Second-order discontinuous problems with nonlinear functional boundary conditions on the half-line

Rodrigo López Pouso and **Jorge Rodríguez–López**✉

Departamento de Estatística, Análise Matemática e Optimización, Instituto de Matemáticas,
Universidade de Santiago de Compostela, 15782, Facultade de Matemáticas,
Campus Vida, Santiago, Spain

Abstract. We establish sufficient conditions for the existence of extremal solutions for a second-order problem on the half-line with discontinuous right-hand side and nonlinear boundary conditions. Our results are new even in the classical case of continuous nonlinearities. We illustrate their applicability with an example.

Keywords: discontinuous differential equations, upper and lower solutions, half line problems, fixed point theorems, extremal solutions.

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1   Introduction

We study the existence of solutions of the nonlinear equation on the half-line

$$x''(t) = f(t, x, x'), \qquad t \in [0, \infty), \tag{1.1}$$

coupled with the functional boundary conditions

$$L(x(0), x'(0), x) = 0, \qquad x'(+\infty) := \lim_{t \to +\infty} x'(t) = B, \tag{1.2}$$

where $B \in \mathbb{R}$ and $L : \mathcal{C}([0, \infty)) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and it is nonincreasing in the second and third variables.

Boundary value problems on unbounded intervals are becoming popular in the literature because of their applications to model real world problems in engineering or chemistry, see [1]. For instance, second-order nonlinear differential problems arise in the investigation of radial solutions for elliptic equations.

The functional boundary conditions considered here are quite general and were recently studied in [8]. They include the extensively studied Sturm–Liouville conditions [11, 12], integral boundary conditions [2], multipoint [18] and other boundary conditions with, for example, maximum or minimum arguments. Observe that the boundary condition at infinity implies that solutions, if they exist, are unbounded if $B \neq 0$.

✉Corresponding author. Email: jorgerodriguez.lopez@usc.es

The main novelty here is that we allow the function f to satisfy weaker conditions than usual. In particular, it may be discontinuous in the second variable over a countable number of *admissible* curves. This more general case wants new fixed point theorems which generalize the classical Schauder's fixed point theorem [9, 15].

We shall assume the existence of well ordered upper and lower solutions on unbounded domains and a Nagumo condition to control the first derivative in order to obtain existence results for (1.1)–(1.2). Another interesting point in this paper is that we are able to relax the usual definition of lower and upper solutions, cf. [2, 8, 11–13], and therefore our main existence result is new even in the classical case of continuous right-hand sides in (1.1). See Remark 3.2 for details.

Furthermore, we also prove the existence of extremal solutions for (1.1)–(1.2) by adapting the arguments in [5] to unbounded domains. This is also a new result even for a continuous function $f(t, x, x')$.

This paper is organized as it follows: in Section 2 we present the fixed point tools which we will employ later and we introduce the Banach space where we will look for solutions to (1.1)–(1.2) as well as the definition for lower and upper solutions. The integral operator associated to problem (1.1)–(1.2) is also studied here. In Section 3, an existence and localization result is established. Finally, in Section 4, we investigate the existence of extremal solutions between the lower and upper functions, that is, a least and a greatest solution.

2   Definition and preliminaries

First, we present some definitions and fixed point theorems based on multivalued theory which will be the key to work with discontinuous operators.

Let K be a nonempty subset of a normed space $(X, \| \cdot \|)$ and $T : K \longrightarrow X$ an operator, not necessarily continuous.

Definition 2.1. The closed-convex envelope of an operator $T : K \longrightarrow X$ is the multivalued mapping $\mathbb{T} : K \longrightarrow 2^X$ given by

$$\mathbb{T}x = \bigcap_{\varepsilon > 0} \overline{\text{co}}\, T\left(\overline{B}_\varepsilon(x) \cap K\right) \quad \text{for every } x \in K, \tag{2.1}$$

where $\overline{B}_\varepsilon(x)$ denotes the closed ball centered at x and radius ε, and $\overline{\text{co}}$ means closed convex hull.

Remark 2.2. The closed-convex envelope (cc-envelope, for short) is similar to the Krasovskij regularization (see [10]), but here we are 'convexifying' operators instead of nonlinear parts of differential equations.

Remark 2.3. Note that $\mathbb{T}$ is an upper semicontinuous multivalued mapping which assumes closed and convex values (see [4, 15]) provided that TK is a relatively compact subset of X.

Theorem 2.4 ([15, Theorem 3.1]). *Let K be a nonempty, convex and compact subset of X.*

Any mapping $T : K \longrightarrow K$ has at least one fixed point provided that for every $x \in K \cap \mathbb{T}K$ we have

$$\{x\} \cap \mathbb{T}x \subset \{Tx\}, \tag{2.2}$$

where $\mathbb{T}$ denotes the closed-convex envelope of T.

Remark 2.5. Condition (2.2) is equivalent to $\text{Fix}(\mathbb{T}) \subset \text{Fix}(T)$, where $\text{Fix}(S)$ denotes the set of fixed points of the operator S.

Theorem 2.6 ([9, Theorem 2.7]). *Let K be a nonempty, closed and convex subset of X and $T : K \longrightarrow K$ be a mapping such that TK is a relatively compact subset of X and it satisfies condition (2.2). Then T has a fixed point in K.*

Now we present some definitions and results concerning the problem (1.1)–(1.2). Consider the space

$$X = \left\{ x \in \mathcal{C}^1([0,\infty)) : \lim_{t \to \infty} \frac{x(t)}{e^{\theta t}} \in \mathbb{R}, \ \theta > 0 \text{ and } \lim_{t \to \infty} x'(t) \in \mathbb{R} \right\}$$

endowed with a Bielecki norm type in $\mathcal{C}^1([0,\infty))$,

$$\|x\| := \max\left\{ \|x\|_0, \|x\|_1 \right\},$$

where

$$\|x\|_0 = \sup_{0 \le t < \infty} \frac{|x(t)|}{e^{\theta t}} \quad \text{and} \quad \|x\|_1 = \sup_{0 \le t < \infty} |x'(t)|.$$

It is clear that $(X, \|\cdot\|)$ is a Banach space and it was employed in [8] where problem (1.1)–(1.2) was studied. For convenience we denote

$$Y = \left\{ x \in \mathcal{C}([0,\infty)) : \lim_{t \to \infty} \frac{x(t)}{e^{\theta t}} \in \mathbb{R}, \ \theta > 0 \right\}.$$

Our approach is based on lower and upper solutions method [7] and fixed point theory. Thus we define the lower and upper solutions for problem (1.1)–(1.2) and we present a Nagumo condition which gives some *a priori* bound on the first derivative for all possible solutions of the differential equation (1.1) between the lower and the upper solutions.

Definition 2.7. A function $\alpha \in Y$ is said to be a lower solution for the problem (1.1)–(1.2) if the following conditions are satisfied.

(*i*) For any $t_0 \in (0,\infty)$, either $D_-\alpha(t_0) < D^+\alpha(t_0)$,
or there exists an open interval I_0 such that $t_0 \in I_0$, $\alpha \in W^{2,1}(I_0)$ and

$$\alpha''(t) \ge f(t, \alpha(t), \alpha'(t)) \quad \text{for a.a. } t \in I_0.$$

(*ii*) $D^+\alpha(0) \in \mathbb{R}$ and $L(\alpha(0), D^+\alpha(0), \alpha) \le 0$.

(*iii*) There exists $N \in \mathbb{N}$ such that $\alpha \in W^{2,1}((N,\infty))$ and $\alpha'(+\infty) \le B$.

Similarly $\beta \in Y$ is an upper solution for (1.1)–(1.2) if it satisfies the inequalities in the reverse order.

Proposition 2.8. *Let $\bar{\alpha}, \bar{\beta} \in Y$ be such that $\bar{\alpha} \le \bar{\beta}$ and define*

$$r > \max\left\{ \sup_{t \in [0,\infty)} \frac{\bar{\beta}(t) - \bar{\alpha}(0)}{e^{\theta t}}, \ \sup_{t \in [0,\infty)} \frac{\bar{\beta}(0) - \bar{\alpha}(t)}{e^{\theta t}} \right\}.$$

Assume there exist a continuous function $\bar{N} : [0,\infty) \to (0,\infty)$ and $\bar{M} \in L^1([0,\infty))$ such that

$$\int_r^\infty \frac{1}{\bar{N}(s)} ds = +\infty. \tag{2.3}$$

Define $E := \{(t, x, y) \in [0, \infty) \times \mathbb{R}^2 : \bar{\alpha}(t) \le x \le \bar{\beta}(t)\}$. Then, there exists $R > 0$ such that for every function $f : E \to \mathbb{R}$ satisfying for a.e. $t \in [0, \infty)$ and all $(x, y) \in \mathbb{R}^2$ with $(t, x, y) \in E$,

$$|f(t, x, y)| \le \bar{M}(t)\bar{N}(|y|),$$

and for every solution x of (1.1) such that $\bar{\alpha} \le x \le \bar{\beta}$, we have

$$\|x\|_1 < R.$$

Proof. Let $R > r$ be big enough such that

$$\int_r^R \frac{1}{\bar{N}(s)} \, ds > \int_0^\infty \bar{M}(s) \, ds. \tag{2.4}$$

Let x be a solution of (1.1) and $t \in [0, \infty)$ such that $x'(t) > R$.

If $|x'(t)| > r$ for all $t \in [0, \infty)$, then for $T > 0$ big enough we have

$$\frac{\bar{\beta}(T) - \bar{\alpha}(0)}{e^{\theta T}} \ge \frac{x(T) - x(0)}{e^{\theta T}} \ge \frac{x(T) - x(0)}{T} = \frac{\int_0^T x'(s) \, ds}{T} > r \ge \frac{\bar{\beta}(T) - \bar{\alpha}(0)}{e^{\theta T}},$$

a contradiction.

Therefore there exist $t_0 < t_1$ (or $t_1 < t_0$) such that $x'(t_0) = r$, $x'(t_1) = R$ and $r \le x'(s) \le R$ in $[t_0, t_1]$ (or $[t_1, t_0]$). Then we have

$$\int_r^R \frac{1}{\bar{N}(s)} ds = \int_{t_0}^{t_1} \frac{x''(s)}{\bar{N}(x'(s))} ds = \int_{t_0}^{t_1} \frac{f(s, x(s), x'(s))}{\bar{N}(x'(s))} ds \le \int_{t_0}^{t_1} \bar{M}(s) \, ds \le \int_0^\infty \bar{M}(s) \, ds,$$

a contradiction, so we deduce that $x'(t) < R$. In the same way we prove that $x'(t) > -R$. $\square$

Remark 2.9. Observe that condition (2.3) in Proposition 2.8 could just be replaced by condition (2.4). However, the first one is easier to check in practice.

Remark 2.10. Notice that a better condition about $\bar{N}$, which allows a quadratic growth with respect to the third variable of the nonlinear term f for the differential equation (1.1), is commonly employed in the literature (see, e.g. [2, 11, 12, 14]), namely,

$$\int_r^\infty \frac{s}{\bar{N}(s)} ds = +\infty.$$

Unfortunately, this type of conditions require harder assumptions about $\bar{M}$ such as

$$\sup_{0 \le t < \infty} (1 + t)^k \bar{M}(t) < +\infty \quad \text{for some } k > 1.$$

In particular, the previous hypothesis avoids that $\bar{M}$ could be singular at $t = 0$.

Lemma 2.11. *Let $h \in L^1([0, \infty))$. Then $x \in X$ is the unique solution of the problem*

$$\begin{aligned} x''(t) &= h(t), \qquad t \in [0, \infty), \\ x(0) &= A, \\ x'(+\infty) &= B, \end{aligned}$$

with $A, B \in \mathbb{R}$, if and only if,

$$x(t) = A + \int_0^t \left(B - \int_s^\infty h(r) \, dr \right) ds.$$

Proof. It is immediate, see [8, Lemma 2.3]. □

For applying the fixed point theorems is necessary to guarantee that certain sets are relatively compact. Nevertheless, the classical Ascoli–Arzelà theorem fails due to the non-compactness of the infinite interval $[0, \infty)$, so this difficulty is overcome by the following result, see [1].

Lemma 2.12. *Let $A \subset X$. The set A is said to be relatively compact if the following conditions hold:*

(a) *A is uniformly bounded in X;*

(b) *the functions belonging to A are equicontinuous on any compact interval of $[0, \infty)$;*

(c) *the functions f from A are equiconvergent at $+\infty$, i.e., given $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that $\|f(t) - f(+\infty)\| < \varepsilon$ for any $t > T(\varepsilon)$ and $f \in A$.*

Now we shall construct a modified problem for proving the existence of solutions for (1.1)–(1.2) under well-ordered lower and upper solutions.

Suppose that there exist α and β lower and upper solutions for (1.1)–(1.2), respectively, such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, \infty)$ and $\alpha, \beta \in W^{1,\infty}((0, \infty))$, and let

$$r > \max \left\{ \sup_{t \in [0,\infty)} \frac{\beta(t) - \alpha(0)}{e^{\theta t}}, \sup_{t \in [0,\infty)} \frac{\beta(0) - \alpha(t)}{e^{\theta t}} \right\}.$$

Assume that for $f : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ the following conditions hold:

($H1$) compositions $t \in I \mapsto f(t, x(t), y(t))$ are measurable whenever $x(t)$ and $y(t)$ are measurable;

($H2$) there exist a continuous function $N : [0, \infty) \to (0, \infty)$ and $M \in L^1([0, \infty))$ such that

$$\int_r^\infty \frac{1}{N(s)} ds = +\infty,$$

and for a.a. $t \in [0, \infty)$, all $x \in [\alpha(t), \beta(t)]$ and all $y \in \mathbb{R}$, we have $|f(t, x, y)| \leq M(t)N(|y|)$.

Consider the modified problem

$$\begin{cases} x''(t) = f(t, \varphi(t, x), \delta_R((\varphi(t, x))')), \\ x(0) = \varphi(0, x(0) - L(x(0), x'(0), x)), \\ x'(+\infty) = B \end{cases} \tag{2.5}$$

where

$$\varphi(t, x) = \max \left\{ \min \left\{ x, \beta(t) \right\}, \alpha(t) \right\} \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}, \tag{2.6}$$

and

$$\delta_R(y) = \max \left\{ \min \left\{ y, R \right\}, -R \right\} \quad \text{for all } y \in \mathbb{R}, \tag{2.7}$$

with R given by Proposition 2.8.

Notice that for $t \in [0, \infty)$,

$$\varphi(t, x(t)) = x(t) + (\alpha - x)^+(t) - (x - \beta)^+(t),$$

where $(u)^+(t) = \max\{u(t), 0\}$. Hence, we have $\varphi(\cdot, x) \in W^{1,\infty}((0, \infty))$ and

$$(\varphi(t, x(t)))' = \frac{d}{dt}\varphi(t, x(t)) = \begin{cases} \alpha'(t), & \text{if } x(t) < \alpha(t), \\ x'(t), & \text{if } \alpha(t) \le x(t) \le \beta(t), \\ \beta'(t), & \text{if } x(t) > \beta(t). \end{cases}$$

Furthermore, if $\{x_n\} \subset X$ is such that $x_n \to x$ in X, then

$$\lim_{n\to\infty} (\varphi(t, x_n(t)))' = (\varphi(t, x(t)))',$$

see [7, 17].

The operator $T : X \to X$ associated to the modified problem (2.5) is defined as

$$Tx(t) = \varphi(0, x(0) - L(x(0), x'(0), x)) + \int_0^t \left(B - \int_s^\infty f(r, \varphi(r, x), \delta_R((\varphi(r, x))')) \, dr \right) ds. \quad (2.8)$$

In order to achieve an existence result for problem (1.1)–(1.2) we shall prove that the operator T has a fixed point by applying Theorem 2.6. In this direction we present some previous lemmas.

Lemma 2.13. *Assume that conditions* $(H1)$ *and* $(H2)$ *hold. Then the operator T is well defined.*

Proof. Given $x \in X$, we shall show that $Tx \in X$. From $(H2)$, (2.6) and (2.7) we have

$$\lim_{t\to\infty} \frac{(Tx)(t)}{e^{\theta t}} \le \lim_{t\to\infty} \frac{\beta(0)}{e^{\theta t}} + \lim_{t\to\infty} \frac{\int_0^t \left(B - \int_s^\infty f(r, \varphi(r, x), \delta_R((\varphi(r, x))')) \, dr \right) ds}{e^{\theta t}}$$

$$\le \lim_{t\to\infty} \frac{\int_0^t \left(B + \int_s^\infty \tilde{M}(r) \, dr \right) ds}{e^{\theta t}}$$

$$\le \lim_{t\to\infty} \frac{\int_0^t (B + k_1) \, ds}{e^{\theta t}} \le \lim_{t\to\infty} \frac{(B + k_1)t}{e^{\theta t}} = 0,$$

where $\tilde{M}(t) = \max_{s \in [0,R]} N(s)M(t)$ and $k_1 = \int_0^\infty \tilde{M}(r) \, dr$.

Moreover,

$$\lim_{t\to\infty} (Tx)'(t) = \lim_{t\to\infty} \left(B - \int_t^\infty f(s, \varphi(s, x), \delta_R((\varphi(s, x))')) \, ds \right) = B < +\infty.$$

Therefore T is well defined. $\qquad\square$

Lemma 2.14. *Assume that conditions* $(H1)$ *and* $(H2)$ *hold. Then TX is relatively compact.*

Proof. It is a standard consequence of Lemma 2.12, see, e.g., [8, Lemma 2.8]. $\qquad\square$

We shall allow the function f to be discontinuous in the second variable over some curves satisfying a 'transversality' condition. These curves were introduced in [15], but as far as we know this is the first time that such type of discontinuity conditions were presented for boundary value problems on infinity intervals.

Definition 2.15. An admissible discontinuity curve for the differential equation (1.1) is a $W^{2,1}$ function $\gamma : [a, b] \subset [0, \infty) \longrightarrow \mathbb{R}$ satisfying one of the following conditions:

either $\gamma''(t) = f(t, \gamma(t), \gamma'(t))$ for a.a. $t \in [a,b]$ (and we then say that γ is viable for the differential equation),

or there exist $\varepsilon > 0$ and $\psi \in L^1(a,b)$, $\psi(t) > 0$ for a.a. $t \in [a,b]$, such that
   either

$$\gamma''(t) + \psi(t) < f(t,y,z) \quad \text{for a.a. } t \in [a,b], \text{ all } y \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon] \tag{2.9}$$
$$\text{and all } z \in [\gamma'(t) - \varepsilon, \gamma'(t) + \varepsilon],$$

   or

$$\gamma''(t) - \psi(t) > f(t,y,z) \quad \text{for a.a. } t \in [a,b], \text{ all } y \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon] \tag{2.10}$$
$$\text{and all } z \in [\gamma'(t) - \varepsilon, \gamma'(t) + \varepsilon].$$

We say that the admissible discontinuity curve γ is inviable for the differential equation if it satisfies (2.9) or (2.10).

Now we state three technical results that we need in the proof of condition (2.2) for the operator T. Their proofs can be lookep up in [15].

In the sequel m denotes the Lebesgue measure in $\mathbb{R}$.

Lemma 2.16. *Let $a, b \in \mathbb{R}$, $a < b$, and let $g, h \in L^1(a,b)$, $g \geq 0$ a.e., and $h > 0$ a.e. on (a,b).*

For every measurable set $J \subset (a,b)$ such that $m(J) > 0$ there is a measurable set $J_0 \subset J$ such that $m(J \setminus J_0) = 0$ and for all $\tau_0 \in J_0$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0, t] \setminus J} g(s)\,ds}{\int_{\tau_0}^t h(s)\,ds} = 0 = \lim_{t \to \tau_0^-} \frac{\int_{[t, \tau_0] \setminus J} g(s)\,ds}{\int_t^{\tau_0} h(s)\,ds}.$$

Corollary 2.17. *Let $a, b \in \mathbb{R}$, $a < b$, and let $h \in L^1(a,b)$ be such that $h > 0$ a.e. on (a,b).*

For every measurable set $J \subset (a,b)$ such that $m(J) > 0$ there is a measurable set $J_0 \subset J$ such that $m(J \setminus J_0) = 0$ and for all $\tau_0 \in J_0$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0, t] \cap J} h(s)\,ds}{\int_{\tau_0}^t h(s)\,ds} = 1 = \lim_{t \to \tau_0^-} \frac{\int_{[t, \tau_0] \cap J} h(s)\,ds}{\int_t^{\tau_0} h(s)\,ds}.$$

Corollary 2.18. *Let $a, b \in \mathbb{R}$, $a < b$, and let f, $f_n : [a,b] \to \mathbb{R}$ be absolutely continuous functions on $[a,b]$ ($n \in \mathbb{N}$), such that $f_n \to f$ uniformly on $[a,b]$ and for a measurable set $A \subset [a,b]$ with $m(A) > 0$ we have*

$$\lim_{n \to \infty} f_n'(t) = g(t) \quad \text{for a.a. } t \in A.$$

If there exists $M \in L^1(a,b)$ such that $|f'(t)| \leq M(t)$ a.e. in $[a,b]$ and also $|f_n'(t)| \leq M(t)$ a.e. in $[a,b]$ ($n \in \mathbb{N}$), then $f'(t) = g(t)$ for a.a. $t \in A$.

Now we present the result which gives the main difference between our existence results and the classical ones. It provides the proof for condition (2.2) what allows to avoid the continuity of the operator T and thus the continuity of f.

Lemma 2.19. *Assume that conditions (H1), (H2) and*

(H3) *there exist admissible discontinuity curves $\gamma_n : I_n = [a_n, b_n] \longrightarrow \mathbb{R}$ ($n \in \mathbb{N}$) such that $\alpha \leq \gamma_n \leq \beta$ on $[0, \infty)$ and their derivatives are uniformly bounded, and for all $y \in \mathbb{R}$ and for a.a. $t \in [0, \infty)$ the function $x \mapsto f(t, x, y)$ is continuous on $[\alpha(t), \beta(t)] \setminus \bigcup_{\{n : t \in I_n\}} \{\gamma_n(t)\}$;*

(H4) *for a.a. $t \in [0, \infty)$ and all $x \in [\alpha(t), \beta(t)]$, the mapping $y \mapsto f(t, x, y)$ is continuous;*

hold.

  Then the operator T satisfies condition (2.2) for all $x \in X$, i.e., $\mathrm{Fix}(\mathbb{T}) \subset \mathrm{Fix}(T)$ where $\mathbb{T}$ is the cc-envelope of T.

Proof. Without loss of generality, suppose that $R > 0$ as given by Proposition 2.8 satisfies

$$R > \max \left\{ \sup_{t \in [0,\infty)} |\alpha'(t)|, \ \sup_{t \in [0,\infty)} |\beta'(t)|, \max_{t \in I_n} |\gamma_n'(t)| \right\}. \tag{2.11}$$

  Consider the closed and convex subset of X,

$$K = \left\{ x \in X : \alpha(0) \le x(0) \le \beta(0), \ |x'(t) - x'(s)| \le \int_s^t \tilde{M}(r) \, dr \quad (0 \le s \le t \le +\infty) \right\}, \tag{2.12}$$

where $\tilde{M}(t) = \max_{s \in [0,R]} N(s) M(t)$. It is clear that $TX \subset K$ and then $\mathbb{T}X \subset K$.

  The proof follows the ideas of [15, Theorem 4.4]. Thus, we fix $x \in K$ and consider the following three cases.

Case 1: $m(\{t \in I_n : x(t) = \gamma_n(t)\}) = 0$ for all $n \in \mathbb{N}$. Then T is continuous at x.

  The assumption implies that for a.a. $t \in [0, \infty)$ the mapping $f(t, \cdot, \cdot)$ is continuous at $(\varphi(t, x(t)), (\varphi(t, x(t))')$. Hence if $\|x_k - x\| \to 0$ in K, then

$$f(t, \varphi(t, x_k(t)), \delta_R((\varphi(t, x_k(t)))')) \to f(t, \varphi(t, x(t)), \delta_R((\varphi(t, x(t)))')) \quad \text{for a.a. } t \in [0, \infty),$$

as one can easily check by considering all possible combinations of the cases $x(t) \in [\alpha(t), \beta(t)]$, $x(t) > \beta(t)$ or $x(t) < \alpha(t)$, and $|x'(t)| \le R$ or $|x'(t)| > R$.

  Moreover,

$$\left| f(t, \varphi(t, x(t)), \delta_R((\varphi(t, x(t)))')) \right| \le \tilde{M}(t) \tag{2.13}$$

for a.a. $t \in [0, \infty)$, hence $\|Tx_k - Tx\| \to 0$, by Lebesgue's dominated convergence theorem.

Case 2: $m(\{t \in I_n : x(t) = \gamma_n(t)\}) > 0$ for some $n \in \mathbb{N}$ such that γ_n is inviable. In this case we can prove that $x \notin \mathbb{T}x$.

  We shall show that there exist $\rho, \tilde{\varepsilon} > 0$ such that for every finite family $x_i \in \overline{B}_{\tilde{\varepsilon}}(x)$ and $\lambda_i \in [0, 1]$ $(i = 1, 2, \ldots, m)$, with $\sum \lambda_i = 1$, we have $\|x - \sum \lambda_i Tx_i\| \ge \rho$. In particular, it is sufficient that $\|x - \sum \lambda_i Tx_i\|_1 \ge \rho$.

  First, we fix some notation. Let us assume that for some $n \in \mathbb{N}$ we have $m(\{t \in I_n : x(t) = \gamma_n(t)\}) > 0$ and there exist $\varepsilon > 0$ and $\psi \in L^1(I_n)$, $\psi(t) > 0$ for a.a. $t \in I_n$, such that (2.10) holds with γ replaced by γ_n. (The proof is similar if we assume (2.9) instead of (2.10), so we omit it.)

  We denote $J = \{t \in I_n : x(t) = \gamma_n(t)\}$, and we observe that $m(\{t \in J : \gamma_n(t) = \beta(t)\}) = 0$. Indeed, if $m(\{t \in J : \gamma_n(t) = \beta(t)\}) > 0$, then from (2.10) it follows that $\beta''(t) - \psi(t) > f(t, \beta(t), \beta'(t))$ on a set of positive measure, which is a contradiction with the definition of upper solution.

  Now we distinguish between two sub-cases.

Case 2.1: $m(\{t \in J : x(t) = \gamma_n(t) = \alpha(t)\}) > 0$. Since $m(\{t \in J : \gamma_n(t) = \beta(t)\}) = 0$, we deduce that $m(\{t \in J : x(t) = \alpha(t) \ne \beta(t)\}) > 0$, so there exists $n_0 \in \mathbb{N}$ such that

$$m \left(\left\{ t \in J : x(t) = \alpha(t), \ x(t) < \beta(t) - \frac{1}{n_0} \right\} \right) > 0.$$

We denote $A = \{t \in J : x(t) = \alpha(t), \, x(t) < \beta(t) - 1/n_0\}$ and we deduce from Lemma 2.16 that there is a measurable set $J_0 \subset A$ with $m(J_0) = m(A) > 0$ such that for all $\tau_0 \in J_0$ we have

$$\lim_{t \to \tau_0^+} \frac{2 \int_{[\tau_0, t] \setminus A} \tilde{M}(s) \, ds}{(1/4) \int_{\tau_0}^t \psi(s) \, ds} = 0 = \lim_{t \to \tau_0^-} \frac{2 \int_{[t, \tau_0] \setminus A} \tilde{M}(s) \, ds}{(1/4) \int_t^{\tau_0} \psi(s) \, ds}. \tag{2.14}$$

By Corollary 2.17 there exists $J_1 \subset J_0$ with $m(J_0 \setminus J_1) = 0$ such that for all $\tau_0 \in J_1$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0, t] \cap J_0} \psi(s) \, ds}{\int_{\tau_0}^t \psi(s) \, ds} = 1 = \lim_{t \to \tau_0^-} \frac{\int_{[t, \tau_0] \cap J_0} \psi(s) \, ds}{\int_t^{\tau_0} \psi(s) \, ds}. \tag{2.15}$$

Let us now fix a point $\tau_0 \in J_1$. From (2.14) and (2.15) we deduce that there exist $t_- < \tau_0$ and $t_+ > \tau_0$, $t_\pm$ sufficiently close to τ_0 so that the following inequalities are satisfied:

$$2 \int_{[\tau_0, t_+] \setminus A} \tilde{M}(s) \, ds < \frac{1}{4} \int_{\tau_0}^{t_+} \psi(s) \, ds, \tag{2.16}$$

$$\int_{[\tau_0, t_+] \cap A} \psi(s) \, ds \geq \int_{[\tau_0, t_+] \cap J_0} \psi(s) \, ds > \frac{1}{2} \int_{\tau_0}^{t_+} \psi(s) \, ds, \tag{2.17}$$

$$2 \int_{[t_-, \tau_0] \setminus A} \tilde{M}(s) \, ds < \frac{1}{4} \int_{t_-}^{\tau_0} \psi(s) \, ds, \tag{2.18}$$

$$\int_{[t_-, \tau_0] \cap A} \psi(s) \, ds > \frac{1}{2} \int_{t_-}^{\tau_0} \psi(s) \, ds. \tag{2.19}$$

Finally, we define a positive number

$$\rho = \min \left\{ \frac{1}{4} \int_{t_-}^{\tau_0} \psi(s) \, ds, \frac{1}{4} \int_{\tau_0}^{t_+} \psi(s) \, ds \right\}, \tag{2.20}$$

and we are now in a position to prove that $x \notin \mathbb{T}x$. It is sufficient to prove the following claim.

Claim – Let $\tilde{\varepsilon} > 0$ be defined as $\tilde{\varepsilon} = \min\{\varepsilon/e^{\theta b_n}, 1/n_0\}$, where ε is given by our assumptions over γ_n and $1/n_0$ by the definition of the set A, and let ρ be as in (2.20). For every finite family $x_i \in \overline{B}_{\tilde{\varepsilon}}(x)$ and $\lambda_i \in [0, 1]$ ($i = 1, 2, \ldots, m$), with $\sum \lambda_i = 1$, we have $\|x - \sum \lambda_i Tx_i\|_1 \geq \rho$.

Let x_i and λ_i be as in the Claim and, for simplicity, denote $y = \sum \lambda_i Tx_i$. For a.a. $t \in J = \{t \in I_n = [a_n, b_n] : x(t) = \gamma_n(t)\}$ we have

$$y''(t) = \sum_{i=1}^m \lambda_i (Tx_i)''(t) = \sum_{i=1}^m \lambda_i f(t, \varphi(t, x_i(t)), \delta_R((\varphi(t, x_i(t)))')). \tag{2.21}$$

On the other hand, for every $i \in \{1, 2, \ldots, m\}$ and for a.a. $t \in J$ we have

$$\frac{\varepsilon}{e^{\theta b_n}} \geq \tilde{\varepsilon} \geq \frac{|x(t) - x_i(t)|}{e^{\theta t}} \geq \frac{|x(t) - x_i(t)|}{e^{\theta b_n}} = \frac{|\gamma_n(t) - x_i(t)|}{e^{\theta b_n}}, \tag{2.22}$$

so $|\gamma_n(t) - x_i(t)| \leq \varepsilon$. Moreover

$$\left| \gamma_n'(t) - x_i'(t) \right| = \left| x'(t) - x_i'(t) \right| \leq \varepsilon, \tag{2.23}$$

for a.a. $t \in J$.

Since $\gamma_n(t) \in [\alpha(t), \beta(t)]$, for a.a. $t \in A$ we have $|\varphi(t, x_i(t)) - \gamma_n(t)| \leq |x_i(t) - \gamma_n(t)|$, and $|(\varphi(t, x_i(t)))' - \gamma_n'(t)| \leq |x_i'(t) - \gamma_n'(t)|$ because if $x_i(t) < \alpha(t)$, then $(\varphi(t, x_i(t)))' = \alpha'(t) = \gamma_n'(t)$.

Hence, from (2.10) it follows that

$$\gamma_n''(t) - \psi(t) > f(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))')$$

for a.a. $t \in A$ and for all $x_i(t)$ satisfying (2.22) and (2.23).

Moreover, since for a.a. $t \in A$ we have $|\gamma_n'(t)| < R$ and $|x_i'(t) - \gamma_n'(t)| < \varepsilon$, without loss of generality we can suppose $|(\varphi(t, x_i(t)))'| \le R$ and thus

$$\gamma_n''(t) - \psi(t) > f(t, \varphi(t, x_i(t)), \delta_R((\varphi(t, x_i(t)))')) \tag{2.24}$$

for a.a. $t \in A$.

Therefore the assumptions on γ_n ensure that for a.a. $t \in A$ we have

$$y''(t) = \sum_{i=1}^{m} \lambda_i f(t, \varphi(t, x_i(t)), \delta_R((\varphi(t, x_i(t)))')) < \sum_{i=1}^{m} \lambda_i (\gamma_n''(t) - \psi(t)) = x''(t) - \psi(t). \tag{2.25}$$

Now we compute

$$\begin{aligned}
y'(\tau_0) - y'(t_-) &= \int_{t_-}^{\tau_0} y''(s)\, ds = \int_{[t_-,\tau_0] \cap A} y''(s)\, ds + \int_{[t_-,\tau_0] \setminus A} y''(s)\, ds \\
&< \int_{[t_-,\tau_0] \cap A} x''(s)\, ds - \int_{[t_-,\tau_0] \cap A} \psi(s)\, ds \\
&\quad + \int_{[t_-,\tau_0] \setminus A} \tilde{M}(s)\, ds \qquad \text{(by (2.25), (2.21) and (2.13))} \\
&= x'(\tau_0) - x'(t_-) - \int_{[t_-,\tau_0] \setminus A} x''(s)\, ds - \int_{[t_-,\tau_0] \cap A} \psi(s)\, ds \\
&\quad + \int_{[t_-,\tau_0] \setminus A} \tilde{M}(s)\, ds \\
&\le x'(\tau_0) - x'(t_-) - \int_{[t_-,\tau_0] \cap A} \psi(s)\, ds + 2\int_{[t_-,\tau_0] \setminus A} \tilde{M}(s)\, ds \\
&< x'(\tau_0) - x'(t_-) - \frac{1}{4} \int_{t_-}^{\tau_0} \psi(s)\, ds \qquad \text{(by (2.18) and (2.19))},
\end{aligned}$$

hence $\|x - y\|_1 \ge y'(t_-) - x'(t_-) \ge \rho$ provided that $y'(\tau_0) \ge x'(\tau_0)$.

Similar computations with t_+ instead of t_- show that if $y'(\tau_0) \le x'(\tau_0)$ then we also have $\|x - y\|_1 \ge \rho$. The claim is proven.

Case 2.2: $m(\{t \in J : \gamma_n(t) \in (\alpha(t), \beta(t))\}) > 0$.

The set $\{t \in J : \gamma_n(t) \in (\alpha(t), \beta(t))\}$ can be written as the following countable union

$$\bigcup_{n \in \mathbb{N}} \left\{ t \in J : \alpha(t) + \frac{1}{n} < x(t) < \beta(t) - \frac{1}{n} \right\},$$

so there exists some $n_0 \in \mathbb{N}$ such that $m(\{t \in J : \alpha(t) + 1/n_0 < x(t) < \beta(t) - 1/n_0\}) > 0$. Now we denote $A = \{t \in J : \alpha(t) + 1/n_0 < x(t) < \beta(t) - 1/n_0\}$. Since A is a set of positive measure we can argue as in Case 2.1 for obtaining inequalities (2.16)–(2.19) and we are in position to prove the Claim again.

Let x_i and λ_i be as in the Claim and, for simplicity, denote $y = \sum \lambda_i T x_i$. Then for every $i \in \{1, 2, \ldots, m\}$ and all $t \in A$ we have $x_i(t) \in (\alpha(t), \beta(t))$, so $\varphi(t, x_i(t)) = x_i(t)$ and $(\varphi(t, x_i(t)))' = x_i'(t)$ and thus

$$|\varphi(t, x_i(t)) - \gamma_n(t)| = |x_i(t) - x(t)| \le \varepsilon \quad \text{and} \quad |(\varphi(t, x_i(t)))' - \gamma_n'(t)| = |x_i'(t) - x'(t)| \le \varepsilon,$$

for all $t \in A$.

Hence, from (2.10) it follows that

$$\gamma_n''(t) - \psi(t) > f(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))')$$

for a.a. $t \in A$ and all $x_i \in \overline{B}_{\tilde{\varepsilon}}(x)$.

Now the proof of the Claim follows exactly as in Case 2.1.

Case 3: $m(\{t \in I_n : x(t) = \gamma_n(t)\}) > 0$ only for some of those $n \in \mathbb{N}$ such that γ_n is viable. Let us prove that in this case the relation $x \in \mathbb{T}x$ implies $x = Tx$.

Note first that $x \in \mathbb{T}x$ implies that x satisfies the boundary conditions in (2.5), because every element in $\mathbb{T}x$ is, roughly speaking, a limit of convex combinations of functions y satisfying

$$\begin{cases} y''(t) = f(t, \varphi(t, x), \delta_R((\varphi(t, x))')), \\ y(0) = \varphi(0, x(0) - L(x(0), x'(0), x)), \\ y'(+\infty) = B, \end{cases}$$

and L is continuous.

Now it only remains to show that $x \in \mathbb{T}x$ implies that x satisfies the ODE in (2.5).

Let us consider the subsequence of all viable admissible discontinuity curves in the conditions of Case 3, which we denote again by $\{\gamma_n\}_{n \in \mathbb{N}}$ to avoid overloading notation. We have $m(J_n) > 0$ for all $n \in \mathbb{N}$, where

$$J_n = \{t \in I_n : x(t) = \gamma_n(t)\}.$$

For each $n \in \mathbb{N}$ and for a.a. $t \in J_n$ we have $\gamma_n''(t) = f(t, \gamma_n(t), \gamma_n'(t))$ and from $\alpha \leq \gamma_n \leq \beta$ and $|\gamma_n'(t)| < R$ it follows that $\gamma_n''(t) = f(t, \varphi(t, \gamma_n(t)), \delta_R((\varphi(t, \gamma_n(t)))'))$, so γ_n is viable for (2.5). Then for a.a. $t \in J_n$ we have

$$x''(t) = \gamma_n''(t) = f(t, \varphi(t, \gamma_n(t)), \delta_R((\varphi(t, \gamma_n(t)))')) = f(t, \varphi(t, x(t)), \delta_R((\varphi(t, x(t)))')),$$

and therefore

$$x''(t) = f(t, \varphi(t, x(t)), \delta_R((\varphi(t, x(t)))')) \quad \text{a.e. in } J = \bigcup_{n \in \mathbb{N}} J_n. \tag{2.26}$$

Now we assume that $x \in \mathbb{T}x$ and we prove that it implies that

$$x''(t) = f(t, \varphi(t, x(t)), \delta_R((\varphi(t, x(t)))')) \quad \text{a.e. in } [0, \infty) \setminus J,$$

thus showing that $x = Tx$.

Since $x \in \mathbb{T}x$ then for each $k \in \mathbb{N}$ we can choose $\varepsilon = \rho = 1/k$ to guarantee that we can find functions $x_{k,i} \in B_{1/k}(x)$ and coefficients $\lambda_{k,i} \in [0, 1]$ ($i = 1, 2, \dots, m(k)$) such that $\sum \lambda_{k,i} = 1$ and

$$\left\| x - \sum_{i=1}^{m(k)} \lambda_{k,i} T x_{k,i} \right\| < \frac{1}{k}.$$

Let us denote $y_k = \sum_{i=1}^{m(k)} \lambda_{k,i} T x_{k,i}$, and notice that $y_k' \to x'$ uniformly in $[0, \infty)$ and $\|x_{k,i} - x\| \leq 1/k$ for all $k \in \mathbb{N}$ and all $i \in \{1, 2, \dots, m(k)\}$. Note also that

$$y_k''(t) = \sum_{i=1}^{m(k)} \lambda_{k,i} f(t, \varphi(t, x_{k,i}(t)), \delta_R((\varphi(t, x_{k,i}(t)))')) \quad \text{for a.a. } t \in [0, \infty). \tag{2.27}$$

For a.a. $t \in [0, \infty) \setminus J$ we have that either $x(t) \in [\alpha(t), \beta(t)]$, and then $f(t, \varphi(t, \cdot), \delta_R((\varphi(t, \cdot))'))$ is continuous at $x(t)$, so for any $\varepsilon > 0$ there is some $k_0 = k_0(t) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq k_0$, we have

$$|f(t, \varphi(t, x_{k,i}(t)), \delta_R((\varphi(t, x_{k,i}(t)))')) - f(t, \varphi(t, x(t)), \delta_R((\varphi(t, x(t)))'))| < \varepsilon$$
$$\text{for all } i \in \{1, 2, \ldots, m(k)\},$$

or $x(t) < \alpha(t)$ (analogously if $x(t) > \beta(t)$), so there is some $k_0 = k_0(t) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq k_0$ we have $x_{k,i}(t) < \alpha(t)$ for all $i \in \{1, 2, \ldots, m(k)\}$ and then $\varphi(t, x(t)) = \alpha(t) = \varphi(t, x_{k,i}(t))$, which implies

$$|f(t, \varphi(t, x_{k,i}(t)), \delta_R((\varphi(t, x_{k,i}(t)))')) - f(t, \varphi(t, x(t)), \delta_R((\varphi(t, x(t)))'))| = 0$$
$$\text{for all } i \in \{1, 2, \ldots, m(k)\}.$$

Now we deduce from (2.27) that $y_k''(t) \to f(t, \varphi(t, x(t)), \delta_R((\varphi(t, x(t)))'))$ for a.a. $t \in [0, \infty) \setminus J$, and then Corollary 2.18 guarantees for each $n \in \mathbb{N}$ that $x''(t) = f(t, \varphi(t, x(t)), \delta_R((\varphi(t, x(t)))'))$ for a.a. $t \in [0, n] \setminus J$. Thus $x''(t) = f(t, \varphi(t, x(t)), \delta_R((\varphi(t, x(t)))'))$ for a.a. $t \in [0, \infty) \setminus J$. Combining this result with (2.26), we see that x solves (2.5), which implies that x is a fixed point of T.   □

Remark 2.20. Admissible discontinuity curves may be defined in infinite intervals as the union of that in Definition 2.15.

3   Existence results

We establish an existence and localization result for (1.1)–(1.2).

Theorem 3.1. *Suppose that there exist α and β lower and upper solutions to (1.1)–(1.2), respectively, such that $\alpha \leq \beta$ on $[0, \infty)$ and $\alpha, \beta \in W^{1,\infty}((0, \infty))$. Assume that conditions (H1)–(H4) hold. Then problem (1.1)–(1.2) has at least a solution $x \in X$ and there exists $R > 0$ such that $\alpha(t) \leq x(t) \leq \beta(t)$ and $|x'(t)| < R$ for all $t \in [0, \infty)$.*

Proof. For simplicity, we divide the proof in several steps. First, we will prove that the modified problem (2.5) has at least a solution, that is, we will ensure that the operator T defined as in (2.8) has a fixed point. Later, we will show that it is a solution for problem (1.1)–(1.2).

Step 1. Problem (2.5) has at least a solution $x \in X$.

By Lemma 2.13, the operator T is well defined. Consider the closed and convex set D defined as

$$D = \{x \in X : \|x\| \leq \rho\},$$

where

$$\rho := \max \left\{ \max\{|\alpha(0)|, |\beta(0)|\} + \frac{B + k_1}{\theta e}, |B| + k_1 \right\}$$

and $k_1 = \int_0^\infty \tilde{M}(r) \, dr$.

For $x \in D$, we have

$$\|Tx\|_0 = \sup_{t \in [0,\infty)} \frac{|(Tx)(t)|}{e^{\theta t}} \leq \sup_{t \in [0,\infty)} \left(\max\{|\alpha(0)|, |\beta(0)|\} + \frac{(B + k_1)t}{e^{\theta t}} \right) \leq \rho,$$

and

$$\|Tx\|_1 = \sup_{t \in [0,\infty)} \left| (Tx)'(t) \right| \leq \sup_{t \in [0,\infty)} \left| B - \int_t^\infty f(s, \varphi(s,x), \delta_R((\varphi(s,x))')) \, ds \right| \leq |B| + k_1 \leq \rho.$$

Therefore, $TD \subset D$. In addition, TD is relatively compact, by Lemma 2.14, and T satisfies condition (2.2), by Lemma 2.19. Then Theorem 2.6 guarantees that the operator T has at least a fixed point $x \in D$.

Step 2. Every solution of (2.5) *satisfies* $\alpha(t) \leq x(t) \leq \beta(t)$ *for all* $t \in [0,\infty)$.

Let $x \in X$ be a solution of (2.5). Suppose that there exists $t \in [0,\infty)$ such that $\alpha(t) > x(t)$. Then

$$\inf_{t \in [0,\infty)} \{x(t) - \alpha(t)\} < 0.$$

It does not happen at 0, since

$$x(0) = \varphi\left(0, x(0) - L(x(0), x'(0), x)\right) \geq \alpha(0).$$

If the infimum is attained as t tends to infinity, there exists $T > 0$ such that

$$x(t) - \alpha(t) < 0 \quad \text{for all } t \in [T, \infty),$$

and $\alpha \in W^{2,1}((T,\infty))$. Hence,

$$x''(t) = f(t, \varphi(t,x), \delta_R((\varphi(t,x))')) = f(t, \alpha(t), \alpha'(t)) \leq \alpha''(t) \quad \text{for a.a. } t \in [T, \infty), \qquad (3.1)$$

and so $x - \alpha$ is a concave function on $[T, \infty)$.

Then there are two options: either there exists $t_0 > T$ such that t_0 is a relative minimum (in this case the reasoning is analogous to that we do below when the infimum is attained at $t_0 \in (0,\infty)$) or there exists $\tilde{T} > T$ such that $(x - \alpha)'(\tilde{T}) < 0$ and, by (3.1),

$$(x - \alpha)'(t) \leq (x - \alpha)'(\tilde{T}) \quad \text{for all } t \geq \tilde{T},$$

which implies

$$\lim_{t \to \infty} \left(x'(t) - \alpha'(t)\right) = x'(+\infty) - \alpha'(+\infty) < 0.$$

However, by the definition of α,

$$0 > x'(+\infty) - \alpha'(+\infty) = B - \alpha'(+\infty) \geq 0,$$

a contradiction.

Hence there exist $t_0 \in (0,\infty)$ such that

$$\min_{t \in [0,\infty)} (x(t) - \alpha(t)) = x(t_0) - \alpha(t_0) < 0.$$

Then we have

$$x'(t_0) - D_-\alpha(t_0) \leq x'(t_0) - D^+\alpha(t_0)$$

so, by the definition of lower solution, there exists an open interval I_0 such that $t_0 \in I_0$ and

$$\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)) \quad \text{for a.a. } t \in I_0.$$

Further $x'(t_0) = \alpha'(t_0)$ and for all $r > 0$ there exists $t_r \in (t_0 - r, t_0)$ such that $x'(t_r) < \alpha'(t_r)$.

On the other hand, by the continuity of $x - \alpha$, there exists $\varepsilon > 0$ such that for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ we have $x(t) - \alpha(t) < 0$. Then,

$$x''(t) = f(t, \varphi(t,x), \delta_R((\varphi(t,x))')) = f(t, \alpha(t), \alpha'(t)) \leq \alpha''(t) \quad \text{for a.a. } t \in [t_0 - \varepsilon, t_0] \cap I_0.$$

Thus, the function $x'(t) - \alpha'(t)$ is nonincreasing on $(t_0 - \varepsilon, t_0) \cap I_0$, so for $t \in (t_0 - \varepsilon, t_0) \cap I_0$,

$$x'(t) - \alpha'(t) \geq x'(t_0) - \alpha'(t_0) = 0,$$

a contradiction.

Step 3. Every solution x of problem (2.5) *satisfies* $|x'(t)| < R$ *for all* $t \in [0, \infty)$.

It is an immediate consequence of the Nagumo condition, see Proposition 2.8.

Step 4. Every solution x of problem (2.5) *is a solution to* (1.1)–(1.2).

Let x be a solution of the modified problem (2.5). It is enough to show that

$$\alpha(0) \leq x(0) - L(x(0), x'(0), x) \leq \beta(0).$$

Suppose, to the contrary, that

$$\alpha(0) > x(0) - L(x(0), x'(0), x).$$

Then, $x(0) = \varphi(0, x(0) - L(x(0), x'(0), x)) = \alpha(0)$. Therefore, since $\alpha \leq x$ and $x'(0) \geq D^+\alpha(0)$, from the monotonicity properties of L, we get the contradiction

$$0 > x(0) - L(x(0), x'(0), x) - \alpha(0) = -L(x(0), x'(0), x) \geq -L(\alpha(0), D^+\alpha(0), \alpha) \geq 0.$$

Analogously, we can prove that $x(0) - L(x(0), x'(0), x) \leq \beta(0)$.   □

Remark 3.2. It is usual in the literature (see [8, 11–13]) to consider upper and lower solutions with more strict boundary conditions at infinity, that is, $\alpha'(+\infty) < B$ and $\beta'(+\infty) > B$. Moreover, in the recent paper [13, Remark 3.3], the authors observe that it is unknown if this strict inequality can be weakened. Observe that this difficulty was overcome in our previous theorem.

To finish this section, we illustrate our existence result with an example which shows its applicability.

Example 3.3. Let $\{q_n\}_{n \in \mathbb{N}}$ be an enumeration of all rational numbers. Define $\phi : \mathbb{R} \to \mathbb{R}$ as

$$\phi(u) = \sum_{n : q_n < u} 2^{-n}.$$

Notice that ϕ is continuous at the irrational points and discontinuous at the rational numbers, see [16, Prop. 2, p. 108-109]. Moreover, $\phi(u) \in (0,1)$ for each $u \in \mathbb{R}$.

Consider the problem (1.1)–(1.2) with the following functional boundary conditions

$$L(x(0), x'(0), x) = 2(x(0))^3 - x'(0) - \int_0^\eta x(t)\, dt = 0,$$

$$x'(+\infty) = 0,$$

where $0 < \eta \leq 1$, and the function

$$f(t, x, y) = \frac{1}{1 + t^2}\phi(t - x) + \min\left\{\frac{1}{\sqrt{t}}, \frac{1}{t^2}\right\} y \cos(2\pi y),$$

for all $x, y \in \mathbb{R}$ and $t \in [0, \infty)$.

First, the functions $M(t) = \min\left\{1/\sqrt{t}, 1/t^2\right\}$ and $N(y) = 1 + y$ satisfy condition $(H2)$.

For all $y \in \mathbb{R}$ and for a.a. $t \in [0, \infty)$ the function $x \mapsto f(t, x, y)$ is continuous on $\mathbb{R} \setminus \bigcup_{\{n : t \in I_n\}} \{\gamma_n(t)\}$ where for each $n \in \mathbb{N}$,

$$\gamma_n(t) = t + q_n \quad \text{for all } t \in I_n = [\max\{0, -q_n\}, \infty).$$

Notice that these curves can be defined in compact domains as in Definition 2.15 by writing the infinite interval $[\max\{0, -q_n\}, \infty)$ as a countable union of compact intervals.

The curves γ_n are inviable admissible discontinuity curves. Indeed, for $\varepsilon > 0$ small enough we have

$$\gamma''(t) = 0 < \frac{1}{2} \min\left\{\frac{1}{\sqrt{t}}, \frac{1}{t^2}\right\} < f(t, y, z)$$

for a.a. $t \in I_n$, all $y \in [\gamma_n(t) - \varepsilon, \gamma_n(t) + \varepsilon]$ and $z \in [\gamma_n'(t) - \varepsilon, \gamma_n'(t) + \varepsilon]$.

It is easy to check that the functions $\alpha(t) = -t - 1$ and $\beta(t) = 0$ are, respectively, lower and upper solutions for this problem.

Therefore, Theorem 3.1 ensures that it has a solution between α and β.

Observe that the function f is not monotone in the third argument, changes sign and it is discontinuous in the second variable over a set of curves which is dense in $[0, \infty) \times \mathbb{R}$. Moreover, it is singular at $t = 0$, and even in the case of being continuous in the second variable, it would fall outside the scope of the results in [8] because the function M is not a possible bound for the nonlinearities considered there, see Remark 2.10.

Multiplicity results can also be derived by means of degree theory and the existence of two pairs of lower and upper solutions as done, for example, in [2, 11, 13].

4 Existence of extremal solutions

In this section, we provide sufficient conditions for the existence of extremal solutions for the following problem

$$\begin{cases} x''(t) = f(t, x, x'), & t \in [0, \infty), \\ L(x(0), x'(0)) = 0, & x'(+\infty) = B, \end{cases} \tag{4.1}$$

where $B \in \mathbb{R}$ and L is continuous and nonincreasing in the second argument.

Since problem (4.1) is a particular case of (1.1)–(1.2) (by removing the functional dependence in the boundary conditions), the existence of solutions is guaranteed by Theorem 3.1 when well ordered lower and upper solutions exist. Now we establish the existence of extremal solutions between them.

Theorem 4.1. *Under the conditions of Theorem 3.1, problem* (4.1) *has extremal solutions between α and β.*

Proof. Consider the set of solutions for problem (4.1) located between the lower and upper solution

$$S = \{x \in [\alpha, \beta] : x \text{ solution of } (4.1)\} = \{x \in X : x \text{ solution of } (2.5)\} = \{x \in X : x = Tx\}.$$

By Lemma 2.19,

$$S = \{x \in X : x \in \mathbb{T}x\} = (I - \mathbb{T})^{-1}(\{0\}),$$

which implies that S is a closed subset of X due to $\mathbb{T}$ is upper semicontinuous and $\{0\}$ is a closed set, see [3, Lemma 17.4]. Hence, since TX is relatively compact and $S \subset TX$, S is a compact set.

Define $x_{\min}(t) = \inf\{x(t) : x \in S\}$ for $t \in [0, \infty)$. The evaluation map $\delta_t : X \to \mathbb{R}$ given by $\delta_t(x) = x(t)$ is a continuous map and then $\delta_t(S) = \{x(t) : x \in S\}$ is compact. Thus, for each $t_0 \in [0, \infty)$, there exists a function $x_0 \in S$ such that $x_0(t_0) = x_{\min}(t_0)$ and $x_{\min}$ is a continuous function on $[0, \infty)$.

Let us see that $x_{\min}$ is a solution of (2.5). It is clear that $x_{\min}$ will be the least solution. By the upper semicontinuity of the operator $\mathbb{T}$ and the condition $\text{Fix}(\mathbb{T}) = \text{Fix}(T)$, the limit in X of a sequence of elements in S must be a solution of (2.5).

Given $T > 0$ and $\varepsilon > 0$ we shall prove that there exists $v \in S$ such that

$$|v(t) - x_{\min}(t)| \leq \varepsilon \quad \text{for all } t \in [0, T].$$

Then, a sequence of elements in S converges pointwise to $x_{\min}$ and by the compactness of S, up to a subsequence, it converges in S.

Following [5, Theorem 4.1], the idea is to construct an upper solution for problem (4.1) and to apply Theorem 3.1 in order to obtain the function $v \in S$ looked for.

By the equicontinuity of S and the continuity of $x_{\min}$ on $[0, T]$, there exists $\delta > 0$ such that $t, s \in [0, T]$ with $|t - s| < \delta$ implies

$$|x(t) - x(s)| < \varepsilon/2 \quad \text{for all } x \in S \cup \{x_{\min}\}.$$

Let $\{t_0, t_1, \ldots, t_n\} \subset [0, T]$ such that $t_0 = 0$, $t_n = T$ and $t_{i+1} - t_i < \delta$ for $i = 0, 1, \ldots, n-1$. Choose a function $x_0 \in S$ such that $x_0(0) = x_{\min}(0)$ and denote $\beta_0 \equiv x_0$.

For each $i \in \{1, 2, \ldots, n-1\}$, define recursively $\beta_i \equiv \beta_{i-1}$ if $\beta_{i-1}(t_i) = x_{\min}(t_i)$ and otherwise, take $x_i \in S$ such that $x_i(t_i) = x_{\min}(t_i)$, define

$$s_i = \inf\{t \in [t_{i-1}, t_i] : x_i(s) < \beta_{i-1}(s) \quad \text{for all } s \in [t, t_i]\}$$

and the function

$$\beta_i(t) = \begin{cases} \beta_{i-1}(t) & \text{if } t \in [0, s_i], \\ x_i(t) & \text{if } t \in (s_i, \infty). \end{cases}$$

Then $\beta_{n-1}(0) = x_0(0)$ and $\beta'_{n-1}(0) \leq x'_0(0)$, so from the monotonicity hypothesis about L and the fact that $x_0 \in S$, we have

$$L(\beta_{n-1}(0), \beta'_{n-1}(0)) \geq L(x_0(0), x'_0(0)) = 0,$$

and it is immediate to check that β_{n-1} is an upper solution for problem (4.1).

From Theorem 3.1 it follows that there exists $v \in S$ such that $\alpha(t) \leq v(t) \leq \beta_{n-1}(t)$ for $t \in [0, \infty)$ and, by the construction of β_{n-1} and the definition of $x_{\min}$, we have that $v(t_i) = x_{\min}(t_i)$ for $i = 0, 1, \ldots, n-1$. Hence, for each $t \in [0, T]$ there is $i \in \{0, 1, \ldots, n-1\}$ such that $t \in [t_i, t_{i+1}]$, so

$$|v(t) - x_{\min}(t)| \leq |v(t) - v(t_i)| + |x_{\min}(t_i) - x_{\min}(t)| < \varepsilon.$$

A similar reasoning shows that problem (4.1) has the greatest solution between α and β. $\quad\square$

Remark 4.2. We note that the existence of extremal solutions in addition to information about the set of solutions for problem (4.1) it provides a method to achieve new existence results for problems where the nonlinearity f has a functional dependence, see [6].

Acknowledgements

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