



The solution of some persistent $p : -q$ resonant center problems

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Abstract. The notion of $p : -q$ resonant center was introduced recently and studied by several authors. In this paper we generalize the notion of a persistent center to a persistent $p : -q$ resonant center and find conditions for existence of a persistent $p : -q$ resonant center for several $p : -q$ resonant systems with quadratic nonlinearities. To prove the sufficiency of the obtained conditions we use either the Darboux theory of integrability or look for a formal first integral of the required form or we use the method based on the blow-up transformation.

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1 Introduction

An essential part of the theory of systems of ODE's is devoted to studying the so-called center-focus problem of two dimensional analytic systems of ordinary differential equations of the form

$$\dot{u} = -v + P(u, v), \quad \dot{v} = u + Q(u, v), \quad (1.1)$$

where u, v are real variables and $P(u, v), Q(u, v)$ are analytic functions whose series expansions start from terms of degree at least two. This is the problem of distinguishing between a center (all trajectories in a neighborhood of the singular point at the origin are ovals) and a focus (all trajectories in a neighborhood of the singular point at the origin are spirals). Most works on the subject are devoted to investigation of polynomial vector fields. By the Poincaré–Lyapunov

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theorem [29, 34], system (1.1) has a center at the origin if and only if there exists an analytic first integral of the form

$$\Phi(u, v) = u^2 + v^2 + \sum_{j+k \geq 3} \phi_{j,k} u^j v^k. \quad (1.2)$$

The theorem says that the qualitative picture of trajectories in a neighborhood of the singular point is related to local integrability of the system: the singular point is a center if and only if there exists an analytic first integral of the form (1.2).

Although the problem of distinguishing between a center and a focus has been studied in many works (see [3, 20, 35, 37, 42] and the references therein) it is completely solved only for quadratic systems (P and Q in (1.1) are homogeneous quadratic polynomials [13, 28]) and for systems with P and Q being homogeneous cubic polynomials [36]. An extensive bibliography about the center problem can be found in [21].

One of the tools to study the problem of distinguishing between a center and a focus is the Poincaré return map, which we compute after introducing polar coordinates in system (1.1). The difficulty in the study of the center problem using this method arises from the complexity in computing the irreducible decomposition of the variety of the ideal generated by the Lyapunov quantities that are the coefficients of the Poincaré first return map. Since it is easier to study complex varieties than real ones we complexify the real system as follows.

Setting $x = u + iv$ system (1.1) becomes the equation

$$\dot{x} = ix + \tilde{F}(x, \bar{x}).$$

Adjoining to this equation its complex conjugate we have the system

$$\dot{x} = ix + \tilde{F}(x, \bar{x}), \quad \dot{\bar{x}} = -i\bar{x} + \overline{\tilde{F}(x, \bar{x})}.$$

Consider $y := \bar{x}$ as a new variable and $\tilde{G} = \overline{\tilde{F}}$ as a new function. Then, from the latter system we obtain the system of two complex differential equations which we can write in the form

$$\dot{x} = ix + \tilde{F}(x, y), \quad \dot{y} = -iy + \tilde{G}(x, y), \quad (1.3)$$

where x, y are complex variables and $\tilde{F}(x, y)$ and $\tilde{G}(x, y)$ are complex analytic functions whose series expansions start from degree at least two. After the change of time $\tau = it$ and rewriting t instead of τ , system (1.3) becomes

$$\dot{x} = x + F(x, y), \quad \dot{y} = -y + G(x, y). \quad (1.4)$$

Following the Poincaré–Lyapunov theorem and [13] we can extend the concept of a center to complex systems of the form (1.4). We say that system (1.4) has center at the origin if it admits a formal first integral of the form

$$\Phi(x, y) = xy + \sum_{j+k \geq 3} \varphi_{j,k} x^j y^k.$$

In such case we also say that system (1.4) has $1 : -1$ resonant center. In [19] the following generalization of the center problem was proposed. Consider differential systems in \mathbb{C}^2 with a $p : -q$ resonant elementary singular point, i.e.,

$$\dot{x} = px + P(x, y), \quad \dot{y} = -qy + Q(x, y), \quad (1.5)$$

where $p, q \in \mathbb{N}$ and $P(x, y)$ and $Q(x, y)$ are polynomials of the form

$$P(x, y) = \sum_{\substack{j+k \geq 1 \\ j \geq -1, k \geq 0}} a_{jk} x^{j+1} y^k$$

and

$$Q(x, y) = \sum_{\substack{j+k \geq 1 \\ j \geq -1, k \geq 0}} b_{kj} x^k y^{j+1}.$$

Determine when the elementary singular point located at the origin is a resonant center where the definition of a resonant center comes from Dulac [13].

Definition 1.1. The singular point O of a complex system (1.5) is a $p : -q$ resonant center if there exists a local analytic first integral of the form

$$\Psi(x, y) = x^q y^p + \sum_{\substack{j+k \geq p+q+1 \\ j, k \in \mathbb{Z}, j, k \geq 0}} \phi_{j-q, k-p} x^j y^k. \quad (1.6)$$

The simplest case is when P and Q in (1.5) are quadratic polynomials and this case has been studied by several authors (see e.g. [4–6, 12, 17, 25, 31, 38, 40, 43] and references therein). For P and Q being cubic polynomials some results can be found in [1, 5, 10, 22, 27, 32, 39, 41] and for quartic polynomials in [11, 16, 30]. The case where P and Q are homogeneous quintic polynomials has been studied in [14, 23, 24].

In this paper we will use the concept of (weakly) persistent center which was introduced in [7]. In [2] the authors generalized the notion of persistent center and weakly persistent center for complex planar differential systems. In [33] these notions were extended to linearizable persistent centers and linearizable weakly persistent centers for complex planar differential systems.

Definition 1.2. The origin O is a (weakly) persistent center of system (1.4) if it is a center of the system

$$\dot{x} = x + \lambda F(x, y), \quad \dot{y} = -y + \mu G(x, y), \quad x, y \in \mathbb{C}$$

for all $\lambda, \mu \in \mathbb{C}$ ($\lambda = \mu \in \mathbb{C}$).

We now extend the notion of (weakly) persistent center to a (weakly) persistent $p : -q$ resonant center and introduce the following generalization of a $p : -q$ resonant center.

Definition 1.3. The origin O is called a persistent $p : -q$ resonant center (weakly persistent $p : -q$ resonant center) of system (1.5) if it is a $p : -q$ resonant center of the system

$$\dot{x} = px + \lambda P(x, y), \quad \dot{y} = -qy + \mu Q(x, y), \quad (1.7)$$

for all $\lambda, \mu \in \mathbb{C}$ ($\lambda = \mu \in \mathbb{C}$).

In [2, Theorem 2.1] the following Theorem was proven for $p = q = 1$.

Theorem 1.4 ([2]). The origin is a $p : -q$ resonant center of system (1.7) for all $\lambda, \mu \in \mathbb{C}$ satisfying $\lambda\mu = 0$, if it is a $p : -q$ resonant center of system (1.7) for all $\lambda, \mu \in \mathbb{C} \setminus \{0\}$.

To prove the above theorem one just has to rewrite the proof of [2, Proof of Theorem 2.1] and change “center” to “ $p : -q$ resonant center”.

In this paper we seek for systems having $p : -q$ resonant center within systems of the form (1.7), where P and Q are quadratic polynomials and (p, q) is either $(1, 2)$, $(1, 3)$, $(1, 4)$ or $(2, 3)$. Such systems are written as

$$\begin{aligned}\dot{x} &= px + a_{10}x^2 + a_{01}xy + a_{-12}y^2 \\ \dot{y} &= -qy + b_{2,-1}x^2 + b_{10}xy + b_{01}y^2,\end{aligned}\tag{1.8}$$

where $x, y, a_{ij}, b_{ji} \in \mathbb{C}$. To find necessary conditions we use the approach described in the next section. Then, using several methods we prove the existence of a first integral of the form (1.6).

2 Preliminaries

To determine if system (1.5) has a resonant center at the origin, by Definition 1.1 we look for a formal first integral of the form (1.6) satisfying the identity

$$\dot{\Psi} := \frac{\partial \Psi}{\partial x}(px + P(x, y)) + \frac{\partial \Psi}{\partial y}(-qy + Q(x, y)) \equiv 0.$$

Similar as in case of a regular center the (formal) series for $\dot{\Psi}$ reduces to

$$\dot{\Psi} = g_{q,p}(x^q y^p)^2 + g_{2q,2p}(x^q y^p)^3 + g_{3q,3p}(x^q y^p)^4 + \dots,$$

where $g_{kq, kp}$ is called the k -th saddle quantity (or k -th focus quantity [35]). Saddle quantities are polynomials in the coefficients a_{ij}, b_{ji} of system (1.5). We see that by Definition 1.1 system (1.5) has a resonant center at the origin if and only if

$$g_{kq, kp}(a, b) = 0, \quad \forall k \in \mathbb{N}.$$

Thus, to obtain conditions for resonant center at the origin of system (1.5) we have to find the set of all parameters (a, b) where all polynomials $g_{kq, kp}$ vanish, i.e. we need to find the variety of the ideal $\langle g_{kq, kp} : k = 1, 2, \dots \rangle^1$.

If we restrict our attention to the systems (1.7), then, for any fixed λ and μ we can easily compute $g_{kq, kp} = g_{kq, kp}(\lambda, \mu, a, b)$ and obtain saddle quantities

$$g_{kq, kp} = \sum_{m, n} g_{kq, kp}^{(m, n)}(a, b) \lambda^m \mu^n,$$

which can be considered as polynomials in λ and μ . Furthermore, the coefficient $g_{kq, kp}^{(m, n)}(a, b)$ in the term with $\lambda^m \mu^n$ plays an important role in the analysis of the persistent resonant centers. We call it the $k_{(m, n)}$ -th persistent saddle quantity. If the origin is a center of system (1.7) for all $\lambda, \mu \in \mathbb{C}$, then it is by Definition 1.3 a persistent center of system (1.5).

¹Variety of the ideal generated by polynomials f_1, \dots, f_s is the set of common zeros of polynomial system $f_1 = 0, \dots, f_s = 0$, i.e.

$$\mathbf{V}(\langle f_1, \dots, f_s \rangle) = \{a = (a_1, \dots, a_n) \in k^n : f_i(a) = 0, \text{ for every } i = 1, \dots, s\}.$$

Because of Theorem 1.4 we can for (1.7) always assume that $\lambda\mu \neq 0$. We now define the following sets of polynomials

$$C_k = \left\{ g_{kq, kp}^{(m, n)}(a, b); m, n \in \mathbb{N}_0, m + n = kq + kp \right\}, \quad k = 1, 2, 3, \dots$$

and the ideals

$$\begin{aligned} \mathcal{C}^p &:= \langle C_1, C_2, \dots, C_k, \dots \rangle, \\ \mathcal{C}_K^p &:= \langle C_1, C_2, \dots, C_K \rangle. \end{aligned}$$

The ideals \mathcal{C}^p and \mathcal{C}_K^p are ideals in the polynomial ring $\mathbb{C}[a, b]$. By the Hilbert Basis Theorem (see e.g. [8, Theorem 1.1.6]) any ideal \mathcal{C}^p is finitely generated that means that there exists $N \in \mathbb{N}$ such that for every $k > N$, $\mathcal{C}_k^p = \mathcal{C}_N^p$.

Therefore, in order to find necessary conditions for the existence of a persistent $p : -q$ resonant center for system (1.5) we have to find first few saddle quantities and then to compute the variety of the ideal generated by these saddle quantities.

Note that the variety of the ideal \mathcal{C}^p is always easier to obtain than the (regular) center variety $\mathbf{V}(\langle g_{kq, kp}(a, b) : k \in \mathbb{N} \rangle)$ since the saddle quantities, $g_{kq, kp}^{(m, n)}(a, b)$ are split compared to (regular) saddle quantities $g_{kq, kp}(a, b)$. Also note that if system has persistent $p : -q$ resonant center, then it also has $p : -q$ resonant center which will be useful fact in the next section where for some cases the (regular) $p : -q$ resonant center problem has been solved, already.

In the following section we present the main results of the paper. We find necessary and sufficient conditions for some persistent $p : -q$ resonant quadratic systems. For proving the sufficiency of the obtained conditions we mainly use the Darboux theory of integrability which is one of the main methods for proving the existence of first integrals for polynomial systems of differential equations on \mathbb{C}^2 (or \mathbb{R}^2). We recall briefly some results related to this theory. We consider systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (2.1)$$

where $x, y \in \mathbb{C}$, P and Q are polynomials without constant terms that have no nonconstant common factor, and $m = \max(\deg(P), \deg(Q))$. By the definition a Darboux factor of system (2.1) is a polynomial $f(x, y)$ such that

$$\frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q = Kf,$$

where $K(x, y)$ is a polynomial of degree at most $n - 1$ ($K(x, y)$ is called the cofactor). The polynomial f defines an invariant algebraic curve $f = 0$ of system (2.1). A simple computation shows that if there are Darboux factors f_1, f_2, \dots, f_k with the cofactors K_1, K_2, \dots, K_k satisfying

$$\sum_{i=1}^k \alpha_i K_i = 0,$$

then $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$, is a Darboux first integral of (2.1), and if

$$\sum_{i=1}^k \alpha_i K_i + \tilde{P}'_x + \tilde{Q}'_y = 0$$

then the equation admits the Darboux integrating factor

$$M = f_1^{\alpha_1} \cdots f_k^{\alpha_k}. \quad (2.2)$$

The definition of Darboux integrating factor is consistent with the classical definition of an integrating factor. For proving the sufficiency of the conditions concerning the existence of Darboux integrating factor we several times refer to following theorem, or more precisely to part (ii) of the following theorem.

Theorem 2.1 ([4]). *If system*

$$\begin{aligned}\dot{x} &= x + F(x, y) \\ \dot{y} &= -qy + G(x, y)\end{aligned}$$

has a local (reciprocal) integrating factor of the form (2.2), with f_i analytic in x and y and nonzero α_i , then the system is

- *integrable if q is irrational;*
- *integrable or orbitally normalizable if q is a nonzero rational.*

More precisely,

- (i) *if all $f_i(0,0) \neq 0$, then the system is integrable;*
- (ii) *if at most one $f_i(0,0)$ vanishes and the corresponding Darboux factor has one of forms $f_i(x, y) = x + o(x, y)$ and $f_i(x, y) = y + o(x, y)$, then the system is integrable;*
- (iii) *if exactly two factors $f_1(x, y) = x + o(x, y)$ and $f_2(x, y) = y + o(x, y)$ vanish at the origin, then the system is integrable, except when the two coefficients α_1 and α_2 are both integers greater than 1, in which case it is orbitally normalizable;*
- (iv) *if (iii) is satisfied and there exists a Darboux change of one coordinate transforming one of the equations into the normal form $\dot{x} = xh(u)$ or $\dot{y} = -qyh(u)$, where $h(u) = 1 + O(u)$ and $u = x^c y^d$ is the resonant monomial as in Case II or Case III [4, Theorem 4.3], then the system is normalizable.*

3 Main results

In this section we consider the problem of persistent $p : -q$ resonant center of system (1.8) for the following values of p and q :

- a) $p = 1, q = 2$;
- b) $p = 1, q = 3$;
- c) $p = 1, q = 4$;
- d) $p = 1, q = 5$;
- e) $p = 2, q = 3$.

According to Definition 1.3 we look for a systems with resonant center within the family

$$\begin{aligned}\dot{x} &= px + \lambda(a_{10}x^2 + a_{01}xy + a_{-12}y^2) \\ \dot{y} &= -qy + \mu(b_{2,-1}x^2 + b_{10}xy + b_{01}y^2)\end{aligned}\tag{3.1}$$

for all $\lambda, \mu \in \mathbb{C}$.

a) Case $p = 1, q = 2$.

Theorem 3.1. *System (1.8) has a persistent $1 : -2$ resonant center at the origin if and only if one of the following four conditions holds:*

1. $b_{01} = b_{2,-1} = a_{-12} = a_{10} = 0$;
2. $b_{01} = b_{2,-1} = a_{01} = 0$;
3. $b_{10} = b_{2,-1} = 0$;
4. $a_{-12} = a_{01} = a_{10} = 2b_{10}^2 + b_{2,-1}b_{01} = 0$.

Proof. In order to obtain conditions listed above we compute first four saddle quantities $g_{2,1}, \dots, g_{8,4}$ of system (3.1) and obtain

$$\begin{aligned} g_{2,1} = & \left(-a_{01}a_{10}b_{10} - \frac{1}{2}a_{01}^2b_{2,-1} - \frac{3}{5}a_{10}a_{-12}b_{2,-1} \right) \lambda^2\mu \\ & - \left(\frac{1}{2}a_{10}b_{01}b_{10} - \frac{1}{4}a_{01}b_{01}b_{2,-1} + \frac{1}{20}a_{-12}b_{10}b_{2,-1} \right) \lambda\mu^2 \\ & + \left(\frac{1}{2}b_{01}b_{10}^2 + \frac{1}{4}b_{01}^2b_{2,-1} \right) \mu^3 \end{aligned}$$

and so on. Therefore, we have

$$\begin{aligned} g_{2,1}^{(3,0)} &= 0, \\ g_{2,1}^{(2,1)} &= -a_{01}a_{10}b_{10} - \frac{1}{2}a_{01}^2b_{2,-1} - \frac{3}{5}a_{10}a_{-12}b_{2,-1}, \\ g_{2,1}^{(1,2)} &= -\frac{1}{2}a_{10}b_{01}b_{10} + \frac{1}{4}a_{01}b_{01}b_{2,-1} - \frac{1}{20}a_{-12}b_{10}b_{2,-1}, \\ g_{2,1}^{(0,3)} &= \frac{1}{2}b_{01}b_{10}^2 + \frac{1}{4}b_{01}^2b_{2,-1} \end{aligned}$$

and $C_1 = \{g_{2,1}^{(2,1)}, g_{2,1}^{(1,2)}, g_{2,1}^{(0,3)}\}$. In a similar way we also obtain C_2, C_3 and C_4 and it turns out that

$$C_4^p = \langle C_1, C_2, C_3, C_4 \rangle = C_3^p = \langle C_1, C_2, C_3 \rangle.$$

Hence, using the routine `minAssGTZ` [9] of computer algebra system `SINGULAR` [26] we compute the decomposition of the variety of ideal C_3^p and obtain four components listed in Theorem 3.1. For the sufficiency of these conditions we use [18, 19], where authors solved the resonant center problem for system (1.8) with $(p, q) = (1, 2)$. They found 20 conditions for a resonant center and among them there are also the above listed four conditions corresponding to the persistent resonant centers. Since in [19] the authors showed that in each of the 20 cases there is an analytic first integral of the form (1.6), the proof of this theorem is completed. \square

b) Case $p = 1, q = 3$

Theorem 3.2. *System (1.8) has a persistent $1 : -3$ resonant center at the origin if and only if one of the following five conditions holds:*

1. $b_{10} = a_{-12} = a_{01} = a_{10} = 0$;
2. $b_{10} = b_{2,-1} = 0$;

$$3. a_{-12} = a_{01} = a_{10} = 3b_{10}^2 + 4b_{2,-1}b_{01} = 0;$$

$$4. b_{01} = b_{2,-1} = a_{-12} = a_{10} = 0;$$

$$5. b_{01} = a_{-12} = a_{01} = 0.$$

Proof. The computation of obtained conditions goes in a similar way as in previous case. The sufficiency is ensured using [12] where $1 : -3$ resonant center problem for system (1.8) was solved. \square

c) Case $p = 1, q = 4$

Theorem 3.3. *System (1.8) has a persistent $1 : -4$ resonant center at the origin if and only if one of the following five conditions holds:*

$$1. a_{-12} = a_{01} = a_{10} = 4b_{10}^2 + 9b_{2,-1}b_{01} = 0;$$

$$2. b_{10} = b_{2,-1} = 0;$$

$$3. a_{-12} = a_{01} = a_{10} = 6b_{10}^2 + b_{2,-1}b_{01} = 0;$$

$$4. b_{01} = b_{2,-1} = a_{-12} = a_{10} = 0;$$

$$5. b_{01} = a_{-12} = a_{01} = 0.$$

Proof. The computation of saddle quantities and the corresponding ideals is similar as in previous two cases. The above five persistent resonant center cases of system (1.8) are listed among 55 conditions (proven to be necessary and sufficient) for the existence of a $1 : -4$ resonant center in [17]. Consequently, we have five necessary and sufficient conditions for persistent $1 : -4$ resonant centers. \square

d) Case $p = 1, q = 5$

Theorem 3.4. *System (1.8) has a persistent $1 : -5$ resonant center at the origin if and only if one of the following four conditions holds:*

$$1. b_{10} = b_{2,-1} = a_{10} = 0;$$

$$2. b_{10} = a_{-12} = a_{01} = a_{10} = 0;$$

$$3. b_{01} = b_{10} = b_{2,-1} = 0;$$

$$4. b_{01} = a_{-12} = a_{01} = 0.$$

The $1 : -5$ resonant center problem for quadratic system of the form (1.8) has been not considered, yet. Here we present four conditions for resonant center, which are also conditions for persistent resonant center. The proof of Theorem 3.4 is given in the next section.

e) Case $p = 2, q = 3$

Theorem 3.5. *System (1.8) has a persistent $2 : -3$ resonant center at the origin if one of the following three conditions holds:*

$$1. b_{10} = b_{2,-1} = 0;$$

$$2. a_{-12} = a_{01} = 0;$$

$$3. b_{01} = b_{2,-1} = a_{-12} = a_{10} = 0;$$

The $2 : -3$ resonant center problem (1.5) was solved only when (1.5) is cubic Lotka–Volterra system, see [10,22]. In the quadratic case the problem is still open. Here, we do not give a complete list of all $2 : -3$ resonant center conditions for (1.8), but we present three systems with resonant center, which are also persistent resonant center. The proof of Theorem 3.5 is given in Section 5.

4 Proof of sufficiency of Theorem 3.4

To prove the sufficiency of the conditions, we apply the Darboux theory of integrability to construct the Darboux integrating factor in all cases unless in case 3, where we look for a formal first integral of the form $\Psi(x, y) = \sum_{k=1}^{\infty} f_k(x)y^k$. Below is the case-by-case analysis.

Case 1. In this case system (3.1) has the form:

$$\dot{x} = x + \lambda(a_{01}xy + a_{-12}y^2), \quad \dot{y} = -5y + \mu b_{01}y^2.$$

We find two invariant lines $l_1 = y$ and $l_2 = y - \frac{5}{b_{01}\mu}$, which help us to construct the Darboux integrating factor $M = l_1^{-4/5}l_2^{-(5a_{01}\lambda+6b_{01}\mu)/5b_{01}\mu}$ for $b_{01} \neq 0$ and $\mu \neq 0$. By Theorem 2.1 there exists a first integral of the form (1.6) with $p = 1$ and $q = 5$.

Remark 4.1. By Theorem 1.4 we can conclude, that origin is a center of system also for $\mu = 0$. If $b_{01} = 0$ this case coincides with the Case 3 of this theorem for $a_{10} = 0$. Note that in this case the rational functions $f_k(x)$ become polynomial.

Case 2. In this case system (3.1) is written as:

$$\dot{x} = x, \quad \dot{y} = -5y + \mu(b_{2,-1}x^2 + b_{01}y^2),$$

and it has invariant line $l_1 = x$ and two invariant curves

$$\begin{aligned} l_2 &= \frac{1}{15}ib_{01}^{3/2}b_{2,-1}^{3/2}\mu^3x^3 + \frac{1}{5}ib_{01}^{3/2}\sqrt{b_{2,-1}}\mu^2xy + \frac{1}{15}b_{01}^2b_{2,-1}\mu^3x^2y - \frac{2}{5}b_{01}b_{2,-1}\mu^2x^2 \\ &\quad - i\sqrt{b_{01}b_{2,-1}}\mu x - \frac{b_{01}\mu y}{5} + 1, \\ l_3 &= -\frac{1}{15}ib_{01}^{3/2}b_{2,-1}^{3/2}\mu^3x^3 - \frac{1}{5}ib_{01}^{3/2}\sqrt{b_{2,-1}}\mu^2xy + \frac{1}{15}b_{01}^2b_{2,-1}\mu^3x^2y - \frac{2}{5}b_{01}b_{2,-1}\mu^2x^2 \\ &\quad + i\sqrt{b_{01}b_{2,-1}}\mu x - \frac{b_{01}\mu y}{5} + 1, \end{aligned}$$

which allows us to construct a Darboux integrating factor $M = l_1^4(l_2l_3)^{-1}$. To prove the existence of a first integral of the form (1.6) with $p = 1$ and $q = 5$ we refer to Theorem 2.1.

Case 3. In this case we find only one invariant curve $f_1 = y$, which is not enough to construct Darboux first integral or Darboux integrating factor. Note that conditions in this case are $b_{2,-1} = 0$, $b_{10} = 0$ and $b_{01} = 0$, the corresponding system is

$$\dot{x} = x + \lambda(a_{10}x^2 + a_{01}xy + a_{-12}y^2), \quad \dot{y} = -5y.$$

We look for a formal first integral in the form $\Psi(x, y) = \sum_{k=1}^{\infty} f_k(x)y^k$. The functions f_k are determined recursively by the differential equation

$$a_{-12}\lambda f'_{k-2}(x) - a_{01}\lambda x f'_{k-1}(x) - 5k f_k(x) + x(1 + a_{10}\lambda x) f'_k(x) = 0.$$

For $k = 1, 2, 3, 4$ (setting the integration constant equal to 1) we obtain

$$\begin{aligned} f_1(x) &= \frac{x^5}{(1 + a_{10}\lambda x)^5}, & f_2(x) &= \frac{x^{10} + \dots + \alpha_2}{(1 + a_{10}\lambda x)^{10}}, \\ f_3(x) &= \frac{x^{15} + \dots + \alpha_3}{(1 + a_{10}\lambda x)^{15}}, & f_4(x) &= \frac{x^{20} + \dots + \alpha_4}{(1 + a_{10}\lambda x)^{20}}. \end{aligned}$$

Suppose by induction that $f_k(x) = \frac{p_{5k}(x)}{(1 + a_{10}\lambda x)^{5k}}$, where $p_{5k}(x)$ denotes a polynomial of degree at most $5k$ and $k = 1, \dots, n-1$. In order to complete this task we solve the differential equation

$$f'_n(x) = \frac{5n}{x(1 + a_{10}\lambda x)} f_n(x) + \frac{a_{01}\lambda x f'_{n-1} - a_{-12}\lambda f'_{n-2}}{x(1 + a_{10}\lambda x)}, \quad (4.1)$$

using the induction assumption about the form of f_{n-1} and f_{n-2} .

The general solution of linear differential equation of the form

$$f'(x) = g(x)f(x) + h(x) \quad (4.2)$$

is

$$f(x) = C e^{\int g(x) dx} + e^{\int g(x) dx} \int e^{-\int g(x) dx} h(x) dx. \quad (4.3)$$

In this case $g(x) = \frac{5n}{x(1 + a_{10}\lambda x)}$ and $h(x) = \frac{p_{5n-4}(x)}{x(1 + a_{10}\lambda x)^{5n-3}}$, yielding $e^{\int g(x) dx} = \frac{x^{5n}}{(1 + a_{10}\lambda x)^{5n}}$ and

$$e^{-\int g(x) dx} h(x) = \frac{(1 + a_{10}\lambda x)^{5n}}{x^{5n}} \cdot \frac{p_{5n-4}(x)}{x(1 + a_{10}\lambda x)^{5n-3}} = \frac{p_{5n-1}(x)}{x^{5n+1}}.$$

Rewriting $e^{-\int g(x) dx} h(x)$ as

$$\frac{p_{5n-1}(x)}{x^{5n+1}} = \frac{a_0 + a_1 x + \dots + a_{5n-1} x^{5n-1}}{x^{5n+1}} = \frac{a_0}{x^{5n+1}} + \frac{a_1}{x^{5n}} + \dots + \frac{a_{5n-1}}{x^2},$$

and integrating, yields

$$\int e^{-\int g(x) dx} h(x) dx = \frac{\bar{a}_0}{x^{5n}} + \frac{\bar{a}_1}{x^{5n-1}} + \dots + \frac{\bar{a}_{5n-1}}{x}$$

for some $\bar{a}_0, \bar{a}_1, \dots, \bar{a}_{5n-1}$. Therefore, using (4.3) and choosing integration constant as $C = 1$ we obtain the solution of (4.1)

$$\begin{aligned} f_n(x) &= \frac{x^{5n}}{(1 + a_{10}\lambda x)^{5n}} + \frac{x^{5n}}{(1 + a_{10}\lambda x)^{5n}} \left[\frac{\bar{a}_0}{x^{5n}} + \frac{\bar{a}_1}{x^{5n-1}} + \dots + \frac{\bar{a}_{5n-1}}{x} \right] \\ &= \frac{[1 \cdot x^{5n} + \bar{a}_{5n-1} x^{5n-1} + \dots + \bar{a}_0]}{(1 + a_{10}\lambda x)^{5n}} = \frac{\bar{p}_{5n}}{(1 + a_{10}\lambda x)^{5n}}, \end{aligned}$$

where \bar{p}_{5n} denotes a polynomial of degree at most $5n$. Therefore, it exists analytic first integral of the form $\Psi(x, y) = \sum_{k=1}^{\infty} f_k(x)y^k$ whose power series expansion is of the form $x^5 y + \sum_{i+j>6}^{\infty} \alpha_{ij} x^i y^j$.

Case 4. The system

$$\dot{x} = x + \lambda a_{10}x^2, \quad \dot{y} = -5y + \mu(b_{2,-1}x^2 + b_{10}xy) \quad (4.4)$$

has two invariant lines $l_1 = x$ and $l_2 = 1 + a_{10}x\lambda$, which allow to construct a Darboux integrating factor of the form $M = l_1^4 l_2^{-(6a_{10}\lambda + b_{10}\mu)/a_{10}\mu}$, for $a_{10} \neq 0$ and $\mu \neq 0$. Thus, for $a_{10} \neq 0$ and $\mu \neq 0$ system (4.4) has a first integral of the form (1.6) with $p = 1$ and $q = 5$ according to Theorem 2.1.

In case $\mu = 0$ we again refer to Theorem 1.4 and we can conclude, that origin is a center of system also for $\mu = 0$. On the other hand if $\mu = 0$, system (4.4) becomes subcase of system in Case 3 of this theorem, which is already proven to be integrable.

In case $a_{10} = 0$ the corresponding system (4.4) is

$$\dot{x} = x, \quad \dot{y} = -5y + \mu(b_{2,-1}x^2 + b_{10}xy).$$

We look for a formal first integral in the form $\Psi(x, y) = \sum_{k=5}^{\infty} f_k(y)x^k$. The functions $f_k(y)$ are determined recursively by the differential equation

$$b_{2,-1}\mu f'_{k-2}(y) + b_{10}\mu y f'_{k-1}(y) + k f_k(y) - 5y f'_k(y) = 0, \quad k = 5, 6, 7, \dots, \quad (4.5)$$

where $f_3(y) = f_4(y) = 0$. We seek for polynomial solutions of (4.5). We claim that a particular solution to (4.5) for $k \geq 5$ is a linear polynomial. For $k = 5$ this is trivial to check, since (4.5) becomes

$$5f_5(y) - 5y f'_5(y) = 0,$$

yielding $f_5(y) = C_5 y$, where $C_5 \in \mathbb{R}$. For sake of simplicity we choose $C_5 = 1$. For $k = 6$ (4.5) becomes

$$b_{10}\mu y \cdot 1 + 6f_6(y) - 5y f'_6(y) = 0,$$

yielding $f_6(y) = C_6 y^{\frac{6}{5}} - y\mu b_{10}$, where $C_6 \in \mathbb{R}$. Choosing $C_6 = 0$ one obtains $f_6(y) = -y\mu b_{10}$. For $k = 7$ (4.5) becomes

$$b_{2,-1}\mu - b_{10}^2\mu^2 y + 7f_7(y) - 5y f'_7(y) = 0,$$

yielding $f_7(y) = C_7 y^{\frac{7}{5}} - \frac{1}{7}\mu b_{2,-1} + \frac{1}{2}y\mu^2 b_{10}^2$, where $C_7 \in \mathbb{R}$. Choosing $C_7 = 0$ one obtains the linear polynomial $f_7(y) = -\frac{1}{7}\mu b_{2,-1} + \frac{1}{2}y\mu^2 b_{10}^2$. Inductively, if $f_5(y), f_6(y), \dots, f_{k-2}(y)$ and $f_{k-1}(y)$ are linear polynomials, then clearly (4.5) takes the form

$$A_k + B_k y + k f_k(y) - 5y f'_k(y) = 0,$$

where A_k and B_k are some constants. Note that this is a linear ODE of first order, whose nonhomogeneous part is being a linear polynomial $A_k + B_k y$. This clearly yields a particular solution for $f_k(y)$ in form of a linear polynomial. In particular, it is trivial to check that for $k \geq 7$ the solution to (4.5) takes the form

$$f_k(y) = C_k y^{\frac{k}{5}} + \frac{(-1)^{k-5}}{(k-5)!} (b_{10}\mu)^{k-5} y + \frac{(-1)^{k-6}}{k \cdot (k-7)!} \mu^{k-6} b_{10}^{k-7} b_{2,-1}.$$

Now, setting $C_5 = 1$, $C_6 = 0$ and $C_k = 0$ for $k \geq 7$ proves the existence of an analytic first integral of the form $\Psi(x, y) = \sum_{k=5}^{\infty} f_k(y)x^k$ which is of the form (1.6) with $p = 1$ and $q = 5$.

5 Proof of sufficiency of Theorem 3.5

Case 1. We consider the system

$$\begin{aligned}\dot{x} &= 2x + \lambda a_{10}x^2 \\ \dot{y} &= -3y + \mu (b_{2,-1}x^2 + b_{10}xy + b_{01}y^2),\end{aligned}\tag{5.1}$$

where $a_{10}, b_{2,-1}, b_{10}, b_{01} \in \mathbb{C}$.

After the blow-up transformation (see [15] where this transformation is shown that it is useful to obtain the sufficiency)

$$(x, y) \longmapsto (z, y) = \left(\frac{x}{y}, y \right)$$

we obtain the following system

$$\begin{aligned}\dot{z} &= 5z - \mu b_{01}yz + (\lambda a_{10} - \mu b_{10})yz^2 - \mu b_{2,-1}yz^3 = P(z, y) \\ \dot{y} &= -3y + \mu b_{01}y^2 + \mu b_{10}y^2z + \mu b_{2,-1}y^2z^2 = Q(z, y).\end{aligned}\tag{5.2}$$

We now look for the first integral of the system (5.2) of the form

$$\Psi(z, y) = \sum_{k=5}^{\infty} f_k(z) y^k.$$

We compute $\dot{\Psi} = \frac{\partial \Psi(z, y)}{\partial z} P(z, y) + \frac{\partial \Psi(z, y)}{\partial y} Q(z, y)$ and for each $k \geq 5$ set the coefficient of power y^k to zero. Setting $f_4(z) = 0$ this yields for $k \geq 5$ the following recurrence differential equation for $f_k(z)$ and $f_{k-1}(z)$

$$\begin{aligned}0 &= (k-1)\mu(b_{01} + zb_{10} + z^2b_{2,-1})f_{k-1}(z) - 3kf_k(z) \\ &\quad + 5zf'_k(z) - z(\mu b_{01} + z(\mu b_{10} - \lambda a_{10}) + z^2\mu b_{2,-1})f'_{k-1}(z).\end{aligned}$$

For $k = 5, 6, 7, 8, 9$ we find

$$\begin{aligned}f_5(z) &= z^3, \\ f_6(z) &= z^3 \left(\frac{2}{3}\mu b_{01} - z \left(\mu b_{10} + \frac{3}{2}\lambda a_{10} \right) - \frac{2}{7}\mu b_{2,-1}z^2 \right), \\ f_7(z) &= z^3 p_4(z), \\ f_8(z) &= z^3 q_5(z), \\ f_9(z) &= z^3 r_6(z),\end{aligned}$$

where $p_4(z)$, $q_5(z)$ and $r_6(z)$ are some polynomials of degree at most 4, 5 and 6, respectively. So, we assume that

$$f_k(z) = z^3 R_{k-3}(z),$$

where $R_{k-3}(z) = \sum_{j=0}^{k-3} \rho_j z^j$ denotes a polynomial of degree at most $k-3$. We prove this by induction. We have to solve the following differential equation

$$f'_n(z) = \frac{3n}{5z} f_n(z) + \alpha(z) \cdot f'_{n-1}(z) - (n-1) \frac{\beta(z)}{z} f_{n-1}(z),\tag{5.3}$$

where

$$\alpha(z) = \frac{\mu b_{2,-1} z^2 + (\mu b_{10} - \lambda a_{10}) z + \mu b_{01}}{5},$$

$$\beta(z) = \frac{\mu (b_{01} + b_{10} z + b_{2,-1} z^2)}{5}.$$

Suppose

$$f_k(z) = z^3 R_{k-3}(z) = \sum_{j=0}^{k-3} \rho_j z^{j+3}, \quad \text{for } k = 5, 6, \dots, n-1.$$

Then $f'_k(z) = \sum_{j=0}^{k-3} (j+3) \rho_j z^{j+2}$, and

$$\begin{aligned} & \alpha(z) \cdot f'_{n-1}(z) - (n-1) \frac{\beta(z)}{z} f_{n-1}(z) \\ &= \frac{\mu b_{2,-1} z^2 + (\mu b_{10} - \lambda a_{10}) z + \mu b_{01}}{5} \cdot \sum_{j=0}^{n-4} (j+3) \rho_j z^{j+2} \\ & \quad - (n-1) \frac{\mu (b_{01} + b_{10} z + b_{2,-1} z^2)}{5} \sum_{j=0}^{n-4} \rho_j z^{j+2}. \end{aligned} \quad (5.4)$$

It is very important to see that the coefficient to the highest power n in expression (5.4) vanishes

$$\frac{\mu b_{2,-1} z^2}{5} \cdot (n-4+3) \rho_{n-4} z^{n-4+2} - (n-1) \frac{\mu (b_{2,-1} z^2)}{5z} \rho_{n-4} z^{n-4+3} = 0.$$

Also, note that the lowest power of expression (5.4) is obviously z^2 . This implies that differential equation (5.3) becomes

$$f'_n(z) = \frac{3n}{5z} f_n(z) + z^2 W_{n-3}(z), \quad (5.5)$$

where $W_{n-3}(z)$ is a polynomial of degree at most $n-3$. From differential equation (5.5) according to (4.3) we have

$$g(z) = \frac{3n}{5z}, \quad h(z) = z^2 W_{n-3}(z) = w_0 z^2 + w_1 z^3 + w_2 z^4 + \dots + w_{n-3} z^{n-1}.$$

A direct integration yields

$$\begin{aligned} e^{\int g(z) dz} &= z^{\frac{3n}{5}}, \\ z^{\frac{3n}{5}} \int z^2 W_{n-3}(z) \cdot z^{-\frac{3n}{5}} dz &= z^3 \sum_{k=0}^{n-3} \frac{5z^k w_k}{5(k+3) - 3n} = z^3 \cdot Q_{n-3}(z), \end{aligned}$$

since $z^{\frac{3n}{5}} \int w_k z^{k+2} \cdot z^{-\frac{3n}{5}} dz = w_k z^{\frac{3n}{5}} \int z^{k+2-\frac{3n}{5}} dz = \frac{5w_k z^{k+3}}{5(k+3)-3n}$, and finally

$$f_n(z) = Cz^{\frac{3n}{5}} + z^3 Q_{n-3}(z).$$

For $C = 0$ we obtain $f_n(z) = z^3 Q_{n-3}(z)$, where $Q_{n-3}(z)$ is a polynomial of degree $n-3$, which completes the proof.

We proved that the formal first integral of (5.2) is of the form

$$\Psi(z, y) = \sum_{k=5}^{\infty} z^3 R_{k-3}(z) y^k = z^3 y^5 + \sum_{k=6}^{\infty} z^3 R_{k-3}(z) y^k.$$

Setting $R_{k-3}(z) = \sum_{j=0}^{k-3} \rho_j z^j$ and applying the inverse blow-up transformation $z \mapsto \frac{x}{y}$, $y \mapsto y$ yields

$$\begin{aligned} \tilde{\Psi}(x, y) &= \Psi\left(\frac{x}{y}, y\right) = \frac{x^3}{y^3} y^5 + \sum_{k=6}^{\infty} \frac{x^3}{y^3} \frac{\left(\sum_{j=0}^{k-3} \rho_j x^j y^{k-3-j}\right)}{y^{k-3}} y^k \\ &= x^3 y^2 + x^3 \sum_{K=0}^{\infty} \left(\sum_{j=0}^{K+3} \rho_j x^j y^{K+3-j}\right) \\ &= x^3 y^2 + \psi_{4,2} x^4 y^2 + \psi_{5,1} x^5 y + \psi_{6,0} x^6 + \psi_{7,0} x^7 + h.o.t., \end{aligned}$$

which is a formal first integral of (5.1) of the required form.

Case 2. The corresponding system has the form

$$\begin{aligned} \dot{x} &= 2x + \lambda(a_{10}x^2 + a_{01}xy + a_{-12}y^2) \\ \dot{y} &= -3y + \mu b_{01}y^2, \end{aligned} \tag{5.6}$$

where $a_{10}, a_{01}, a_{-12}, b_{01} \in \mathbb{C}$.

Using blow-up transformation

$$(x, y) \mapsto (z, y) = \left(\frac{x}{y}, y\right)$$

we obtain the following system

$$\begin{aligned} \dot{z} &= 5z + \lambda(a_{-12}y + a_{01}yz + a_{10}yz^2) - \mu b_{01}yz = P(z, y) \\ \dot{y} &= -3y + \mu b_{01}y^2 = Q(z, y). \end{aligned} \tag{5.7}$$

We look for the first integral of the form

$$\Psi(z, y) = \sum_{k=5}^{\infty} f_k(z) y^k.$$

Again we compute $\dot{\Psi} = \frac{\partial \Psi(z, y)}{\partial z} P(z, y) + \frac{\partial \Psi(z, y)}{\partial y} Q(z, y)$ and for each $k \geq 5$ we set the coefficient of power y^k to zero. For $k \geq 5$ this yields the following recurrence differential equation for $f_k(z)$ and $f_{k-1}(z)$

$$(k-1)\mu b_{01}f_{k-1}(z) - 3kf_k(z) + 5zf'_k(z) + (\lambda(a_{-12} + a_{01}z + a_{10}z^2) - \mu b_{01}z)f'_{k-1}(z) = 0.$$

For $k = 5, 6, 7, 8, 9$ we find

$$\begin{aligned} f_5(z) &= z^3, \\ f_6(z) &= \frac{3}{8}a_{-12}\lambda z^2 + (a_{01}\lambda + \frac{2}{3}b_{01}\mu)z^3 - \frac{3}{2}a_{10}\lambda z^4, \\ f_7(z) &= p_5(z), \\ f_8(z) &= q_6(z), \\ f_9(z) &= r_7(z), \end{aligned}$$

where $p_5(z)$, $q_6(z)$ and $r_7(z)$ are polynomials of degree at most 5, 6 and 7, respectively. Now we assume that

$$f_k(z) = R_{k-2}(z),$$

where $R_{k-2}(z) = \sum_{j=0}^{k-2} \rho_j z^j$ denotes polynomial of degree at most $k-2$. We prove this by induction. We have to solve the following differential equation

$$f'_n(z) = \frac{3n}{5z} f_n(z) + \frac{1}{5z} (\mu b_{01} z - \lambda(a_{-12} + a_{01}z + a_{10}z^2)) \cdot f'_{n-1}(z) - \frac{n-1}{5z} \mu b_{01} \cdot f_{n-1}(z). \quad (5.8)$$

Suppose $f_k(z) = R_{k-2}(z) = \sum_{j=0}^{k-2} \rho_j z^j$, for $k = 5, 6, \dots, n-1$. Then $f'_k(z) = \sum_{j=0}^{k-2} \rho_j j z^{j-1}$ and

$$\begin{aligned} & \frac{1}{5z} (\mu b_{01} z - \lambda(a_{-12} + a_{01}z + a_{10}z^2)) \cdot f'_{n-1}(z) - \frac{n-1}{5z} \mu b_{01} \cdot f_{n-1}(z) \\ &= \frac{1}{5z} (\mu b_{01} z - \lambda(a_{-12} + a_{01}z + a_{10}z^2)) \sum_{j=0}^{n-3} \rho_j j z^{j-1} - \frac{n-1}{5z} \mu b_{01} \sum_{j=0}^{n-3} \rho_j z^j \\ &= \frac{1}{5z} W_{n-2}(z), \end{aligned}$$

where $W_{n-2}(z)$ is a polynomial of degree at most $n-2$. This is the case, since the term $(\mu b_{01} z - \lambda(a_{-12} + a_{01}z + a_{10}z^2)) \cdot f'_{n-1}(z)$ contains the highest power in the above expression, $\deg(\mu b_{01} z - \lambda(a_{-12} + a_{01}z + a_{10}z^2)) = 2$ and $\deg(f'_{n-1}(z)) = n-4$. Equation (5.8) becomes

$$f'_n(z) = \frac{3n}{5z} f_n(z) + \frac{1}{5z} W_{n-2}(z), \quad (5.9)$$

which is of the form (4.2) and the corresponding solution is of the form (4.3). From differential equation (5.9) and (4.2) it follows

$$g(z) = \frac{3n}{5z}, \quad h(z) = \frac{1}{5z} W_{n-2}(z) = \frac{1}{5z} (w_0 + w_1 z + w_2 z^2 + \dots + w_{n-2} z^{n-2}).$$

A direct integration yields

$$\begin{aligned} e^{\int g(z) dz} &= z^{\frac{3n}{5}}, \\ z^{\frac{3n}{5}} \int \frac{1}{5z} W_{n-2}(z) \cdot z^{-\frac{3n}{5}} dz &= \sum_{k=0}^{n-2} \frac{w_k z^k}{5k-3n} = Q_{n-2}(z), \end{aligned}$$

since $z^{\frac{3n}{5}} \int \frac{1}{5z} w_k z^k \cdot z^{-\frac{3n}{5}} dz = z^{\frac{3n}{5}} \int \frac{1}{5} w_k z^{k-1-\frac{3n}{5}} dz = \frac{w_k z^k}{5k-3n}$, and finally

$$f_n(z) = C z^{\frac{3n}{5}} + Q_{n-2}(z).$$

For $C = 0$ we finally obtain $f_n(z) = Q_{n-2}(z)$, where $Q_{n-2}(z)$ is a polynomial of degree $n-2$, which completes the proof.

We proved that the formal first integral of (5.7) is of the form

$$\Psi(z, y) = z^3 y^5 + \sum_{k=6}^{\infty} R_{k-2}(z) y^k.$$

Setting $R_{k-2}(z) = \sum_{j=0}^{k-2} \rho_j z^j$ the inverse blow-up transformation $z \mapsto \frac{x}{y}$, $y \mapsto y$ yields

$$\begin{aligned}\tilde{\Psi}(x, y) &= \Psi\left(\frac{x}{y}, y\right) = \frac{x^3}{y^3}y^5 + \sum_{k=6}^{\infty} \left(\sum_{j=0}^{k-2} \rho_j x^j y^{k-j} \right) \\ &= x^3 y^2 + \sum_{k=6}^{\infty} \left(\rho_0 y^k + \rho_1 x y^{k-1} + \rho_2 x^2 y^{k-2} + \cdots + \rho_{k-2} x^{k-2} y^2 \right),\end{aligned}$$

which is a formal integral of system (5.6) of the form (1.6) with $p = 2$ and $q = 3$.

Case 3. The corresponding system has the form

$$\begin{aligned}\dot{x} &= 2x + \lambda a_{01} xy \\ \dot{y} &= -3y + \mu b_{10} xy,\end{aligned}\tag{5.10}$$

where $a_{01}, b_{10} \in \mathbb{C}$.

After the blow-up transformation

$$(x, y) \mapsto (z, y) = \left(\frac{x}{y}, y \right)$$

we obtain the following system

$$\begin{aligned}\dot{z} &= 5z + \lambda a_{01} yz - \mu b_{10} yz^2 = P(z, y) \\ \dot{y} &= -3y + \mu b_{10} y^2 z = Q(z, y).\end{aligned}\tag{5.11}$$

We now look for the first integral of the form

$$\Psi(z, y) = \sum_{k=5}^{\infty} f_k(z) y^k.$$

After computing $\dot{\Psi} = \frac{\partial \Psi(z, y)}{\partial z} P(z, y) + \frac{\partial \Psi(z, y)}{\partial y} Q(z, y)$ we set the coefficient to power y^k for each $k \geq 5$ to zero. This yields for $k \geq 5$ the following recurrence differential equation for $f_k(z)$ and $f_{k-1}(z)$

$$(k-1)\mu b_{10} z f_{k-1}(z) - 3k f_k(z) + 5z f_k'(z) + (\lambda a_{01} z - \mu b_{10} z^2) f_{k-1}'(z) = 0.$$

For $k = 5, 6, 7, 8, 9$ we find

$$\begin{aligned}f_5(z) &= z^3, \\ f_6(z) &= z^3(\lambda a_{01} - \mu b_{10} z), \\ f_7(z) &= z^3 p_2(z), \\ f_8(z) &= z^3 q_3(z), \\ f_9(z) &= z^3 r_4(z),\end{aligned}$$

where $p_2(z)$, $q_3(z)$ and $r_4(z)$ are polynomials of degree at most 2, 3, and 4, respectively. We can assume that

$$f_k(z) = z^3 R_{k-5}(z),$$

where $R_{k-5}(z) = \sum_{j=0}^{k-5} \rho_j z^j$ denotes a polynomial of degree at most $k-5$. Again we use induction to prove this assumption. To this end we solve the following differential equation

$$f_n'(z) = \frac{3n}{5z} f_n(z) + \frac{\mu b_{10} z - \lambda a_{01}}{5} \cdot f_{n-1}'(z) - \frac{n-1}{5} \mu b_{10} \cdot f_{n-1}(z).\tag{5.12}$$

Suppose

$$f_k(z) = z^3 R_{k-5}(z) = \sum_{j=0}^{k-5} \rho_j z^{j+3}, \quad \text{for } k = 5, 6, \dots, n-1.$$

Then $f'_k(z) = \sum_{j=0}^{k-5} (j+3) \rho_j z^{j+2}$, and

$$\begin{aligned} & \frac{\mu b_{10} z - \lambda a_{01}}{5} \cdot f'_{n-1}(z) - \frac{n-1}{5} \mu b_{10} \cdot f_{n-1}(z) \\ &= \frac{\mu b_{10} z - \lambda a_{01}}{5} \cdot \sum_{j=0}^{n-6} (j+3) \rho_j z^{j+2} - \frac{n-1}{5} \mu b_{10} \sum_{j=0}^{n-6} \rho_j z^{j+3}. \end{aligned} \quad (5.13)$$

Now we can see that the highest power in expression (5.13) is $n-3$ and the lowest power of expression (5.13) is obviously z^2 . This implies that differential equation (5.12) becomes

$$f'_n(z) = \frac{3n}{5z} f_n(z) + z^2 W_{n-5}(z), \quad (5.14)$$

where $W_{n-5}(z)$ is some polynomial of degree at most $n-5$. From differential equation (5.14) using (4.2) it follows

$$g(z) = \frac{3n}{5z}, \quad h(z) = z^2 W_{n-5}(z) = w_0 z^2 + w_1 z^3 + w_2 z^4 + \dots + w_{n-5} z^{n-3}.$$

An integration yields

$$\begin{aligned} e^{\int g(z) dz} &= z^{\frac{3n}{5}}, \\ z^{\frac{3n}{5}} \int z^2 W_{n-5}(z) \cdot z^{-\frac{3n}{5}} dz &= z^3 \sum_{k=0}^{n-5} \frac{5w_k z^k}{5(k+3) - 3n} = z^3 \cdot Q_{n-5}(z), \end{aligned}$$

since $z^{\frac{3n}{5}} \int w_k z^{k+2} \cdot z^{-\frac{3n}{5}} dz = z^{\frac{3n}{5}} \int w_k z^{k+2-\frac{3n}{5}} dz = \frac{5w_k z^{k+3}}{5(k+3)-3n}$, yielding

$$f_n(z) = C z^{\frac{3n}{5}} + z^3 Q_{n-5}(z).$$

For $C = 0$ we finally obtain $f_n(z) = z^3 Q_{n-5}(z)$, where $Q_{n-5}(z)$ is a polynomial of degree $n-5$, which completes the proof by induction.

We proved that the formal first integral of (5.11) is of the form

$$\Psi(z, y) = \sum_{k=5}^{\infty} z^3 R_{k-5}(z) y^k = z^3 y^5 + \sum_{k=6}^{\infty} z^3 R_{k-5}(z) y^k.$$

Similar as in previous two cases we set $R_{k-5}(z) = \sum_{j=0}^{k-5} \rho_j z^j$ and apply inverse blow-up transformation $z \mapsto \frac{x}{y}$, $y \mapsto y$ to obtain

$$\tilde{\Psi}(x, y) = \Psi\left(\frac{x}{y}, y\right) = x^3 y^2 + \psi_{3,3} x^3 y^3 + \psi_{4,2} x^4 y^2 + h.o.t.,$$

which is a formal integral of (5.10) of the required form.

6 Conclusions

In this paper we introduce the notion of persistent $p : -q$ resonant center and we solve the problem of $p : -q$ persistent resonant center for some quadratic systems. First we fix p as 1 and we increase q starting with $q = 2$. Although the system is polynomial containing just linear and quadratic terms the computations of saddle quantities become too laborious for $q > 5$. If $p = 2$ and $q = 3$ computations are again very complex, and with increasing of q they become more demanding. On the other hand, note that once we obtain a sufficient number of saddle quantities then the study of persistent resonant centers is much easier than the study of (regular) resonant centers since for persistent centers we use two parameters to “split” the saddle quantities, whereas in the second case we have no splitting. Hence, similar as noted in [7] in the first case we obtain a simpler variety to decompose than in the second case.

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References

- [1] X. CHEN, J. GINÉ, V. G. ROMANOVSKI, D. S. SHAFER, The $1 : -q$ resonant center problem for certain cubic Lotka–Volterra systems, *Appl. Math. Comput.* **218**(2012), No. 23, 11620–11633. <https://doi.org/10.1016/j.amc.2012.05.045>; MR2944006; Zbl 1365.34006
- [2] X. CHEN, V. ROMANOVSKI, W. ZHANG, Persistent centers of complex systems, *Bull. Sci. Math.* **138**(2014), No. 1, 110–123. <https://doi.org/10.1016/j.bulsci.2013.10.003>; MR3245496; Zbl 1291.34061
- [3] C. J. CHRISTOPHER, Invariant algebraic curves and conditions for a centre, *Proc. Roy. Soc. Edinburgh Sect. A* **124**(1994), No. 6, 1209–1229. <https://doi.org/10.1017/S0308210500030213>; MR1313199; Zbl 0821.34023
- [4] C. CHRISTOPHER, P. MARDEŠIĆ, C. ROUSSEAU, Normalizable, integrable and linerizable saddle points in complex quadratic systems in \mathbb{C}^2 , *J. Dyn. Control Syst.* **9**(2003), No. 3, 311–363. <https://doi.org/10.1023/A:1024643521094>; MR1990240; Zbl 1022.37035
- [5] C. CHRISTOPHER, C. ROUSSEAU, Nondegenerate linearizable centers of complex planar quadratic and symmetric cubic systems in \mathbb{C}^2 , *Publ. Mat.* **45**(2001), 95–123. https://doi.org/10.5565/PUBLMAT_45101_04; MR1829579;
- [6] C. CHRISTOPHER, C. ROUSSEAU, Normalizable, integrable and linearizable saddle points in the Lotka–Volterra system, *Qual. Theory Dyn. Syst.* **5**(2004), 11–61. <https://doi.org/10.1007/BF02968129>; MR2197423; Zbl 1100.34022
- [7] A. CIMA, A. GASUL, J. C. MEDRADO, On persistent centers, *Bull. Sci. Math.* **133**(2009), 644–657. <https://doi.org/10.1016/j.bulsci.2008.08.007>; MR2561367; Zbl 1183.34041

- [8] D. COX, J. LITTLE, D. O'SHEA, *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra*, Springer, 3rd edition, New York, 2007. MR2290010; Zbl 1118.13001
- [9] W. DECKER, S. LAPLAGNE, G. PFISTER, H. A. SCHONEMANN, *SINGULAR 3-1 library for computing the prime decomposition and radical of ideals, primdec.lib*, 2010.
- [10] D. DOLIĆANIN, J. GINÉ, R. OLIVEIRA, V. G. ROMANOVSKI, The center problem for a $2 : -3$ resonant cubic Lotka–Volterra system, *Appl. Math. Comput.* **220**(2013), 12–19. <https://doi.org/10.1016/j.amc.2013.06.007>; MR3091826; Zbl 1329.34064
- [11] G. DONG, Linearizability of homogeneous quartic polynomial systems with $1 : -2$ resonance, *J. Math. Anal. Appl.* **396**(2012), No. 1, 215–224. <https://doi.org/10.1016/j.jmaa.2012.06.013>; MR2956956; Zbl 1259.34026
- [12] M. DUKARIĆ, B. FERČEC, J. GINÉ, The solution of the $1 : -3$ resonant center problem in the quadratic case, *Appl. Math. Comput.* **237**(2014), 501–511. <https://doi.org/10.1016/j.amc.2014.03.147>; MR3201150; Zbl 1334.34067
- [13] H. DULAC, Détermination et intégration d'une certain classe d'équations différentielle ayant pour point singulier un centre, *Bull. Sci. Math. Sér. (2)* **32**(1908), 230–252. Zbl 39.0374.01
- [14] B. FERČEC, X. CHEN, V. ROMANOVSKI, Integrability conditions for complex systems with homogeneous quintic nonlinearities, *J. Appl. Anal. Comput.* **1**(2011), No. 1, 9–20. <https://doi.org/10.11948/2011002>; MR2881844 ; Zbl 1304.34055
- [15] B. FERČEC, J. GINÉ, Blow-up method to prove formal integrability for planar differential systems, *J. Appl. Anal. Comput.* **8**(2018), No. 6, 1833–1850. <https://doi.org/10.11948/2018.1833>;
- [16] B. FERČEC, J. GINÉ, Y. LIU, V. G. ROMANOVSKI, Integrability conditions for Lotka–Volterra planar complex quartic systems having homogeneous nonlinearities, *Acta Appl. Math.* **124**(2013), 107–122. <https://doi.org/10.1007/s10440-012-9772-5>; MR3029242; Zbl 1280.34034
- [17] B. FERČEC, J. GINÉ, M. MENCINGER, R. OLIVEIRA, The center problem for a $1 : -4$ resonant quadratic system, *J. Math. Anal. Appl.* **420**(2014), No. 2, 1568–1591. <https://doi.org/10.1016/j.jmaa.2014.06.060>; MR3240094; Zbl 1301.34043
- [18] J. P. FRANÇOISE, A. FRONVILLE Computer algebra and bifurcation theory of vector fields of the plane, *Proceedings of the 3rd Catalan Days on Applied Mathematics* (1996), 57–63, Univ. Lleida, Spain. MR1609601; Zbl 0901.68088
- [19] A. FRONVILLE, A. P. SADOVSKI, H. ŻOŁĄDEK, Solution of the $1 : -2$ resonant center problem in the quadratic case, *Fund. Math.* **157**(1998), No. 2–3, 191–207. MR1636887; Zbl 0943.34018
- [20] J. GINÉ, The nondegenerate center problem and the inverse integrating factor, *Bull. Sci. Math.* **130**(2006), No 2, 152–161. <https://doi.org/10.1016/j.bulsci.2005.09.001>; MR2200643; Zbl 1102.34001

- [21] J. GINÉ, On some open problems in planar differential systems and Hilbert's 16th problem, *Chaos Solitons Fractals* **31**(2007), No. 5, 1118–1134. <https://doi.org/10.1016/j.chaos.2005.10.057>; MR2261479; Zbl 1149.34023
- [22] J. GINÉ, C. CHRISTOPHER, M. PREŠERN, V. G. ROMANOVSKI, N. L. SHCHEGLOVA, The resonant center problem for a $2 : -3$ resonant cubic Lotka–Volterra system, in: V. P. Gert et al (Eds.), *Computer algebra in scientific computing. 14th international workshop, CASC 2012, Maribor, Slovenia, September 3–6, 2012. Proceedings*, Lecture Notes in Computer Science, Vol. 7442, Springer, Berlin, 2012, pp. 129–142. <https://doi.org/10.1007/978-3-642-32973-911>; Zbl 1375.34057
- [23] J. GINÉ, V. G. ROMANOVSKI, Integrability conditions for Lotka–Volterra planar complex quintic systems, *Nonlinear Anal. Real World Appl.* **11**(2010), No. 3, 2100–2105. <https://doi.org/10.1016/j.nonrwa.2009.06.002>; MR2646619; Zbl 1194.34003
- [24] J. GINÉ, C. VALLS, Integrability conditions of a resonant saddle perturbed with homogeneous quintic nonlinearities, *Nonlinear Dynam.* **81**(2015), No. 4, 2021–2030. <https://doi.org/10.1007/s11071-015-2122-1>; MR3382796; Zbl 1348.34064
- [25] S. GRAVEL, P. THIBAUT, Integrability and linearizability of the Lotka–Volterra system with a saddle point with rational hyperbolicity ratio, *J. Differential Equations* **184**(2002), No 1, 20–47. <https://doi.org/10.1006/jdeq.2001.4128>; MR1929144; Zbl 1054.34049
- [26] G. M. GREUEL, G. PFISTER, H. A. SCHÖNEMANN, SINGULAR 3.0 *A Computer Algebra System for Polynomial Computations*, Centre for Computer Algebra, University of Kaiserslautern, 2005. <http://www.singular.uni-kl.de>.
- [27] Z. HU, V. G. ROMANOVSKI, D. S. SHAFER, $1 : -3$ resonant centers on \mathbb{C}^2 with homogeneous cubic nonlinearities, *Comput. Math. Appl.* **56**(2008), No. 8, 1927–1940. <https://doi.org/10.1016/j.camwa.2008.04.009>; MR2466695; Zbl 1165.34334
- [28] W. KAPTEYN, New investigations on the midpoints of integrals of differential equations of the first degree. *Nederl. Akad. Wetensch. Verslag Afd. Natuurk.* **20**(1912) 1354–1365; **21**(1912) 27–33 (Dutch).
- [29] A. M. LIAPUNOV, *Stability of motion*. With a contribution by V. Pliss. Translated by F. Abramovici and M. Shimshoni. Academic Press, New York, 1966. MR0208093; Zbl 0161.06303
- [30] C. LIU, G. CHEN, G. CHEN, Integrability of Lotka–Volterra type systems of degree 4, *J. Math. Anal. Appl.* **388**(2012), No. 2, 1107–1116. <https://doi.org/10.1016/j.jmaa.2011.10.056>; MR2869810; Zbl 1259.34007
- [31] C. LIU, G. CHEN, C. LI, Integrability and linearizability of the Lotka–Volterra systems, *J. Differential Equations* **198**(2004), No. 2, 301–320. [https://doi.org/10.1016/S0022-0396\(03\)00196-7](https://doi.org/10.1016/S0022-0396(03)00196-7); MR2038583; Zbl 1046.34075
- [32] C. LIU, Y. LI, The integrability of a class of cubic Lotka–Volterra systems, *Nonlinear Anal. Real World Appl.* **19**(2014), 67–74. <https://doi.org/10.1016/j.nonrwa.2014.02.007>; MR3206659; Zbl 1300.34008

- [33] M. MENCINGER, B. FERČEC, W. FERNANDES, R. OLIVEIRA, Linearizability problem of persistent centers, *Electron. J. Qual. Theory Differ. Equ.* **2018**, No. 37, 1–27. <https://doi.org/10.14232/ejqtde.2018.1.37>; MR3811500;
- [34] H. POINCARÉ, Mémoire sur les courbes définies par une équation différentielle (in French), *J. Math. Pures Appl. (9)* (Sér. 3) **7**(1881) 375–422; (Sér. 3) **8**(1882), 251–296; (Sér. 4) **1**(1885) 167–244; (Sér. 4) **2**(1886), 151–217. Zbl 14.0666.01
- [35] V. G. ROMANOVSKI, D. S. SHAFER, *The center and cyclicity problems: a computational algebra approach*. Birkhäuser, Boston, 2009. <https://doi.org/10.1007/978-0-8176-4727-8>; MR2500203; Zbl 1192.34003
- [36] K. S. SIBIRSKII, On the number of limit cycles in the neighborhood of a singular point (in Russian), *Differencial'nye Uravnenija* **1**(1965), 53–66. MR0188542;
- [37] D. WANG, Polynomial systems from certain differential equations, *J. Symbolic Comput.* **28**(1999), No. 1–2, 303–315. <https://doi.org/10.1006/jsco.1998.0278>; MR1709908; Zbl 0948.34018
- [38] Q. WANG, W. HUANG, Integrability and linearizability for Lotka-Volterra systems with the $3 : -q$ resonant saddle point, *Adv. Difference Equ.* **2014**, 2014:23, 15 pp. <https://doi.org/10.1186/1687-1847-2014-23>; MR3213923; Zbl 1351.34033
- [39] Q. WANG, W. HUANG, H. WU, Linear recursion formulas of generalized focus quantities and applications, *Appl. Math. Comput.* **219**(2013), No. 10, 5233–5240. <https://doi.org/https://doi.org/10.1016/j.amc.2012.11.031>; MR3009481; Zbl 1286.34049
- [40] Q. WANG, Y. LIU, Linearizability of the polynomial differential systems with a resonant singular point, *Bull. Sci. Math.* **132**(2008), No. 2, 97–111. <https://doi.org/10.1016/j.bulsci.2006.07.005>; MR2387820; Zbl 1138.34022
- [41] Q. WANG, H. WU, Integrability and linearizability for a class of cubic Kolmogorov systems, *Ann. Differential Equations* **26**(2010), No. 4, 442–449. MR2779856; Zbl 1240.34199
- [42] H. ŻOŁĄDEK, Quadratic systems with center and their perturbations, *J. Differential Equations* **109**(1994), No. 2, 223–273. <https://doi.org/10.1006/jdeq.1994.1049>; MR1273302; Zbl 0797.34044
- [43] H. ŻOŁĄDEK, The problem of center for resonant singular points of polynomial vector fields, *J. Differential Equations* **137**(1997), No. 1, 94–118. <https://doi.org/10.1006/jdeq.1997.3260>; MR1451537; Zbl 0885.34034