



A note on a second order PDE with critical nonlinearity

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Abstract. In this work, we are interested in a nonlinear PDE of the form: $-\Delta u = K(x)u^{\frac{n+2}{n-2}}$, $u > 0$ on Ω and $u = 0$ on $\partial\Omega$, where $n \geq 3$ and Ω is a regular bounded domain of \mathbb{R}^n . Following the results of [K. Sharaf, *Appl. Anal.* **96**(2017), No. 9, 1466–1482] and [K. Sharaf, On an elliptic boundary value problem with critical exponent, *Turk. J. Math.*, to appear], we provide a full description of the loss of compactness of the problem and we establish a general index account formula of existence result, when the flatness order of the function K at any of its critical points lies in $(1, \infty)$.

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1 Introduction and main results

In this work, we consider the existence of smooth solutions of

$$\begin{cases} -\Delta u = K(x) u^{\frac{n+2}{n-2}}, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \quad (1.1)$$

where $n \geq 3$, Ω is a regular bounded domain of \mathbb{R}^n and K is a given function on $\bar{\Omega}$.

The original interest of such problem grew out of prescribing scalar curvature equations, see for example [1, 3, 7–9, 11, 12, 15, 16, 22] and the references therein.

Equation (1.1) can be expressed as a variational problem in $H_0^1(\Omega)$. However, the variational structure presents a loss of compactness since the exponent $\frac{n+2}{n-2}$ is critical and $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ is not compact.

The first contributions to (1.1) concern the case $K = 1$, where Bahri-Coron and Pohozaev proved that the resolution of (1.1) depends on the topology of the domain Ω , see [4] and [17]. For $K \neq 1$, many conditions on K were provided to ensure existence of solutions of (1.1), see for example [6, 13, 14, 18–21].

Recently in [19] and [21], we studied problem (1.1) and provided existence and compactness results under the following four conditions:

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$$(A) \quad \frac{\partial K}{\partial \nu}(x) < 0, \quad \forall x \in \partial\Omega.$$

Here ν is the unit outward normal vector on $\partial\Omega$.

(f) $_{\beta}$ K is a C^1 -positive function such that at any critical point y of K , there exists a real number $\beta = \beta(y)$ satisfying the following expansion:

$$K(x) = K(y) + \sum_{k=1}^n b_k |(x-y)_k|^{\beta} + o(|x-y|^{\beta}),$$

for all $x \in B(y, \rho_0)$ where ρ_0 is a positive fixed constant, $b_k = b_k(y) \in \mathbb{R} \setminus \{0\}$, $\forall k = 1, \dots, n$, and

$$\begin{cases} -\frac{n-2}{n} \frac{c_1}{K(y)} \sum_{k=1}^n b_k(y) + c_2 \frac{n-2}{2} H(y, y) \neq 0, & \forall y \in \mathcal{K}_{n-2}, \\ \sum_{k=1}^n b_k(y) \neq 0, & \forall y \in \mathcal{K}_{<n-2}. \end{cases}$$

Here

$$\begin{aligned} \mathcal{K}_{n-2} &:= \{y \in \Omega, \nabla K(y) = 0 \text{ and } \beta(y) = n-2\}, \\ \mathcal{K}_{<n-2} &:= \{y \in \Omega, \nabla K(y) = 0 \text{ and } \beta(y) < n-2\}, \end{aligned}$$

$$c_1 = \int_{\mathbb{R}^n} \frac{|z_1|^{n-2}}{(1+|z|^2)^n} dz, \quad c_2 = \int_{\mathbb{R}^n} \frac{dz}{(1+|z|^2)^{\frac{n+2}{2}}},$$

and $H(\cdot, \cdot)$ is the regular part of the Green function $G(\cdot, \cdot)$ of $(-\Delta)$ under the zero-Dirichlet boundary condition. Let us denote also

$$\begin{aligned} \mathcal{K}_{n-2}^+ &:= \left\{ y \in \mathcal{K}_{n-2}, -\frac{n-2}{n} \frac{c_1}{K(y)} \sum_{k=1}^n b_k(y) + c_2 \frac{n-2}{2} H(y, y) > 0 \right\}, \\ \mathcal{K}_{>n-2} &:= \{y \in \Omega, \nabla K(y) = 0 \text{ and } \beta(y) > n-2\}. \end{aligned}$$

For any $\tau_p := (y_{\ell_1}, \dots, y_{\ell_p}) \in (\mathcal{K}_{n-2}^+ \cup \mathcal{K}_{>n-2})^p$, $p \geq 1$, such that $y_{\ell_i} \neq y_{\ell_j}$, $\forall 1 \leq i \neq j \leq p$, we set the matrix $M(\tau_p) = (m_{ij})_{1 \leq i, j \leq p}$ defined by

$$\begin{aligned} m_{ii} &= m(y_{\ell_i}, y_{\ell_i}) \\ &= \begin{cases} -\frac{1}{K(y_{\ell_i})^{\frac{n-2}{2}}} \left(\frac{n-2}{n} \frac{c_1}{K(y_{\ell_i})} \sum_{k=1}^n b_k(y_{\ell_i}) - c_2 \frac{n-2}{2} H(y_{\ell_i}, y_{\ell_i}) \right) & \text{if } \beta(y_{\ell_i}) = n-2, \\ \frac{n-2}{2} \frac{c_2}{K(y_{\ell_i})^{\frac{n-2}{2}}} H(y_{\ell_i}, y_{\ell_i}) & \text{if } \beta(y_{\ell_i}) > n-2, \end{cases} \end{aligned}$$

$\forall i = 1, \dots, p$ and

$$m_{ij} = m(y_{\ell_i}, y_{\ell_j}) = -\frac{n-2}{2} c_2 \frac{G(y_{\ell_i}, y_{\ell_j})}{\left(K(y_{\ell_i})K(y_{\ell_j})\right)^{\frac{n-2}{4}}}, \quad \text{for } 1 \leq i \neq j \leq p.$$

(B) Assume that the least eigenvalue $\rho(\tau_p)$ of $M(\tau_p)$ is not zero.

The last assumption is

$$(C) \quad \begin{cases} \beta(y) \in (1, n-2] & \forall y \text{ s.t. } \nabla K(y) = 0, \\ \text{or} \\ \beta(y) \in [n-2, \infty) & \forall y \text{ s.t. } \nabla K(y) = 0. \end{cases}$$

Thus, it becomes of interest to study the equation (1.1) in the mixed case situation; that is when there exists some critical points y of K having $\beta(y) < n-2$ and other having $\beta(y) \geq n-2$ and therefore get global compactness and existence results under $(f)_\beta$ -condition for β varies in $(1, \infty)$. Define

$$\mathcal{C}_{<n-2}^\infty := \left\{ (y_{\ell_1}, \dots, y_{\ell_p}) \in \mathcal{K}_{<n-2}^p, p \geq 1, y_{\ell_i} \neq y_{\ell_j}, \forall i \neq j \text{ and } -\sum_{k=1}^n b_k(y_{\ell_i}) > 0, \forall i = 1, \dots, p \right\},$$

$$\mathcal{C}_{\geq n-2}^\infty := \left\{ (y_{\ell_1}, \dots, y_{\ell_p}) \in (\mathcal{K}_{n-2}^+ \cup \mathcal{K}_{>n-2})^p, p \geq 1, y_{\ell_i} \neq y_{\ell_j}, \forall i \neq j \text{ and } \rho(y_{\ell_1}, \dots, y_{\ell_p}) > 0 \right\}.$$

The first result of this paper describes the loss of compactness and the concentration phenomenon of the problem (1.1).

For $a \in \Omega$ and $\lambda \gg 1$, let $P\delta_{(a,\lambda)}$ be the almost solution of the Yamabe-type problem defined in the next section.

Theorem 1.1. *Assume that (1.1) has no solution. Under conditions (A), (B) and $(f)_\beta, \beta > 1$, the critical points at infinity of the associated variational problem (see definition (2.1)) are:*

$$(y_{\ell_1}, \dots, y_{\ell_p})_\infty := \sum_{j=1}^p \frac{1}{K(y_{\ell_j})^{\frac{n-2}{2}}} P\delta_{(y_{\ell_j}, \infty)},$$

where $(y_{\ell_1}, \dots, y_{\ell_p}) \in \mathcal{C}_{<n-2}^\infty \cup \mathcal{C}_{\geq n-2}^\infty \cup \mathcal{C}_{<n-2}^\infty \times \mathcal{C}_{\geq n-2}^\infty$. The index of $(y_{\ell_1}, \dots, y_{\ell_p})_\infty$ is $i(y_{\ell_1}, \dots, y_{\ell_p})_\infty = p-1 + \sum_{i=1}^p n - \tilde{i}(y_{\ell_i})$, where $\tilde{i}(y) = \#\{b_k(y), 1 \leq k \leq n, \text{ s. t. } b_k(y) < 0\}$.

The above characterization allow us to derive a global index formula of existence.

Theorem 1.2. *Let Ω be a regular bounded domain of $\mathbb{R}^n, n \geq 3$ and let $K : \bar{\Omega} \rightarrow \mathbb{R}$ be a given function satisfying (A), (B) and $(f)_\beta, \beta \in (1, \infty)$. If*

$$\sum_{(y_{\ell_1}, \dots, y_{\ell_p}) \in \mathcal{C}_{<n-2}^\infty \cup \mathcal{C}_{\geq n-2}^\infty \cup (\mathcal{C}_{<n-2}^\infty \times \mathcal{C}_{\geq n-2}^\infty)} (-1)^{i(y_{\ell_1}, \dots, y_{\ell_p})_\infty} \neq 1,$$

then (1.1) has a solution.

Remark 1.3. For an explicit example of function K satisfying the hypotheses of Theorem 1.2, let Ω be the unit ball \mathbb{B}^n of $\mathbb{R}^n, n \geq 4$ and let β be a real larger than $n-2$. For any $X \in \mathbb{R}^n$, we define

$$f_1(X) = 1 - \sum_{k=1}^n |x_k|^\beta \quad \text{and} \quad f_2(x) = \sum_{k=1}^n |x_k|^{\frac{3}{2}}.$$

For any integer $k_0 \geq 2$, we denote $y_{k_0} = (\frac{1}{k_0}, 0, \dots, 0)$. Let θ be the cut-off function defined by:

$$\theta(t) = 1 \quad \text{if } t < \frac{1}{4k_0}, \quad \theta(t) = 0 \quad \text{if } t > \frac{1}{2k_0} \quad \text{and} \quad \theta'(t) < 0 \quad \text{if } \frac{1}{4k_0} < t < \frac{1}{2k_0}.$$

Now let $K : \overline{\mathbb{B}^n} \rightarrow \mathbb{R}$, such that $\forall X \in \overline{\mathbb{B}^n}$:

$$K(X) = \theta(\|X - y_{k_0}\|)f_1(X - y_{k_0}) + \theta(\|X + y_{k_0}\|)f_1(X + y_{k_0}) \\ + \theta(\|X\|)f_2(X) - \left(1 - \theta(\|X - y_{k_0}\|) - \theta(\|X + y_{k_0}\|) - \theta(\|X\|)\right)\|X\|^2.$$

Observe that K admits three critical points y_{k_0} , $-y_{k_0}$ and $0_{\mathbb{R}^n}$. By construction K satisfies $(f)_\beta$ condition near its critical points with

$$\beta(y_{k_0}) = \beta(-y_{k_0}) = \beta > n - 2 \quad \text{and} \quad \beta(0_{\mathbb{R}^n}) = \frac{3}{2} < n - 2.$$

According to the result of Theorem 1.1, $0_{\mathbb{R}^n}$ does not give a critical point at infinity since $-\sum_{k=1}^n b_k(0_{\mathbb{R}^n}) = -n < 0$. However y_{k_0} and $-y_{k_0}$ correspond to two critical points at infinity $(y_{k_0})_\infty$ and $(-y_{k_0})_\infty$ respectively. In addition, the pair $(y_{k_0}, -y_{k_0})$ corresponds to a critical point at infinity if and only if $\rho(y_{k_0}, -y_{k_0}) > 0$ where ρ is the least eigenvalue of the matrix

$$M = \frac{n-2}{2} \begin{pmatrix} H(y_{k_0}, y_{k_0}) & -G(y_{k_0}, -y_{k_0}) \\ -G(y_{k_0}, -y_{k_0}) & H(-y_{k_0}, -y_{k_0}) \end{pmatrix}$$

It is easy to see that $\rho(y_{k_0}, -y_{k_0}) > 0$ if and only if

$$H(y_{k_0}, y_{k_0})H(-y_{k_0}, -y_{k_0}) - G^2(y_{k_0}, -y_{k_0}) > 0,$$

since $Tr(M) > 0$. We know from [5, Remark 3, p. 72], that $G(X, Y) \rightarrow -\infty$ if $|X - Y| \rightarrow 0$. Thus for k_0 large enough, $\rho(y_{k_0}, -y_{k_0}) < 0$. Therefore, the only critical points at infinity in our statement are,

$$(y_{k_0})_\infty \text{ and } (-y_{k_0})_\infty \text{ with } \tilde{i}(y_{k_0}) = \tilde{i}(-y_{k_0}) = n.$$

It follows that the function K satisfies the index formula of Theorem 1.2 and the assumption (B). Concerning the assumption (A), observe that outside $B(y_{k_0}, \frac{1}{2k_0}) \cup B(-y_{k_0}, \frac{1}{2k_0}) \cup B(0_{\mathbb{R}^n}, \frac{1}{2k_0})$, the function K is equals to $-\|X\|^2$. Therefore, $DK(X) = -2X$ and on the boundary of \mathbb{B}^n , $\nu_X = X$ and hence

$$\frac{\partial K}{\partial \nu}(X) = \langle Dk(X), \nu_X \rangle = -2.$$

Our argument follows the critical points at infinity theory of A. Bahri [2]. In the next section, we will state the general framework of the variational structure of (1.1). After that we will characterize the critical points at infinity and prove Theorems 1.1 and 1.2.

2 General framework

Equation (1.1) is equivalent to finding the critical points of the following functional

$$J(u) = \frac{\int_{\Omega} |\nabla u|^2}{\left(\int_{\Omega} K(x) u^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}}, \quad u \in \Sigma^+.$$

Here

$$\Sigma = \left\{ u \in H_1^0(\Omega), \text{ s.t. } \|u\|_{H_1^0(\Omega)} = \left(\int_{\Omega} |\nabla u(x)|^2 dx\right)^{\frac{1}{2}} = 1 \right\},$$

and

$$\Sigma^+ = \{ u \in \Sigma, u > 0 \}.$$

It is known that J fails the Palais–Smale condition. The sequences which violate the (P. S) condition has been analyzed as follows. For $a \in \Omega$ and $\lambda > 0$, define

$$\delta_{a,\lambda}(x) = c_0 \left(\frac{\lambda}{1 + \lambda^2|x - a|^2} \right)^{\frac{n-2}{2}}, \quad (2.1)$$

where c_0 is a fixed positive constant. The family $\delta_{a,\lambda}$, $a \in \Omega$ and $\lambda > 0$ are the only solutions of

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}}, \\ u > 0 \quad \text{in } \mathbb{R}^n. \end{cases} \quad (2.2)$$

Define $P\delta_{a,\lambda}$ on Ω be the unique solution of

$$\begin{cases} -\Delta u = \delta_{a,\lambda}^{\frac{n+2}{n-2}} \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \quad (2.3)$$

By the maximum principal and regularity arguments, $P\delta_{a,\lambda}$ is smooth and positive on Ω . For $\varepsilon > 0$ and $p \in \mathbb{N}^*$, let $V(p, \varepsilon)$ be the set of all functions $u \in \Sigma^+$ such that there exists $(a_1, \dots, a_p) \in \Omega^p$, $\lambda_1, \dots, \lambda_p > \varepsilon^{-1}$ and $\alpha_1, \dots, \alpha_p > 0$ satisfying

$$\left| u - \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \right| < \varepsilon,$$

with $|J(u)^{\frac{n}{n-2}} \alpha_i^{\frac{4}{n-2}} K(a_i) - 1| < \varepsilon$ and $\varepsilon_{ij} = (\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2)^{\frac{-(n-2)}{2}} < \varepsilon \quad \forall i \neq j$.

For any sequence $(u_k)_k$ in Σ^+ failing the (P.S) condition, there exists an extracted subsequence $(u_{k_\ell})_\ell$ such that $u_{k_\ell} \in V(p, \varepsilon_{k_\ell}), \forall \ell \in \mathbb{N}$. Here $p \in \mathbb{N}^*$ and $\varepsilon_{k_\ell} \rightarrow 0$ when $\ell \rightarrow +\infty$. See [4] and [23].

The following parametrization of $V(p, \varepsilon)$ was given in [4]. For any $u \in V(p, \varepsilon)$, u can be written as

$$\sum_{i=1}^p \bar{\alpha}_i P\delta_{\bar{a}_i, \bar{\lambda}_i} + v,$$

where $v \in H_0^1(\Omega)$ and satisfies

$$(V_0) \quad \langle v, \psi \rangle = 0 \quad \text{for } \psi \in \left\{ P\delta_{a_i, \lambda_i}, \frac{\partial P\delta_{a_i, \lambda_i}}{\partial \lambda_i}, \frac{\partial P\delta_{a_i, \lambda_i}}{\partial a_i}, i = 1, \dots, p \right\},$$

$\langle \cdot, \cdot \rangle$ denotes the inner product on $H_0^1(\Omega)$ associated to the norm $\| \cdot \|$, and $\bar{\alpha}_i, \bar{a}_i, \bar{\lambda}_i, i = 1, \dots, p$ are the unique solution of

$$\min_{\sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \in V(p, \varepsilon)} \left\| u - \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \right\|.$$

In the following, we show that the v -part of u is negligible with respect to the concentration phenomenon. See [2, 4].

There is a \mathcal{C}^1 -map which to each $(\alpha_i, a_i, \lambda_i)$ such that $\sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i}$ belongs to $V(p, \varepsilon)$ associates $\bar{v} = \bar{v}(\alpha_i, a_i, \lambda_i)$ such that \bar{v} is the unique solution of the following minimization problem

$$\min \left\{ J \left(\sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} + v \right), v \in H_0^1(\Omega) \text{ and satisfies } (V_0) \right\}.$$

Moreover, there exists a change of variables $v - \bar{v} \rightarrow V$ such that

$$J \left(\sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} + v \right) = J \left(\sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} + \bar{v} \right) + \|V\|^2.$$

The following definition is extracted from [2].

Definition 2.1 ([2]). A critical point at infinity of J is a limit of a non-compact flow line $u(s)$ of the gradient vector field $(-\partial J)$. By the above argument, $u(s)$ can be written as:

$$u(s) = \sum_{i=1}^p \alpha_i(s) P\delta_{a_i(s), \lambda_i(s)} + v(s).$$

Denoting by $y_i = \lim_{s \rightarrow +\infty} a_i(s)$ and $\alpha_i = \lim_{s \rightarrow +\infty} \alpha_i(s)$, we then denote by

$$\sum_{i=1}^p \alpha_i P\delta_{y_i, \infty} \text{ or } (y_1, \dots, y_p)_\infty$$

such a critical point at infinity.

3 Critical points at infinity

In this section we prove Theorems 1.1 and 1.2. We start by the following result which describes the concentration phenomenon of the variational structure associated to the problem (1.1).

Theorem 3.1. *Under the assumptions (A), (B) and $(f)_\beta$, $\beta > 1$. There exists a decreasing bounded pseudo-gradient W in $V(p, \varepsilon)$, $p \geq 1$, satisfying the following:*

There exists $c > 0$ such that for any $u = \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \in V(p, \varepsilon)$ we have

$$(i) \quad \langle \partial J(u), W(u) \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^{\min(n, \beta)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right),$$

$$(ii) \quad \left\langle \partial J(u + \bar{v}), W(u) + \frac{\partial \bar{v}}{\partial (\alpha_i, a_i, \lambda_i)}(W(u)) \right\rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^{\min(n, \beta)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right).$$

Moreover, the only case where $\lambda_i(t)$, $i = 1, \dots, p$, tends to ∞ is when $a_i(t)$ goes to y_{ℓ_i} , $\forall i = 1, \dots, p$ such that $(y_{\ell_1}, \dots, y_{\ell_p}) \in \mathcal{C}_{<n-2}^\infty \cup \mathcal{C}_{\geq n-2}^\infty \cup (\mathcal{C}_{<n-2}^\infty \times \mathcal{C}_{\geq n-2}^\infty)$.

Here $(\mathcal{C}_{<n-2}^\infty$ and $\mathcal{C}_{\geq n-2}^\infty)$ are defined in the first section.

Before presenting the proof of Theorem 3.1, we recall the following result which describes the concentration phenomena of the problem when $\beta \in (1, n-2)$, see [19, Section 3].

Theorem 3.2 ([19]). *Under the assumptions of Theorem 3.1 with $\beta \in]1, n-2[$, there exists a decreasing bounded pseudo-gradient W_1 satisfying (i) of Theorem 3.1, for any $u = \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \in V(p, \varepsilon)$ and the only case where $\lambda_i(t)$ goes to $+\infty$, $i = 1, \dots, p$ is when $a_i(t)$ goes to y_{ℓ_i} with $(y_{\ell_1}, \dots, y_{\ell_p}) \in \mathcal{C}_{<n-2}^\infty$.*

Notice that the case of $\beta = n - 2$ was handled also in [19].

Recently we proved the following result which describes the concentration phenomena in the case where $\beta \in [n - 2, +\infty)$.

Theorem 3.3 ([21]). *Under the assumptions of Theorem 3.1 with $\beta \in [n - 2, \infty)$, there exists a decreasing bounded pseudo-gradient W_2 satisfying (i) of Theorem 3.1, for any $u = \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \in V(p, \varepsilon)$ and the only case, where $\lambda_i(t)$, $i = 1, \dots, p$ goes to $+\infty$ is when $a_i(t)$ goes to y_{ℓ_i} with $(y_{\ell_1}, \dots, y_{\ell_p}) \in C_{\geq n-2}^\infty$.*

The complete construction of the required pseudo-gradient W_2 in $V(p, \varepsilon)$ was given in [21]. We provide in the next the construction of W_2 in a specific region $R_{>n-2}^\delta(p, \varepsilon)$ where

$$R_{>n-2}^\delta(p, \varepsilon) := \left\{ u = \sum_{i=1}^p \alpha_i P\delta_{(a_i, \lambda_i)} \in V(p, \varepsilon), a_i \in B(y_{\ell_i}, \rho_0) \right. \\ \left. y_{\ell_i} \in \mathcal{K}_{>n-2}, \lambda_i^{n-2} |a_i - y_{\ell_i}|^{\beta(y_{\ell_i})} < \delta, \forall i = 1, \dots, p \text{ and } y_{\ell_i} \neq y_{\ell_j}, \forall i \neq j \right\}.$$

Here δ is a small positive constant. Let $u = \sum_{i=1}^p \alpha_i P\delta_{(a_i, \lambda_i)} \in R_{>n-2}^\delta(p, \varepsilon)$.

Case 1: If $\rho(y_{\ell_1}, \dots, y_{\ell_p}) > 0$. We use the expansion (3.1) below. Since $J(u) \stackrel{n-2}{\sim} \alpha_j^{\frac{4}{n-2}} K(a_j) = 1 + o(1), \forall j = 1, \dots, p$, (3.1) becomes

$$\left\langle \partial J(u), \alpha_i \lambda_i \frac{\partial P\delta_{(a_i, \lambda_i)}}{\partial \lambda_i} \right\rangle = -2c_2 J(u)^{\frac{2-n}{2}} \left[\sum_{i \neq j} \frac{1}{(K(a_i)K(a_j))^{\frac{n-2}{4}}} \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \frac{n-2}{2} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \right) \right. \\ \left. + \frac{n-2}{2} \frac{1}{K(a_i)^{\frac{n-2}{2}}} \frac{H(y_{\ell_i}, y_{\ell_i})}{\lambda_i^{n-2}} \right] \\ + O\left(|a_i - y_{\ell_i}|^{\beta(y_{\ell_i})}\right) + o\left(\sum_{k \neq j} \left(\varepsilon_{jk} + \frac{H(a_i, a_k)}{(\lambda_i \lambda_k)^{\frac{n-2}{2}}}\right)\right).$$

Observe that as δ small we have,

$$|a_i - y_{\ell_i}|^{\beta(y_{\ell_i})} = o\left(\frac{1}{\lambda_i^{n-2}}\right).$$

Moreover, since $|a_i - a_j| \geq \rho_0, \forall i \neq j$, we have

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\frac{n-2}{2} \frac{1}{(\lambda_i \lambda_j |a_i - a_j|^2)^{\frac{n-2}{2}}} (1 + o(1)).$$

Therefore ,

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-2}{2} \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} = -\frac{n-2}{2} \frac{G(y_{\ell_i}, y_{\ell_j})}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} (1 + o(1)).$$

Thus

$$\left\langle \partial J(u), \alpha_i \lambda_i \frac{\partial P\delta_{(a_i, \lambda_i)}}{\partial \lambda_i} \right\rangle = -2J(u)^{\frac{2-n}{2}} \left[\sum_{i \neq j} \frac{m(y_{\ell_i}, y_{\ell_j})}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} + \frac{m(y_{\ell_i}, y_{\ell_i})}{\lambda_i^{n-2}} \right] + o\left(\sum_{k=1}^p \frac{1}{\lambda_k^{n-2}}\right),$$

where the coefficients $m(y_{\ell_i}, y_{\ell_j}), 1 \leq i, j \leq p$ are defined in the first section.

For any $i = 1, \dots, p$ we set $\dot{\lambda}_i = \lambda_i$. The corresponding pseudo-gradient is

$$W_2(u) = \sum_{i=1}^p \alpha_i \lambda_i \frac{\partial P\delta_{(a_i, \lambda_i)}}{\partial \lambda_i}.$$

From the latest expansion, W_2 satisfies

$$\begin{aligned} & \langle \partial J(u), W_2(u) \rangle \\ &= -2J(u)^{\frac{2-n}{2}} \left(\frac{1}{\lambda_1^{\frac{n-2}{2}}}, \dots, \frac{1}{\lambda_p^{\frac{n-2}{2}}} \right) M(y_{\ell_1}, \dots, y_{\ell_p}) \left(\frac{1}{\lambda_1^{\frac{n-2}{2}}}, \dots, \frac{1}{\lambda_p^{\frac{n-2}{2}}} \right)^t + o \left(\sum_{k=1}^p \frac{1}{\lambda_k^{n-2}} \right) \\ &\leq -\rho(y_{\ell_1}, \dots, y_{\ell_p}) \sum_{i=1}^p \frac{1}{\lambda_i^{n-2}}, \end{aligned}$$

since $\rho(y_{\ell_1}, \dots, y_{\ell_p})$ is the least eigenvalue of $M(y_{\ell_1}, \dots, y_{\ell_p})$. Using the fact that

$$\frac{|\nabla K(a_j)|}{\lambda_j} \sim \frac{|a_j - y_{\ell_j}|^{\beta-1}}{\lambda_j} = O \left(\frac{1}{\lambda_j^{n-2}} \right)$$

and for any $i \neq j$, we have

$$\varepsilon_{ij} \sim \frac{1}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} = O \left(\frac{1}{\lambda_i^{n-2}} \right) + O \left(\frac{1}{\lambda_j^{n-2}} \right),$$

we get

$$\langle \partial J(u), W_2(u) \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^{n-2}} \right) + \sum_{i \neq j} \varepsilon_{ij} \right).$$

Case 2: If $\rho(y_{\ell_1}, \dots, y_{\ell_p}) < 0$. This is the opposite situation of the case 1. Thus

$$W_2(u) = - \sum_{i=1}^p \alpha_i \lambda_i \frac{\partial P\delta_{(a_i, \lambda_i)}}{\partial \lambda_i},$$

satisfies the requirement of Theorem 3.3.

Proof of Theorem 3.1. Let $u = \sum_{i=1}^p \alpha_i P\delta_{a_i, \lambda_i} \in V(p, \varepsilon)$, $p \geq 1$. Following the above two results, the only case that we will consider here is when u can be written as

$$u = \sum_{i=1}^q \alpha_i P\delta_{a_i, \lambda_i} + \sum_{i=q+1}^p \alpha_i P\delta_{a_i, \lambda_i} =: u_1 + u_2,$$

where $1 \leq q < p$ and

$$u_1 \in R_1 := \left\{ u = \sum_{i=1}^s \alpha_i P\delta_{(a_i, \lambda_i)} \in V(s, \varepsilon), s \geq 1, \text{ s.t. } a_i \in B(y_{\ell_i}, \rho_0), \right. \\ \left. \text{with } \beta(y_{\ell_i}) < n - 2, \forall i = 1, \dots, s \right\},$$

$$u_2 \in R_2 := \left\{ u = \sum_{i=1}^s \alpha_i P\delta_{(a_i, \lambda_i)} \in V(s, \varepsilon), s \geq 1, \text{ s.t. } a_i \in B(y_{\ell_i}, \rho_0), \right. \\ \left. \text{with } \beta(y_{\ell_i}) \geq n-2, \forall i = 1, \dots, s \right\}.$$

Let us denote by W_1 the pseudo-gradient given by Theorem 3.2 and W_2 the pseudo-gradient given by Theorem 3.3. In order to construct the required pseudo-gradient W of Theorem 3.1, we distinguish three cases. Let δ be a fixed positive constant small enough.

• **Case 1.**

$$u_1 \in \left\{ u = \sum_{i=1}^s \alpha_i P\delta_{(a_i, \lambda_i)} \in R_1, \text{ s.t., } \lambda_i |a_i - y_{\ell_i}| < \delta, \forall i = 1, \dots, s, \text{ and } (y_{\ell_1}, \dots, y_{\ell_s}) \in C_{<n-2}^\infty \right\}$$

and

$$u_2 \in \left\{ u = \sum_{i=1}^s \alpha_i P\delta_{(a_i, \lambda_i)} \in R_2, \text{ s.t., } \lambda_i |a_i - y_{\ell_i}| < \delta, \forall i = 1, \dots, s, \text{ and } (y_{\ell_1}, \dots, y_{\ell_s}) \in C_{\geq n-2}^\infty \right\}.$$

According to the construction of [19] and [21], the vector fields W_1 and W_2 in these regions are defined as follows:

$$W_1(u_1) := \sum_{i=1}^q \alpha_i \lambda_i \frac{\partial P\delta_{(a_i, \lambda_i)}}{\partial \lambda_i} \quad \text{and} \quad W_2(u_2) = \sum_{i=q+1}^p \alpha_i \lambda_i \frac{\partial P\delta_{(a_i, \lambda_i)}}{\partial \lambda_i}.$$

Observe that all the components λ_i of the corresponding flow lines satisfies the differential equation

$$\dot{\lambda} = \lambda_i, \quad \forall i = 1, \dots, p.$$

In this case, we set

$$W(u) = \tilde{W}_1(u) + \tilde{W}_2(u),$$

where $\tilde{W}_1(u) := W_1(u_1)$ and $\tilde{W}_2(u) := W_2(u_2)$. Following the computation of [19] and [20], we have

$$\left\langle \partial J(u), \alpha_i \lambda_i \frac{\partial P\delta_{a_i, \lambda_i}}{\partial \lambda_i} \right\rangle = -2c_2 \frac{J(u)}{K(a_i)} \sum_{j \neq i} \alpha_j \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \right) \\ + 2\alpha_i^2 \frac{J(u)}{K(a_i)} \begin{cases} \frac{n-2}{2} c_1 \frac{\sum_{k=1}^n b_k(y_{\ell_i})}{\lambda_i^{\beta(y_{\ell_i})}}, & \text{if } \beta(y_{\ell_i}) < n-2 \\ \frac{n-2}{2} c_1 \frac{\sum_{k=1}^n b_k(y_{\ell_i})}{\lambda_i^{\beta(y_{\ell_i})}} - c_2 \frac{H(y_{\ell_i}, y_{\ell_i})}{\lambda_i^{n-2}}, & \text{if } \beta(y_{\ell_i}) = n-2 \\ -c_2 \frac{H(y_{\ell_i}, y_{\ell_i})}{\lambda_i^{n-2}}, & \text{if } \beta(y_{\ell_i}) > n-2 \end{cases} \\ + O(|a_i - y_{\ell_i}|^\beta) + o\left(\sum_{j \neq i} \left(\varepsilon_{ij} + \frac{H(a_i, a_j)}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \right) + \frac{1}{\lambda_i^{n-2}} \right). \quad (3.1)$$

Since we have $|a_i - a_j| \geq \rho_0, \forall i \neq j$, we obtain

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\frac{n-2}{2} \frac{1}{(|a_i - a_j|^2 \lambda_i \lambda_j)^{\frac{n-2}{2}}} + o\left(\frac{1}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \right) \leq -c\varepsilon_{ij}. \quad (3.2)$$

Using the fact that K satisfies $(f)_\beta$ assumption around each y_{ℓ_i} , we derive

$$|\nabla K(a_i)| \sim |a_i - y_{\ell_i}|^{\beta-1}. \quad (3.3)$$

Estimate (3.3) with the fact that $\lambda_i |a_i - y_{\ell_i}| \leq \delta$ yield

$$\frac{\nabla K(a_i)}{\lambda_i} = o\left(\frac{1}{\lambda_i^\beta}\right) \text{ and } |a_i - y_{\ell_i}|^\beta = o\left(\frac{1}{\lambda_i^\beta}\right).$$

We therefore have

$$\langle \partial J(u), \tilde{W}_1(u) \rangle \leq -c \left(\sum_{i=1}^q \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{1 \leq j \neq i \leq q} \varepsilon_{ij} \right) + O \left(\sum_{1 \leq i \leq q, q+1 \leq j \leq p} \varepsilon_{ij} \right), \quad (3.4)$$

and

$$\begin{aligned} & \langle \partial J(u), \tilde{W}_2(u) \rangle \\ & \leq -c \left(\sum_{i=q+1}^p \left(\frac{1}{\lambda_i^{\min(n,\beta)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{q+1 \leq j \neq i \leq p} \varepsilon_{ij} \right) + O \left(\sum_{q+1 \leq i \leq p, 1 \leq j \leq q} \varepsilon_{ij} \right). \end{aligned} \quad (3.5)$$

For any $1 \leq i \leq q$ and for any $q+1 \leq j \leq p$ we claim that

$$\varepsilon_{ij} = o\left(\frac{1}{\lambda_i^\beta}\right) + o\left(\frac{1}{\lambda_j^{n-2}}\right). \quad (3.6)$$

Indeed, since $|a_i - a_j| \geq \rho_0$, we have $\varepsilon_{ij} \sim \frac{1}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}}$. Let $M \gg 1$. If $\lambda_i < M\lambda_j$, then

$$\frac{1}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \leq \frac{M^{\frac{n-2}{2}}}{\lambda_j^{n-2}} = o\left(\frac{1}{\lambda_j^\beta}\right),$$

since $\beta < n-2$ for $1 \leq i \leq q$. If $\lambda_i > M\lambda_j$, then

$$\frac{1}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \leq \frac{1}{M^{\frac{n-2}{2}}} \frac{1}{\lambda_j^{n-2}} = o\left(\frac{1}{\lambda_j^{n-2}}\right), \text{ as } M \text{ large.}$$

Thus (3.6) follows. The inequalities (3.4) and (3.5) with estimates (3.6) yield

$$\langle \partial J(u), W(u) \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^{\min(n,\beta)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right).$$

Observe that through W , $\lambda_i(t)$ tends to ∞ , $\forall i = 1, \dots, p$; it is a concentration phenomenon.

• **Case 2.**

$$u_1 \notin \left\{ u = \sum_{i=1}^s \alpha_i P_{\delta(a_i, \lambda_i)} \in R_1, \text{ s.t.}, \lambda_i |a_i - y_{\ell_i}| < \delta, \forall i = 1, \dots, s, \text{ and } (y_{\ell_1}, \dots, y_{\ell_s}) \in \mathcal{C}_{<n-2}^\infty \right\}.$$

Three possibilities may occur. Either there exists $i_0, 1 \leq i_0 \leq q$ such that $\lambda_{i_0} |a_{i_0} - y_{\ell_{i_0}}| \geq \delta$ or there exists $i_1, 1 \leq i_1 \leq q$ such that $-\sum_{k=1}^n b_k(y_{i_1}) < 0$ or there exist $i \neq j$ such that $y_{\ell_i} = y_{\ell_j}$.

In all possibilities, it was constructed in [19], section 3, a pseudo-gradient W_1 along which the $\max_{1 \leq i \leq q}(\lambda_i(s))$ remains bounded and satisfies

$$\langle \partial J(u_1), W_1(u_1) \rangle \leq -c \left(\sum_{i=1}^q \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{1 \leq j \neq i \leq q} \varepsilon_{ij} \right). \quad (3.7)$$

Therefore, for $\tilde{W}_1(u) = W_1(u_1)$, we set

$$\langle \partial J(u), \tilde{W}_1(u) \rangle \leq -c \left(\sum_{i=1}^q \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{1 \leq j \neq i \leq q} \varepsilon_{ij} \right) + O \left(\sum_{1 \leq i \leq q, q+1 \leq j \leq p} \varepsilon_{ij} \right). \quad (3.8)$$

Denote by i_1 an index such that

$$\lambda_{i_1}^\beta = \inf \{ \lambda_i^\beta, i = 1, \dots, q \}$$

and let us denote

$$L = \left\{ j, 1 \leq j \leq p; \lambda_j^\beta \geq \frac{1}{2} \lambda_{i_1}^\beta \right\}.$$

It is easy to see that we can appear all $-\frac{1}{\lambda_i^\beta}$, $i \in L$ in the upper bound of (3.8). In order to make appear all $-\frac{|\nabla K(a_i)|}{\lambda_i}$, $i \in L$, let us recall the following estimate obtained in [19, Section 3].

$$\begin{aligned} \left\langle \partial J(u), \alpha_i \frac{1}{\lambda_i} \frac{\partial P \delta_{a_i, \lambda_i}}{\partial (a_i)_k} \right\rangle &= -(n-2) \alpha_i^2 J(u) \frac{b_k}{\lambda_i K(a_i)} \beta \operatorname{sign}(a_i - y_{\ell_i})_k |a_i - y_{\ell_i}|_k^{\beta-1} \\ &+ O \left(\sum_{j=2}^{[\beta]} \frac{|a_i - y_{\ell_i}|^{\beta-j}}{\lambda_i^j} \right) + O \left(\frac{1}{\lambda_i^\beta} \right) + O \left(\sum_{j \neq i} \left| \frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} \right| \right). \end{aligned} \quad (3.9)$$

Let

$$Y_i(u) = \sum_{k=1}^n b_k \operatorname{sign}(a_i - y_{\ell_i})_k \frac{1}{\lambda_i} \frac{\partial P \delta_{(a_i, \lambda_i)}}{\partial (a_i)_k}.$$

For each index $i \in L \setminus \{1, \dots, q\}$, we move the concentration point a_i with respect to Y_i . Using (3.9), the corresponding variation of J is given by:

$$\begin{aligned} \langle \partial J(u), Y_i(u) \rangle &\leq -\frac{c_3}{K(a_i)} \alpha_i^2 J(u) \sum_{k=1}^n b_k^2 \frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i} \\ &+ O \left(\sum_{j=2}^{[\beta]} \frac{|a_i - y_{\ell_i}|^{\beta-j}}{\lambda_i^j} \right) + O \left(\frac{1}{\lambda_i^\beta} \right) + O \left(\sum_{j \neq i} \varepsilon_{ij} \right). \end{aligned} \quad (3.10)$$

For any $j = 2, \dots, [\beta]$, we claim that

$$\frac{|a_i - y_{\ell_i}|^{\beta-j}}{\lambda_i^j} = O \left(\frac{1}{\lambda_i^\beta} \right) + o \left(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i} \right). \quad (3.11)$$

Indeed, let $M \gg 1$. If $|\lambda_i(a_i - y_{\ell_i})| \leq M$, we have

$$\frac{|a_i - y_{\ell_i}|^{\beta-j}}{\lambda_i^j} = O \left(\frac{1}{\lambda_i^\beta} \right),$$

and if $|\lambda_i(a_i - y_{\ell_i})| \geq M$, we have

$$\frac{|a_i - y_{\ell_i}|^{\beta-j}}{\lambda_i^j} = o\left(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i}\right), \quad \text{as } M \text{ large enough.}$$

Hence (3.11) follows. From this and (3.3), (3.10) becomes

$$\langle \partial J(u), Y_i(u) \rangle \leq -c \frac{|\nabla K(a_i)|}{\lambda_i} + O\left(\frac{1}{\lambda_i^\beta}\right) + O\left(\sum_{j \neq i} \varepsilon_{ij}\right). \quad (3.12)$$

The inequalities (3.8) and (3.12) with the estimates (3.6) yield for $m > 0$ small enough

$$\begin{aligned} \left\langle \partial J(u), \tilde{W}_1(u) + m \sum_{i \in L \setminus \{1, \dots, q\}} Y_i(u) \right\rangle &\leq -c \left(\sum_{i \in L} \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{1 \leq j \neq i \leq q} \varepsilon_{ij} \right) \\ &\quad + o\left(\sum_{j \neq i} \varepsilon_{ij}\right) + o\left(\sum_{j=1}^p \frac{1}{\lambda_j^\beta}\right). \end{aligned} \quad (3.13)$$

We now appear $-\sum_{i \neq j, i, j \in L} \varepsilon_{ij}$ in the last upper bound. For this, we decrease all λ_i such that $i \in L \setminus \{1, \dots, q\}$. Define

$$Z(u) = \sum_{i \in L \setminus \{1, \dots, q\}} -2^i \lambda_i \frac{\partial P_{\delta_{a_i, \lambda_i}}}{\partial \lambda_i}.$$

Without loss of the generality, we can assume that if $i < j$ then $\lambda_i \leq \lambda_j$. In that case we have

$$2^i \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + 2^j \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \leq -c \varepsilon_{ij}, \quad \forall i \neq j \in L \setminus \{1, \dots, q\}.$$

Therefore,

$$\begin{aligned} \langle \partial J(u), Z(u) \rangle &\leq -c \left(\sum_{i \neq j \in L \setminus \{1, \dots, q\}} \varepsilon_{ij} \right) + O\left(\sum_{i \in L \setminus \{1, \dots, q\}} \frac{1}{\lambda_i^\beta}\right) \\ &\quad + O\left(\sum_{i \in L \setminus \{1, \dots, q\}} \sum_{j=2}^{[\beta]} \frac{|a_i - y_{\ell_i}|^{\beta-j}}{\lambda_i^j}\right). \end{aligned} \quad (3.14)$$

The inequalities (3.13) and (3.14) with the estimate (3.11) yield for $m' > 0$ and small

$$\begin{aligned} &\left\langle \partial J(u), \tilde{W}_1(u) + m \sum_{i \in L \setminus \{1, \dots, q\}} Y_i(u) + m' Z(u) \right\rangle \\ &\leq -c \left(\sum_{i \in L} \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i, j \in L, j \neq i} \varepsilon_{ij} \right) + o\left(\sum_{i \in L, j \notin L} \varepsilon_{ij}\right) + o\left(\sum_{j \notin L} \frac{1}{\lambda_j^\beta}\right). \end{aligned} \quad (3.15)$$

To add the left indices, we denote

$$\tilde{u} = \sum_{i \notin L} \alpha_i P_{\delta_{a_i, \lambda_i}}.$$

It is easy to see that $\tilde{u} \in R_2$. Set

$$\tilde{W}_2(u) = W_2(\tilde{u}).$$

Observe that the $\max_{1 \leq i \leq p} \lambda_i(s)$ does not move along \tilde{W}_2 , since it acts only on the indices $i \notin L$. Using the above techniques (see also [21] for more details), we have

$$\langle \partial J(u), \tilde{W}_2(u) \rangle \leq -c \left(\sum_{i \notin L} \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{i \notin L, j \neq i} \varepsilon_{ij} \right) + O \left(\sum_{i \notin L, j \in L} \varepsilon_{ij} \right). \quad (3.16)$$

For $m'' > 0$ and small, define

$$W(u) = \tilde{W}_1(u) + m \sum_{i \in L \setminus \{1, \dots, q\}} Y_i(u) + m' Z(u) + m'' \tilde{W}_2(u).$$

We obtain from (3.15) and (3.16)

$$\langle \partial J(u), W(u) \rangle \leq -c \left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{j \neq i} \varepsilon_{ij} \right).$$

• **Case 3.**

$$u_2 \notin \left\{ u = \sum_{i=1}^s \alpha_i P \delta_{(a_i, \lambda_i)} \in \mathbb{R}_2, s.t., \lambda_i |a_i - y_{\ell_i}| < \delta, \forall i = 1, \dots, s, \text{ and } (y_{\ell_1}, \dots, y_{\ell_s}) \in \mathcal{C}_{\geq n-2}^\infty \right\}.$$

In this case, either $W_2(u_2) = -\sum_{i=q+1}^p \alpha_i \lambda_i \frac{\partial P \delta_{a_i \lambda_i}}{\partial \lambda_i}$ or $W_2(u_2)$ is as the one defined in [19, Section 3]. In both cases, $\max_{q+1 \leq i \leq p} \lambda_i$ remains bounded along W_2 . Let

$$\tilde{W}_2(u) = W_2(u_2).$$

It satisfies

$$\langle \partial J(u), \tilde{W}_2(u) \rangle \leq -c \left(\sum_{j=q+1}^p \left(\frac{1}{\lambda_j^\beta} + \frac{|\nabla K(a_i)|}{\lambda_i} \right) + \sum_{q+1 \leq j \neq i \leq p} \varepsilon_{ij} \right) + O \left(\sum_{1 \leq j \leq q, q+1 \leq i \leq p} \varepsilon_{ij} \right).$$

Let i_1 be an index such that

$$\lambda_{i_1}^\beta = \inf \{ \lambda_i^\beta, q+1 \leq i \leq p \},$$

and let

$$L = \left\{ j, 1 \leq j \leq p \text{ such that } \lambda_j^\beta \geq \frac{1}{2} \lambda_{i_1}^\beta \right\}.$$

Using the same argument as in Case 2, we obtain for

$$W(u) = \tilde{W}_2 + m \sum_{i \in L \setminus \{q+1, \dots, p\}} Y_i(u) + m' Z(u) + m'' \tilde{W}_1(u),$$

the required estimate of Theorem 3.1.

Finally observe that the Palais–Smale condition is satisfied along the decreasing flow lines of the pseudo-gradient W as long as the concentration points of the flow do not enter in some neighborhood of $(y_{\ell_1}, \dots, y_{\ell_p}) \in \mathcal{C}_{< n-2}^\infty \cup \mathcal{C}_{\geq n-2}^\infty \cup \mathcal{C}_{< n-2}^\infty \times \mathcal{C}_{\geq n-2}^\infty$ since $\max_{1 \leq i \leq p} \lambda_i(t)$ remains bounded in this region. However, if the concentration points are near critical points $(y_{\ell_1}, \dots, y_{\ell_p}) \in \mathcal{C}_{< n-2}^\infty \cup \mathcal{C}_{\geq n-2}^\infty \cup \mathcal{C}_{< n-2}^\infty \times \mathcal{C}_{\geq n-2}^\infty$, $\lambda_i(s)$ increases on the flow line and goes to $+\infty$. Thus, we obtain a critical point at infinity. This finishes the proof of Theorem 3.1. \square

Proof of Theorem 1.1. Using the result of Theorem 3.1, we observe that the only case where $\lambda_i(t)$, $i = 1, \dots, p$, tends to ∞ is when $a_i(t)$ goes to y_{ℓ_i} , $\forall i = 1, \dots, p$, such that $(y_{\ell_1}, \dots, y_{\ell_p}) \in \mathcal{C}_{<n-2}^\infty \cup \mathcal{C}_{\geq n-2}^\infty \cup (\mathcal{C}_{<n-2}^\infty \times \mathcal{C}_{\geq n-2}^\infty)$. Thus, the critical points at infinity of J are in one to one correspondence with the elements $\tau_p = (y_{\ell_1}, \dots, y_{\ell_p})$ such that $\tau_p \in \mathcal{C}_{<n-2}^\infty \cup \mathcal{C}_{\geq n-2}^\infty \cup (\mathcal{C}_{<n-2}^\infty \times \mathcal{C}_{\geq n-2}^\infty)$.

Concerning the Morse index of the critical point at infinity $(\tau_p)_\infty$, it follows from the following expansion of $J(u)$ where u is close to $(\tau_p)_\infty$:

$$J(u) = \left(\sum_{i=1}^p \frac{1}{K(y_{\ell_i})^{\frac{n-2}{2}}} \right)^{\frac{2}{n}} \left(1 - \|h\|_{\mathbb{R}^{p-1}}^2 + \sum_{i=1}^p (|a_i^-|^2 - |a_i^+|^2) \right).$$

Here $h \in \mathbb{R}^{p-1}$, a_i^+ and a_i^- are the coordinates of a_i along the stable and unstable manifold of K at y_{ℓ_i} . \square

Proof of Theorem 1.2. Assume that (1.1) has no solution. Using the result of Theorem 1.1, Σ^+ retracts by deformation on

$$\bigcup_{\tau_p \in \mathcal{C}_{<n-2}^\infty \cup \mathcal{C}_{\geq n-2}^\infty \cup (\mathcal{C}_{<n-2}^\infty \times \mathcal{C}_{\geq n-2}^\infty)} W_u^\infty(\tau_p)_\infty,$$

where $W_u^\infty(\tau_p)_\infty$ denotes the unstable manifold of the critical point at infinity $(\tau_p)_\infty$. Using an Euler–Poincaré characteristic argument, we get after recalling that Σ^+ is a contractible set

$$1 = \sum_{\tau_p \in \mathcal{C}_{<n-2}^\infty \cup \mathcal{C}_{\geq n-2}^\infty \cup (\mathcal{C}_{<n-2}^\infty \times \mathcal{C}_{\geq n-2}^\infty)} (-1)^{i(\tau_p)_\infty}.$$

This is a contradiction. The proof of Theorem 1.2 is thereby completed. \square

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