{10} + 1)\|u_n - u_0\|^2 + o_n(1). \end{aligned} \tag{3.20}$$

where $C_{10} = 2^{\frac{\alpha_4+4}{2}} C_3 \tau_{\alpha_4}^{\alpha_4} T_0^{\frac{\alpha_4-2}{2}} - 2^{\frac{\sigma+12}{2}} A^{-1} C_4 \tau_\sigma^\sigma T_0^{\frac{\sigma-2}{2}}$.
By (3.19) and (3.20), we obtain

$$|Q_1(u_n, u_n - u_0)| \le C_{11}\|u_n - u_0\|^2 + o_n(1), \tag{3.21}$$

where $C_{11} = -2^{\frac{\sigma+12}{2}} A^{-1} C_4 \tau_\sigma^\sigma T_0^{\frac{\sigma-2}{2}} (C_{10} + 1)$. In view of (2.12), (2.28) and (2.36),

$$\begin{aligned} |Q_2(u_n, u_n - u_0)| &\le 2|\zeta'_\mu(\theta_{T_0}(u_n))||\theta_{T_0}(u_n)|\|u_n\|^{-2} Q(u_n)(u_n, u_n - u_0) \\ &\le 2^{\frac{\sigma+12}{2}} C_4 \tau_\sigma^\sigma T_0^{\frac{\sigma-2}{2}} \left(\|u_n - u_0\|^2 + (u_0, u_n - u_0)\right). \end{aligned} \tag{3.22}$$

In combination with (3.18), (3.21) and (3.22), we have

$$|k_{T_0}(u_n)\langle l'(u_n), u_n - u_0\rangle Q(u_n)| \le (C_{11} + C_{10})\|u_n - u_0\|^2 + o_n(1). \tag{3.23}$$

By (2.15), (2.24) and (2.29), we conclude that

$$|k_{T_0}(u_n) l(u_n)\langle Q'(u_n), u_n - u_0\rangle| \le \int_{\mathbb{R}} |\nabla \tilde{G}(t, u_n(t))||u_n(t) - u_0(t)|dt. \tag{3.24}$$

It follows from (2.32), (3.7), (3.16), (3.23) and (3.24) that

$$|\langle \psi'(u_n), u_n - u_0\rangle| \le (C_{11} + 2C_{10})\|u_n - u_0\|^2 + o_n(1). \tag{3.25}$$

In view of (2.45), (3.7), (3.12), (3.13), (3.17) and (3.25), we get

$$\begin{aligned} |\langle J'_{T_0}(u_n), u_n - u_0\rangle| &\ge \|u_n - u_0\|^2 - \left|\langle k'_{T_0}(u_n), u_n - u_0\rangle \int_{\mathbb{R}} W_2(t, u_n)dt\right| \\ &\quad - |\langle \psi'(u_n), u_n - u_0\rangle| + o_n(1) \\ &\ge (1 - C_{11} - 3C_{10})\|u_n - u_0\|^2 + o_n(1). \end{aligned} \tag{3.26}$$

By (2.14) and (3.26), we have

$$|\langle J'_{T_0}(u_n), u_n - u_0\rangle| \ge 2^{-1}\|u_n - u_0\|^2 + o_n(1). \tag{3.27}$$

In both cases, from (3.14) and (3.27), we get $u_n \to u_0$, $n \to \infty$. Thus J_{T_0} satisfies Palais–Smale condition. □

In view of (L_0), the self-adjoint operator of $-d^2/dt^2 + L(t)$ in $L^2(\mathbb{R}, \mathbb{R}^N)$ has a sequence of eigenvalues $\lambda_n \to \infty$. Moreover, the corresponding system of normalized eigenfunctions $\{e_n : n \in \mathbb{N}\}$ forms an orthogonal basis in E. Hereafter, set $E_n = \mathrm{span}\{e_1, \ldots, e_n\}$ and $E_n^\perp$ be the orthogonal complement of E_n in E. With the help of the normalized orthogonal sequence $\{e_n\}_{n=1}^\infty$, define some subspaces as follows:

$$B_n = \{u \in E_n;\ \|u\| \le 1\}, \qquad S^n = \{u \in E_n;\ \|u\| = 1\}$$

and
$$S_+^{n+1} = \{u = w + se_{n+1};\ \|u\| = 1,\ w \in B_n,\ 0 \le s \le 1\}.$$
By these subspaces, we can introduce some continuous maps and minimax sequences of J as follows
$$\Lambda_n = \Big\{\gamma \in C(S^n, E);\ \gamma \text{ is odd}\Big\}, \qquad \Gamma_n = \Big\{\gamma \in C(S_+^{n+1}, E);\ \gamma|_{S^n} \in \Lambda_n\Big\}, \tag{3.28}$$
and
$$b_n = \inf_{\gamma\in\Lambda_n} \max_{u\in S^n} J_{T_0}(\gamma(u)), \qquad c_n = \inf_{\gamma\in\Gamma_n} \max_{u\in S_+^{n+1}} J_{T_0}(\gamma(u)). \tag{3.29}$$
For any $\delta > 0$, set
$$\Gamma_n(\delta) = \Big\{\gamma \in \Gamma_n;\ J_{T_0}(\gamma(u)) \le b_n + \delta,\ u \in S^n\Big\}, \tag{3.30}$$
$$c_n(\delta) = \inf_{\gamma\in\Gamma_n(\delta)} \max_{u\in S_+^{n+1}} J_{T_0}(\gamma(u)). \tag{3.31}$$
By (3.28)–(3.31), it is obvious that $b_n \le c_n \le c_n(\delta)$, $n \in \mathbb{N}$. Next we give some useful estimates for minimax values b_n and $c_n(\delta)$.

Lemma 3.2. *Let (L_0), (H_1), (H_3), (H_4) and (G_1) be satisfied. Then for any $n \in \mathbb{N}$, $b_n < 0$.*

Proof. By (1.6), for any $u \in E_n$ we have
$$\int_{\mathbb{R}} W_1(t,u)dt \ge \int_{\mathbb{R}} b(t)|u|^{\alpha_2}dt - C_2\int_{\mathbb{R}} |u|^{\alpha_3}dt. \tag{3.32}$$
By standard arguments as in [20], for any $u \in E_n \setminus \{0\}$, there exists $\varepsilon_1 > 0$ depending on E_n such that
$$\text{meas}\Big\{t \in \mathbb{R}:\ b(t)|u|^{\alpha_2} \ge \varepsilon_1\|u\|^{\alpha_2}\Big\} \ge \varepsilon_1. \tag{3.33}$$
By (1.7), (2.1), (2.15), (2.24), (2.28), (3.32)–(3.33), for any $u \in E_n \setminus \{0\}$, we get
$$\begin{aligned} J_{T_0}(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W_1(t,u)dt - k_{T_0}(u)\int_{\mathbb{R}} W_2(t,u)dt - \psi(u) \\ &\le \|u\|^2 + C_{12}\|u\|^{\alpha_3} + C_{13}\|u\|^{\alpha_4} + C_{14}\|u\|^{\sigma} - \varepsilon_1^2\|u\|^{\alpha_2}. \end{aligned} \tag{3.34}$$
In view of (3.34), there exist $\epsilon(n) > 0$ and $\kappa(n) > 0$ such that $J_{T_0}(\kappa u) < -\epsilon$, $u \in S^n$. Then we set $\gamma(u) = \kappa u$, $u \in S^n$. By (3.29), we obtain $b_n < 0$. □

Lemma 3.3. *Assume that (L_0), (H_1), (H_2), (H_3), (H_4) and (G_1) hold. Then for any $n \in \mathbb{N}$ and any $\delta > 0$, $c_n(\delta) < 0$.*

Proof. From (3.30) and (3.31), for fixed $n \in \mathbb{N}$, if $0 < \delta < \delta'$, we have $\Gamma_n(\delta) \subset \Gamma_n(\delta')$ and $c_n(\delta) \ge c_n(\delta')$. So we only need to prove $c_n(\delta) < 0$ for any $\delta \in (0, |b_n|)$. For any $\delta \in (0, |b_n|)$, by (3.29), there exists $\gamma_0 \in \Lambda_n$ such that $\max_{u\in S^n} J_{T_0}(\gamma_0(u)) \le b_n + \frac{\delta}{2}$. By the fact that $\gamma_0(S^n)$ is a compact set in E, there exists a positive integer m_0 such that
$$\max_{u\in S^n} J_{T_0}\big((P_{m_0} \circ \gamma_0)u\big) \le b_n + \delta, \tag{3.35}$$
where P_{m_0} denotes the orthogonal projective operator from E to E_{m_0}.

For any $c \in \mathbb{R}$, let $J_{T_0}^c = \{u \in E : J_{T_0}(u) \le c\}$. Choose $\bar{\varepsilon} = -(b_n + \delta)/2 > 0$. By a similar proof as in Lemma 3.2, there exists $\rho_{m_0+1} > 0$ such that if $u \in \bar{B}(0, \rho_0) \cap E_{m_0+1}$,

$J_{T_0}(u) \leq 0$, where $B(x_0, \rho)$ denotes the open ball of radius ρ centred at x_0 in E, and $\bar{B}(x_0, \rho)$ denotes the closure of $B(x_0, \rho)$ in E. Since $J_{T_0} \in C^1(E, \mathbb{R})$ and $J_{T_0}(0) = 0$, dist $(0, J_{T_0}^{-\bar{\varepsilon}}) > 0$. Set $\rho_0' = \min\{\rho_0, \text{ dist } (0, J_{T_0}^{-\bar{\varepsilon}})\}$, then $\rho_0' > 0$. By Deformation Theorem (see Theorem A.4 in [15]), there exists $\varepsilon \in (0, \bar{\varepsilon})$ and a continuous map $\eta \in C([0,1] \times E, E)$ such that

$$\eta(1, u) = u, \quad \text{if } J_{T_0}(u) \notin [-\bar{\varepsilon}, \bar{\varepsilon}], \tag{3.36}$$

and

$$\eta\big(1, J_{T_0}^{\varepsilon} \backslash B(0, \rho_0')\big) \subset J_{T_0}^{-\varepsilon}, \tag{3.37}$$

where $B(0, \rho_0')$ is a neighbourhood of K_0 defined by (iii) in Lemma 2.5.

By (3.28), $P_{m_0} \circ \gamma_0 \in C(S^n, E_{m_0})$. Since E_{n+1} is a metric space with the norm $\|\cdot\|$ and S^n is a closed subset in E_{n+1}, there exists an extension $\widetilde{P_{m_0} \circ \gamma_0} : E_{n+1} \to E_{m_0}$ of $(P_{m_0} \circ \gamma_0)$ by Dugundji extension theorem (see Theorem 4.1 in [7]); furthermore,

$$\big((\widetilde{P_{m_0} \circ \gamma_0}) E_{n+1}\big) \subset \operatorname{co}\big((P_{m_0} \circ \gamma_0) S^n\big), \tag{3.38}$$

where the symbol co denotes the convex hull. Since $(P_{m_0} \circ \gamma_0) S^n$ is a compact set in E_{m_0}, by the definition of convex hull, $\operatorname{co}\big((P_{m_0} \circ \gamma_0) S^n\big)$ is a bounded set in E_{m_0}. Then there exists a constant ν such that $J_{T_0}(u) \leq \nu$, $u \in \operatorname{co}\big((P_{m_0} \circ \gamma_0) S^n\big)$. It follows from (3.38) that

$$J_{T_0}\big((\widetilde{P_{m_0} \circ \gamma_0}) u\big) \leq \nu, \qquad \forall\, u \in E_{n+1}. \tag{3.39}$$

Next we distinguish two cases.

Case 1. $\nu \leq \varepsilon$. Since $\widetilde{P_{m_0} \circ \gamma_0} \in C(E_{n+1}, E_{m_0})$, by (3.39) we have

$$(\widetilde{P_{m_0} \circ \gamma_0}) u \in J_{T_0, m_0}^{\varepsilon}, \qquad \forall\, u \in E_{n+1}, \tag{3.40}$$

where $J_{T_0, m_0}^{\varepsilon} := \{u \in E_{m_0} : \ J_{T_0}(u) \leq \varepsilon\}$. Define a map χ as follows:

$$\chi(u) = \begin{cases} u, & u \notin \bar{B}(0, \rho_0') \cap E_{m_0} \\ u + (\rho_0'^2 - \|u\|^2)^{1/2} e_{m_0+1}, & u \in \bar{B}(0, \rho_0') \cap E_{m_0}. \end{cases} \tag{3.41}$$

It is clear that $\chi \in C(E_{m_0}, E_{m_0+1})$ and

$$\big(\chi \circ (\widetilde{P_{m_0} \circ \gamma_0})\big) u \notin B(0, \rho_0'), \qquad \forall\, u \in E_{n+1}. \tag{3.42}$$

If $u \in E_{n+1}$ and $\|(\widetilde{P_{m_0} \circ \gamma_0}) u\| > \rho_0'$, in view of (3.40) and (3.41), we get

$$\big(\chi \circ (\widetilde{P_{m_0} \circ \gamma_0})\big) u = (\widetilde{P_{m_0} \circ \gamma_0}) u \in J_{m_0}^{\varepsilon}. \tag{3.43}$$

Otherwise, when $u \in E_{n+1}$ and $\|(\widetilde{P_{m_0} \circ \gamma_0}) u\| \leq \rho_0'$, by (3.41) $\|\big(\chi \circ (\widetilde{P_{m_0} \circ \gamma_0})\big) u\| = \rho_0'$. By the definition of ρ_0' and (3.43), we deduce that

$$\big(\chi \circ (\widetilde{P_{m_0} \circ \gamma_0})\big) u \in J_{T_0}^{\varepsilon}, \qquad \forall\, u \in E_{n+1}. \tag{3.44}$$

Define a map $H_0 : E_{n+1} \to E$ as follows:

$$H_0(\cdot) = \eta\Big(1, \big(\chi \circ (\widetilde{P_{m_0} \circ \gamma_0})\big)(\cdot)\Big). \tag{3.45}$$

Next we need to prove $H_0 \in \Gamma_n(\delta)$ and $\max_{u\in S_+^{n+1}} J_{T_0}(H_0(u)) < 0$. First, it is obvious that $H_0 \in C(S_+^{n+1}, E)$. Next we prove $H_0|_{S^n} \in \Lambda_n$. By Dugundji extension theorem, we obtain

$$(\widetilde{P_{m_0} \circ \gamma_0})u = (P_{m_0} \circ \gamma_0)u, \qquad \forall\, u \in S^n. \tag{3.46}$$

From (3.35), $(P_{m_0} \circ \gamma_0)\, u \in J_{T_0}^{-2\bar{\varepsilon}}$, $u \in S^n$. The definition of ρ_0' and $J_{T_0}^{-2\bar{\varepsilon}} \subset J_{T_0}^{-\bar{\varepsilon}}$ imply that

$$\|(P_{m_0} \circ \gamma_0)\, u\| \geq \rho_0', \qquad \forall\, u \in S^n. \tag{3.47}$$

It follows from (3.41), (3.46) and (3.47) that

$$\left(\chi \circ (\widetilde{P_{m_0} \circ \gamma_0})\right)\, u = \chi \circ \left((P_{m_0} \circ \gamma_0)\, u\right) = (P_{m_0} \circ \gamma_0)\, u, \qquad \forall\, u \in S^n. \tag{3.48}$$

Since $(P_{m_0} \circ \gamma_0)\, u \in J_{T_0}^{-2\bar{\varepsilon}}$, $\forall\, u \in S^n$, in view of (3.35)–(3.36), (3.45) and (3.48)

$$H_0(u) = \eta\left(1, (\chi \circ (\widetilde{P_{m_0} \circ \gamma_0}))\, u\right) = (P_{m_0} \circ \gamma_0)\, u, \qquad \forall\, u \in S^n. \tag{3.49}$$

which implies that $H_0|_{S^n} \in \Lambda_n$. Moreover, from (3.30), (3.35) and (3.49), we have $H_0 \in \Gamma_n(\delta)$. Since $S^{n+1} \subset E_{n+1}$, by (3.42) and (3.44), we have $\left(\chi \circ (\widetilde{P_{m_0} \circ \gamma_0})\right)u \notin B(0, \rho_0')$, $\forall\, u \in S_+^{n+1}$ and $\left(\chi \circ (\widetilde{P_{m_0} \circ \gamma_0})\right)u \in J_{T_0}^{\varepsilon}$, $\forall\, u \in S_+^{n+1}$. From (3.37) and (3.45), we deduce that $\max_{u\in S_+^{n+1}} J_{T_0}(H_0(u)) \leq -\varepsilon < 0$, which implies that $c_n(\delta) < 0$ by (3.31).

Case 2. $\nu > \varepsilon$. Let $J_{T_0}|_{E_{m_0}}$ denote the restriction of J_{T_0} on E_{m_0}. By a similar proof as in Lemma 2.5 and Lemma 3.1, we can prove that $J_{T_0}|_{E_{m_0}} \in C^1(E_{m_0}, \mathbb{R})$ and satisfies Palais–Smale condition. Moreover, $J_{T_0}|_{E_{m_0}}$ has no critical point with positive critical values on E_{m_0}. By Noncritical interval theorem (see Theorem 5.1.6 in [5]), $J_{T_0,m_0}^{\varepsilon}$ is a strong deformation retraction of J_{T_0,m_0}^{ν}. So there exists a map ς such that $\varsigma \in C(J_{T_0,m_0}^{\nu}, J_{T_0,m_0}^{\varepsilon})$ and $\varsigma(u) = u$, if $u \in J_{T_0,m_0}^{\varepsilon}$. Define a map from $E_{n+1} \to E$ as follows:

$$\bar{H}_0(\cdot) = \eta\left(1, \left(\chi \circ (\varsigma \circ (\widetilde{P_{m_0} \circ \gamma_0}))\right)(\cdot)\right).$$

By a similar proof as in Case 1, we also obtain $\bar{H}_0 \in \Gamma_n(\delta)$ and $\max_{u\in S_+^{n+1}} J_{T_0}(\bar{H}_0(u)) \leq -\varepsilon < 0$, which leads to $c_n(\delta) < 0$ by (3.31). □

Lemma 3.4. *Suppose that (L_0), (H_1), (H_4) and (G_1) are satisfied. Then there exists a positive constant C_{15} independent of n such that for all n large enough*

$$b_n \geq -C_{15} n^{\frac{-\beta p}{2-p}}. \tag{3.50}$$

Proof. For any $\gamma \in \Lambda_n$ $(n \geq 2)$, when $0 \notin \gamma(S^n)$, the genus $\Pi(\gamma(S^n))$ is well defined and $\Pi(\gamma(S^n)) \geq \Pi(S^n) = n$. From Proposition 7.8 in [15], we have $\gamma(S^n) \cap E_{n-1}^{\perp} \neq \emptyset$. Otherwise, if $0 \in \gamma(S^n)$, then $0 \in \gamma(S^n) \cap E_{n-1}^{\perp}$. So for any $\gamma \in \Lambda_n$ $(n \geq 2)$, $\gamma(S^n) \cap E_{n-1}^{\perp} \neq \emptyset$. Therefore, for any $\gamma \in \Lambda_n$ $(n \geq 2)$, we get

$$\max_{u\in S^n} J_{T_0}(\gamma(u)) \geq \inf_{u\in E_{n-1}^{\perp}} J_{T_0}(u). \tag{3.51}$$

It follows from Hölder inequality, (1.4), (1.7), (2.13), (2.15) and (2.26) that

$$J_{T_0}(u) \geq \frac{1}{4}\|u\|^2 - C_{16}\|u\|_2^p, \qquad \forall\, u \in E. \tag{3.52}$$

If $u \in E_{n-1}^{\perp}$, $\lambda_n \|u\|_2^2 \leq \|u\|^2$. When $u \in E_{n-1}^{\perp}$, by (3.52) we obtain

$$J_{T_0}(u) \geq \frac{1}{4}\|u\|^2 - C_{16}\lambda_n^{-\frac{p}{2}}\|u\|^p. \tag{3.53}$$

By (3.29), (3.51) and (3.53), for $n \geq 2$, we conclude that

$$\begin{aligned} b_n &\geq \inf_{t\geq 0}\left\{\frac{1}{4}t^2 - C_{16}\lambda_n^{-\frac{p}{2}}t^p\right\} \\ &= -C_{17}\lambda_n^{\frac{-p}{2-p}}, \end{aligned} \tag{3.54}$$

where C_{17} is a positive constant independent of n and λ_n. From (G_2) in Theorem 1.1 and (3.54), it is easy to verify that (3.50) holds. □

Lemma 3.5. *Suppose that $c_n = b_n$ for $n \geq n_0$, where $n_0 \in \mathbb{N}$. Then there exists a positive integer n_1 such that*

$$|b_n| \geq C_{18}n^{\frac{2}{2-\sigma}}, \qquad n \geq n_1, \tag{3.55}$$

where C_{18} is a positive constant independent of n.

Proof. For any $n \geq n_0$ and any $\varepsilon \in (0, |b_n|)$, by (3.29) there exists a map $\gamma_1 \in \Gamma_n$ such that

$$\max_{u\in S_+^{n+1}} J_{T_0}(\gamma_1(u)) < c_n + \varepsilon = b_n + \varepsilon < 0. \tag{3.56}$$

In view of $S^{n+1} = S_+^{n+1} \cup (-S_+^{n+1})$, γ_1 can be continuously extended to S^{n+1} as an odd function, also denoted by γ_1, then $\gamma_1 \in \Lambda_{n+1}$. From (3.29), we have

$$b_{n+1} \leq \max_{u\in S^{n+1}} J_{T_0}(\gamma_1(u)) = J_{T_0}(\gamma_1(u_0)) \tag{3.57}$$

for some $u_0 \in S^{n+1}$. When $u_0 \in S_+^{n+1}$, in combination with (3.56) and (3.57), $b_{n+1} \leq J_{T_0}(\gamma_1(u_0)) < b_n + \varepsilon$. We have

$$b_{n+1} < b_n + \varepsilon + C_6|b_{n+1}|^{\frac{\sigma}{2}}, \tag{3.58}$$

where C_6 is given in (2.31). Otherwise, $u_0 \in -S_+^{n+1}$. It follows from (2.31) and (3.56) that

$$\begin{aligned} J_{T_0}(\gamma_1(u_0)) &\leq J_{T_0}(\gamma_1(-u_0)) + C_6|J_{T_0}(\gamma_1(u_0))|^{\frac{\sigma}{2}} \\ &\leq b_n + \varepsilon + C_6|J_{T_0}(\gamma_1(u_0))|^{\frac{\sigma}{2}}. \end{aligned} \tag{3.59}$$

Next we consider two possible cases.

Case 1. $J_{T_0}(\gamma_1(u_0)) \leq |b_{n+1}|$. By (3.57) and (3.59), we get

$$b_{n+1} \leq b_n + \varepsilon + C_6|b_{n+1}|^{\frac{\sigma}{2}}. \tag{3.60}$$

Case 2. $J_{T_0}(\gamma_1(u_0)) > |b_{n+1}|$. From (3.56), there exists $u_1 \in S_+^{n+1}$ such that

$$J_{T_0}(\gamma_1(u_1)) < b_n + \varepsilon < 0. \tag{3.61}$$

In view of $J_{T_0}(\gamma_1(u_0)) > |b_{n+1}|$ and $J_{T_0}(\gamma_1(u_1)) < 0$. Since $(J_{T_0} \circ \gamma_1) \in C(S^{n+1}, \mathbb{R})$ and S^{n+1} is a connected space with the norm $\|\cdot\|$, by the Intermediate Value Theorem (see Theorem 24.3

in [12]), there exists $u_2 \in S^{n+1}$ such that $J_{T_0}(\gamma_1(u_2)) = |b_{n+1}|/2$. By (3.56), $u_2 \in -S_+^{n+1}$. From (2.31) and (3.56), we obtain

$$\begin{aligned} J_{T_0}(\gamma_1(u_2)) &\le J_{T_0}(\gamma_1(-u_2)) + C_6|J_{T_0}(\gamma_1(u_2))|^{\frac{\sigma}{2}} \\ &< b_n + \varepsilon + C_6|J_{T_0}(\gamma_1(u_2))|^{\frac{\sigma}{2}}, \end{aligned}$$

which implies that

$$b_{n+1} \le b_n + \varepsilon + C_6|b_{n+1}|^{\frac{\sigma}{2}}. \tag{3.62}$$

By Lemma 3.2, $b_n < 0$ for any $n \in \mathbb{N}$. It follows from (3.58), (3.60) and (3.62) that

$$|b_n| \le |b_{n+1}| + C_6|b_{n+1}|^{\frac{\sigma}{2}}, \qquad n \ge n_0. \tag{3.63}$$

Next we show that (3.63) implies (3.55). The proof will be done by induction. First, we introduce a useful inequality as follows:

$$(1+x)^{\alpha_0} \ge 1 + \frac{\alpha_0 x}{2}, \qquad x \in [0, \delta], \tag{3.64}$$

where α_0, δ are positive constants and δ depends on α_0. Set $\alpha_0 = 2(\sigma-2)^{-1}$. In view of (3.64), there exists $\bar{n}_0 \in \mathbb{N}$ such that

$$\left(1+\frac{1}{n}\right)^{\frac{2}{\sigma-2}} \ge 1 + \frac{1}{(\sigma-2)n}, \qquad n \ge \bar{n}_0. \tag{3.65}$$

Set

$$C_{18} = \min\left\{ n_1^{\frac{2}{\sigma-2}}|b_{n_1}|,\ \left(\frac{1}{C_6(\sigma-2)}\right)^{\frac{2}{\sigma-2}} \right\}, \tag{3.66}$$

where $n_1 := \max\{n_0,\ \bar{n}_0\}$. We claim (3.55) holds. By (3.66), it is obvious that $|b_{n_1}| \ge C_{18} n_1^{\frac{2}{2-\sigma}}$. Assume that (3.55) holds for $j \ge n_1$. Then we only need to prove (3.55) also holds for $j+1$. If not, we have

$$|b_{j+1}| < C_{18}(j+1)^{\frac{2}{2-\sigma}}. \tag{3.67}$$

Since (3.55) holds for j, by (2.31), (3.63) and (3.67), we have

$$C_{18} j^{\frac{2}{2-\sigma}} \le |b_j| \le |b_{j+1}| + C_6|b_{j+1}|^{\frac{\sigma}{2}} < C_{18}(j+1)^{\frac{2}{2-\sigma}} + C_6 C_{18}^{\frac{\sigma}{2}}(j+1)^{\frac{\sigma}{2-\sigma}}, \tag{3.68}$$

When we divide (3.68) by $C_{18}(j+1)^{\frac{2}{2-\sigma}}$ on both sides, in view of (3.66) we get

$$\left(1+\frac{1}{j}\right)^{\frac{2}{\sigma-2}} < 1 + C_6 C_{18}^{\frac{\sigma-2}{2}} \frac{1}{j+1} < 1 + C_6 C_{18}^{\frac{\sigma-2}{2}} \frac{1}{j} \le 1 + \frac{1}{(\sigma-2)j},$$

which contradicts (3.65). So (3.55) holds. □

By the fact that $b_n < 0$, (G_2), (3.50) and (3.55), it is impossible that $c_n = b_n$ for all large n. Next we can construct critical values of J_{T_0} as follows.

Lemma 3.6. *Suppose that $c_n > b_n$. Then for any $\delta \in (0, c_n - b_n)$, $c_n(\delta)$ given by (3.31) is a critical value of J_{T_0}.*

Proof. We prove this lemma by contradiction. For any $\delta \in (0, c_n - b_n)$, if $c_n(\delta)$ is not a critical value of J_{T_0}, define $\bar{\varepsilon} = (c_n - b_n - \delta)/2$, by Deformation Theorem, there exist a positive constant $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0,1] \times E, E)$ such that

$$\eta(1, u) = u, \quad \text{if } J_{T_0}(u) \notin [c_n(\delta) - \bar{\varepsilon}, c_n(\delta) + \bar{\varepsilon}], \tag{3.69}$$

and

$$\eta\left(1, J_{T_0}^{c_n(\delta)+\varepsilon}\right) \subset J_{T_0}^{c_n(\delta)-\varepsilon}. \tag{3.70}$$

By (3.31), there exists $\gamma_2 \in \Gamma_n(\delta)$ such that

$$\max_{u \in S_+^{n+1}} J_{T_0}(\gamma_2(u)) < c_n(\delta) + \varepsilon. \tag{3.71}$$

Define

$$\bar{\gamma}_2(u) = \eta(1, \gamma_2(u)), \qquad u \in S_+^{n+1}. \tag{3.72}$$

It is evident that $\bar{\gamma}_2 \in C(S_+^{n+1}, E)$. Since $\gamma_2 \in \Gamma_n(\delta)$, by (3.30) we have

$$J_{T_0}(\gamma_2(u)) \le b_n + \delta = c_n - 2\bar{\varepsilon} \le c_n(\delta) - 2\bar{\varepsilon}, \qquad u \in S^n. \tag{3.73}$$

By (3.69), (3.72) and (3.73), we have $\bar{\gamma}_2(u) = \gamma_2(u)$, $u \in S^n$, which yields

$$\bar{\gamma}_2|_{S_n} \in \Lambda_n \quad \text{and} \quad J_{T_0}(\bar{\gamma}_2(u)) = J_{T_0}(\gamma_2(u)) \le b_n + \delta, \qquad u \in S^n. \tag{3.74}$$

In view of (3.74), we obtain $\bar{\gamma}_2 \in \Gamma_n(\delta)$. It follows from (3.70), (3.71) and (3.72) that

$$\max_{u \in S_+^{n+1}} J_{T_0}(\bar{\gamma}_2(u)) = \max_{u \in S_+^{n+1}} J_{T_0}\big(\eta(1, \gamma_2(u))\big) \le c_n(\delta) - \varepsilon,$$

which contradicts (3.31). So $c_n(\delta)$ given by (3.31) is a critical value of J_{T_0}. □

Proof of Theorem 1.1. Since it is impossible that $c_n = b_n$ for all large n, we can choose a subsequence $\{n_j\} \subset \mathbb{N}$ such that $c_{n_j} > b_{n_j}$. In combination with Lemma 3.3, Lemma 3.4 and Lemma 3.6, there exists a sequence of critical points $\{u_{n_j}\}_{j=1}^{\infty}$ of J such that

$$-C_{15} n_j^{-\frac{\beta p}{(2-p)}} \le b_{n_j} < c_{n_j} \le c_{n_j}(\delta_j) = J_{T_0}(u_{n_j}) < 0, \tag{3.75}$$

where $\delta_j \in (0, c_{n_j} - b_{n_j})$. In view of (H_2), (H_4), (2.5), (2.26) and (2.27), $u_{n_j} \ne 0$, $j \in \mathbb{N}$. Next we consider the following two possible cases.

Case 1. $\|u_{n_j}\|^2 > 2T_0$. Combining (2.15), (2.16) and (2.32), we have $k_{T_0}(u_{n_j}) = 0$, $k'_{T_0}(u_{n_j}) = 0$ and $\psi'(u_{n_j}) = 0$. It follows from (2.17) and (2.45) that

$$\bar{I}_{T_0}(u_{n_j}) = \frac{1}{2}\|u_{n_j}\|^2 - \int_{\mathbb{R}} W_1(t, u_{n_j})dt, \qquad \langle J'_{T_0}(u_{n_j}), u_{n_j}\rangle = \|u_{n_j}\|^2 - \int_{\mathbb{R}} (\nabla W_1(t, u_{n_j}), u_{n_j})dt.$$

By (1.5) and two equalities above, we get

$$\begin{aligned} \bar{I}_{T_0}(u_{n_j}) &= \bar{I}_{T_0}(u_{n_j}) - \mu^{-1}\langle J'_{T_0}(u_{n_j}), u_{n_j}\rangle \\ &= 2A\|u_{n_j}\|^2 + \mu^{-1}\int_{\mathbb{R}} \big((\nabla W_1(t, u_{n_j}), u_{n_j}) - \mu W_1(t, u_{n_j})\big)dt \\ &< A\|u_{n_j}\|^2. \end{aligned} \tag{3.76}$$

Case 2. $\|u_{n_j}\|^2 \le 2T_0$. It follows from Lemma 2.3, (2.17), (2.26)–(2.28) and (2.45) that

$$\bar{I}_{T_0}(u_{n_j}) \le \frac{1}{2}\|u_0\|^2 - \int_{\mathbb{R}} W_1(t, u_{n_j})dt + C_3\tau_{\alpha_4}^{\alpha_4}\|u_{n_j}\|^{\alpha_4} + 16C_4\tau_\sigma^\sigma\|u_{n_j}\|^\sigma,$$

and

$$\langle J'_{T_0}(u_{n_j}), u_{n_j}\rangle \ge \|u_{n_j}\|^2 - \int_{\mathbb{R}} (\nabla W_1(t, u_{n_j}), u_{n_j})dt - 9C_3\tau_{\alpha_4}^{\alpha_4}\|u_{n_j}\|^{\alpha_4} - 16(9 - 32A^{-1})C_4\tau_\sigma^\sigma\|u_{n_j}\|^\sigma.$$

By (1.5), (2.13), and two equalities above, we obtain

$$\begin{aligned}
\bar{I}_{T_0}(u_{n_j}) &= \bar{I}_{T_0}(u_{n_j}) - \mu^{-1}\langle J'_{T_0}(u_{n_j}), u_{n_j}\rangle \\
&\le 2A\|u_{n_j}\|^2 + C_1\tau_{\alpha_1}^{\alpha_1}\|u_{n_j}\|^{\alpha_1} + 10C_3\tau_{\alpha_4}^{\alpha_4}\|u_{n_j}\|^{\alpha_4} + 16(10 - 32A^{-1})C_4\tau_\sigma^\sigma\|u_{n_j}\|^\sigma \\
&\le A\|u_{n_j}\|^2. \qquad (3.77)
\end{aligned}$$

In both cases, by (2.12), (2.24), (3.76) or (3.77), we get $l_\mu(u_{n_j}) = 1$ and $l'_\mu(u_{n_j}) = 0$. Moreover, it follows from (2.26) and (2.27) that $J_{T_0}(u_{n_j}) = \bar{I}_{T_0}(u_{n_j}) \le A\|u_{n_j}\|^2 < 0$, which implies that $\|u_{n_j}\| \to 0$, $j \to \infty$ by (3.75). So there exists $j_0 \in \mathbb{N}$ such that $\|u_{n_j}\|^2 < T_0$, $j \ge j_0$. By (2.15)-(2.16), we have $k_{T_0}(u_{n_j}) = 1$ and $k'_{T_0}(u_{n_j}) = 0$ for all $j \ge j_0$, which leads to $\{u_{n_j}\}$ are also critical points of I for all $j \ge j_0$ by (2.5) and (2.11).

Since E is continuously embedded into $L^\infty(\mathbb{R}, \mathbb{R}^N)$ and $\|u_{n_j}\| \to 0$ as $j \to \infty$, then $\max_{t\in\mathbb{R}} |u_{n_j}(t)| \to 0$ as $j \to \infty$. Thus, there exists $j_1 \in \mathbb{N}$ such that $\max_{t\in\mathbb{R}} |u_{n_j}(t)| < r_1$ for all $j \ge j_1$. Set $j_2 = \max\{j_0, j_1\}$. By (2.5) and (2.11), u_{n_j} are also homoclinic solutions of problem (1.1) for each $j \ge j_2$. This completes the proof. □

4 Examples

In this section, we give an example to illustrate our result.

Example 4.1. In problem (1.1), let $L(t) = t^2 + 1$ and $W(t, x) = a(t)\ln(1 + |x|^{3/2})$, $(t, x) \in \mathbb{R} \times \mathbb{R}$, where $a : \mathbb{R} \to \mathbb{R}^+$ is a continuous function such that $a(t) \in L^4(\mathbb{R})$. Moreover, the perturbation term G is given by

$$G(t, x) = b(t)|x|^{\sigma-1}x, \qquad (t, x) \in \mathbb{R} \times (-r_2, r_2),$$

where b is a bounded continuous function on $\mathbb{R}$ and $\sigma > 8/3$. Let $W_1(t, x) = a(t)|x|^{3/2}$ and $W_2(t, x) = a(t)\big(\ln(1 + |x|^{3/2}) - |x|^{3/2}\big)$. Then we choose $p = \mu = 3/2$ and

$$\alpha_1 = \alpha_3 = \alpha_4 = 3, \qquad \alpha_2 = 3/2, \qquad N = 1.$$

Since $-d^2/dt^2 + L(t)$ has eigenvalues $\lambda_n = 2n + 2$ with multiplicity 1 (see [2]), we can choose $\beta = 1$. By Theorem 1.1, problem (1.1) has infinitely many homoclinic solutions. Since the perturbation term G breaks the symmetry of the energy functional, the results in [20,22,34,35] cannot be applied to this example.

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