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Positive solutions for a fourth-order three-point BVP with sign-changing Green's function

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Abstract. This paper is concerned with the following fourth-order three-point boundary value problem

$$\begin{cases} u^{(4)}(t) = f\big(t, u(t)\big), \ t \in [0,1], \\ u'(0) = u''(0) = u'''(\eta) = u(1) = 0, \end{cases}$$

where $\eta \in \big[\frac{1}{3}, 1\big)$. In spite of sign-changing Green's function, for arbitrary positive integer n (≥ 2), we still obtain the existence of at least $n-1$ decreasing positive solutions to the above problem by imposing some suitable conditions on f. The main tool used is the fixed point index theory.

Keywords: fourth-order three-point boundary value problem, sign-changing Green's function, positive solution, existence, fixed point index.

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1 Introduction

Boundary value problems (BVPs for short) of fourth-order ordinary differential equations have received much attention due to their striking applications in engineering, physics, material mechanics, fluid mechanics and so on. Many authors have studied the existence of single or multiple positive solutions to some fourth-order BVPs by using Banach contraction theorem, Guo–Krasnosel'skii fixed point theorem, Leray–Schauder nonlinear alternative, fixed point index theory in cones, monotone iterative technique, the method of upper and lower solutions, degree theory, critical point theorems in conical shells and so forth. However, it is necessary to point out that, in most of the existing literature, the Green's functions involved are nonnegative, which is an important condition in the study of positive solutions of BVPs.

Recently, there have been some works on positive solutions for second-order or third-order BVPs when the corresponding Green's functions are sign-changing. For example, Zhong and

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An [20] studied the existence of at least one positive solution for the following second-order periodic BVP with sign-changing Green's function

$$\begin{cases} u'' + \rho^2 u = f(u), \ 0 < t < T, \\ u(0) = u(T), \ u'(0) = u'(T), \end{cases}$$

where $0 < \rho \leq \frac{3\pi}{2T}$. The main tool used was the fixed point index theory of cone mapping. In 2008, for the singular third-order three-point BVP with an indefinitely signed Green's function

$$\begin{cases} u'''(t) = a(t)f(t, u(t)), \ 0 < t < 1, \\ u(0) = u(1) = u''(\eta) = 0, \end{cases}$$

where $\eta \in \left(\frac{17}{24}, 1\right)$, Palamides and Smyrlis [14] discussed the existence of at least one positive solution. Their technique was a combination of the Guo–Krasnosel'skii fixed point theorem and properties of the corresponding vector field. In 2012, by applying the Guo–Krasnosel'skii and Leggett–Williams fixed point theorems, Sun and Zhao [17, 18] obtained the existence of single or multiple positive solutions for the following third-order three-point BVP with sign-changing Green's function

$$\begin{cases} u'''(t) = f(t, u(t)), \ t \in [0, 1], \\ u'(0) = u(1) = u''(\eta) = 0, \end{cases}$$

where $\eta \in \left(\frac{1}{2}, 1\right)$. For relevant results, one can refer to [3,4,9,10,13,15,16,19].

It is worth mentioning that there are other type of achievements on either sign-changing or vanishing Green's functions which prove the existence of sign-changing solutions, positive in some cases, see [1,2,5,7,8,12].

Motivated and inspired by the above-mentioned works, in this paper, we are concerned with the following fourth-order three-point BVP with sign-changing Green's function

$$\begin{cases} u^{(4)}(t) = f\big(t, u(t)\big), \ t \in [0, 1], \\ u'(0) = u''(0) = u'''(\eta) = u(1) = 0. \end{cases} \tag{1.1}$$

By imposing some suitable conditions on f and η, we obtain the existence of at least $n-1$ decreasing positive solutions to the BVP (1.1) for arbitrary positive integer n (≥ 2).

To end this section, we state some knowledge of the classical fixed point index for compact maps [6].

Let K be a cone in a Banach space X. If Ω is a bounded open subset of K (in the relative topology) we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary relative to K. When D is an open bounded subset of X we write $D_K = D \cap K$, an open subset of K.

Theorem 1.1. *Let D be an open bounded set with $D_K \neq \emptyset$ and $\overline{D}_K \neq K$. Assume that $T : \overline{D}_K \to K$ is a compact map such that $x \neq Tx$ for $x \in \partial D_K$. Then the fixed point index $i_K(T, D_K)$ has the following properties.*

(1) *If there exists $e \in K \backslash \{0\}$ such that $x \neq Tx + \lambda e$ for all $x \in \partial D_K$ and all $\lambda > 0$, then $i_K(T, D_K) = 0$.*

(2) *If $\|Tx\| \leq \|x\|$ for $x \in \partial D_K$, then $i_K(T, D_K) = 1$.*

(3) *Let D^1 be open in X with $\overline{D^1} \subset D_K$. If we have $i_K(T, D_K) = 1$ and $i_K(T, D^1_K) = 0$, then T has a fixed point in $D_K \backslash \overline{D^1_K}$. The same result holds if $i_K(T, D_K) = 0$ and $i_K(T, D^1_K) = 1$.*

2 Preliminaries

Let $X = C[0,1]$ be equipped with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Then X is a Banach space.

Lemma 2.1. *Let $\eta \in (0,1)$. Then for any given $y \in X$, the BVP*

$$\begin{cases} u^{(4)}(t) = y(t),\ t \in [0,1], \\ u'(0) = u''(0) = u'''(\eta) = u(1) = 0 \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds,\ t \in [0,1],$$

where

$$G(t,s) = \frac{1}{6}\begin{cases} 3(1-t)(1+t-s)s, & s \le \min\{\eta, t\}, \\ 3s - 3s^2 + s^3 - t^3, & t \le s \le \eta, \\ (t-s)^3 - (1-s)^3, & \eta < s \le t, \\ -(1-s)^3, & s > \max\{\eta, t\}. \end{cases}$$

Proof. In view of $u^{(4)}(t) = y(t),\ t \in [0,1]$ and $u'''(\eta) = 0$, we have

$$u'''(t) = \int_\eta^t y(\tau)d\tau, \qquad t \in [0,1],$$

which together with the boundary condition $u''(0) = 0$ implies that

$$u''(t) = \int_0^t \int_\eta^\tau y(s)dsd\tau, \qquad t \in [0,1].$$

If $t \in [0,\eta]$, then

$$\begin{aligned} u''(t) &= -\left[\int_0^t \int_\tau^t y(s)dsd\tau + \int_0^t \int_t^\eta y(s)dsd\tau\right] \\ &= -\left[\int_0^t \int_0^s y(s)d\tau ds + \int_t^\eta \int_0^t y(s)d\tau ds\right] \\ &= -\left[\int_0^t sy(s)ds + \int_t^\eta ty(s)ds\right], \end{aligned}$$

so, in view of the boundary condition $u'(0) = 0$, we have

$$\begin{aligned} u'(t) &= -\left[\int_0^t \int_0^\tau sy(s)dsd\tau + \int_0^t \int_\tau^\eta \tau y(s)dsd\tau\right] \\ &= -\left[\int_0^t \int_0^\tau sy(s)dsd\tau + \int_0^t \int_\tau^t \tau y(s)dsd\tau + \int_0^t \int_t^\eta \tau y(s)dsd\tau\right] \\ &= -\left[\int_0^t \int_s^t sy(s)d\tau ds + \int_0^t \int_0^s \tau y(s)d\tau ds + \int_t^\eta \int_0^t \tau y(s)d\tau ds\right] \\ &= \frac{1}{2}\left[\int_0^t \left(s^2 - 2ts\right) y(s)ds - \int_t^\eta t^2 y(s)ds\right], \end{aligned}$$

and so, by the boundary condition $u(1) = 0$, we get

$$
\begin{aligned}
u(t) =& -\frac{1}{2}\left[\int_t^1\int_0^\tau \left(s^2-2\tau s\right)y(s)dsd\tau - \int_t^1\int_\tau^\eta \tau^2 y(s)dsd\tau\right]\\
=& -\frac{1}{2}\left[\int_t^1\int_0^t \left(s^2-2\tau s\right)y(s)dsd\tau + \int_t^1\int_t^\tau \left(s^2-2\tau s\right)y(s)dsd\tau\right.\\
&\left. - \int_t^\eta\int_\tau^\eta \tau^2 y(s)dsd\tau + \int_\eta^1\int_\eta^\tau \tau^2 y(s)dsd\tau\right]\\
=& -\frac{1}{2}\left[\int_0^t\int_t^1 \left(s^2-2\tau s\right)y(s)d\tau ds + \int_t^1\int_s^1 \left(s^2-2\tau s\right)y(s)d\tau ds\right.\\
&\left. - \int_t^\eta\int_t^s \tau^2 y(s)d\tau ds + \int_\eta^1\int_s^1 \tau^2 y(s)d\tau ds\right]\\
=& \frac{1}{6}\left[\int_0^t 3(1-t)(1+t-s)sy(s)ds + \int_t^\eta \left(3s-3s^2+s^3-t^3\right)y(s)ds - \int_\eta^1 (1-s)^3 y(s)ds\right]\\
=& \int_0^1 G(t,s)y(s)ds.
\end{aligned}
$$

Similarly, when $t \in (\eta, 1]$, we may obtain that

$$
u''(t) = -\int_0^\eta sy(s)ds + \int_\eta^t (t-s)y(s)ds,
$$

$$
u'(t) = \frac{1}{2}\left[\int_0^\eta \left(s^2-2ts\right)y(s)ds + \int_\eta^t (t-s)^2 y(s)ds\right]
$$

and

$$
\begin{aligned}
u(t) =& \frac{1}{6}\left\{\int_0^\eta 3(1-t)(1+t-s)sy(s)ds + \int_\eta^t \left[(t-s)^3-(1-s)^3\right]y(s)ds - \int_t^1 (1-s)^3 y(s)ds\right\}\\
=& \int_0^1 G(t,s)y(s)ds.
\end{aligned}
$$

□

Lemma 2.2. *Let $\eta \in (0,1)$. Then*

$$
\int_0^\eta G(t,s)ds + \int_\eta^1 G(t,s)ds \geq 0, \qquad t \in [0,1]
$$

if and only if $\eta \in \left[\frac{1}{3}, 1\right)$.

Proof. Since

$$
\int_0^\eta G(t,s)ds + \int_\eta^1 G(t,s)ds = \frac{1}{24}\left(t^4-4\eta t^3+4\eta-1\right), \qquad t \in [0,1],
$$

we only need to prove that $g(t) := t^4 - 4\eta t^3 + 4\eta - 1 \geq 0$ for $t \in [0,1]$ if and only if $\eta \in \left[\frac{1}{3}, 1\right)$.

First, if $\eta \in \left[\frac{1}{3}, 1\right)$, then we have

$$
\begin{aligned}
g(t) &= (1-t)\left[-\left(t^3+t^2+t+1\right)+4\eta\left(t^2+t+1\right)\right]\\
&\geq (1-t)\left[-\left(t^3+t^2+t+1\right)+\frac{4}{3}\left(t^2+t+1\right)\right]\\
&= \frac{1}{3}(1-t)^2\left(3t^2+2t+1\right)\\
&\geq 0, \qquad t \in [0,1].
\end{aligned}
$$

Next, we will show that if $g(t) \geq 0$ for $t \in [0,1]$, then $\eta \in \left[\frac{1}{3}, 1\right)$. Suppose on the contrary that $\eta \in \left(0, \frac{1}{3}\right)$. In view of $g'(t) = 4t^2(t - 3\eta) > 0$, $t \in (3\eta, 1]$, we know that

$$g(t) < g(1) = 0, \qquad t \in (3\eta, 1),$$

which is a contradiction. This indicates that if $g(t) \geq 0$ for $t \in [0,1]$, then $\eta \in \left[\frac{1}{3}, 1\right)$. □

In the remainder of this paper, we always assume that $\eta \in \left[\frac{1}{3}, 1\right)$.

Now, if we let

$$h(x) = -x^4 + 4x^3 - 6x^2 + 4\eta - 1, \qquad x \in [0,1],$$

then it is easy to know that $h(x)$ is strictly decreasing on $[0,1]$, which together with $h(0) > 0$ and $h(\eta) < 0$ implies that there exists a unique $x_0 \in (0, \eta)$ such that $h(x_0) = 0$. Obviously, x_0 is dependent on η. In fact, a direct calculation shows that

$$x_0 = \frac{2\alpha - \alpha^2 + \sqrt{8\alpha - \alpha^4}}{2\alpha},$$

where

$$\alpha = \sqrt{2\beta}, \qquad \beta = \sqrt[3]{1+\gamma} + \sqrt[3]{1-\gamma} \quad \text{and} \quad \gamma = \frac{\sqrt{81 - 192(1-\eta)^3}}{9}.$$

For example, if we choose $\eta = \frac{1}{3}$, then $x_0 \approx 0.2572437$.

From now on, we suppose that $\theta \in (0, x_0]$ is a constant.

Lemma 2.3. *$G(t,s)$ satisfies the following properties.*

(1) *$G(t,s) \geq 0$ for $(t,s) \in [0,1] \times [0,\eta]$ and $G(t,s) \leq 0$ for $(t,s) \in [0,1] \times (\eta, 1]$.*

(2) *$\int_\theta^\eta G(0,s)ds + \int_\eta^1 G(0,s)ds \geq 0$.*

Proof. Since (1) is obvious, we only prove (2). Noting that $h(x)$ is strictly decreasing on $[0,1]$ and $\theta \in (0, x_0]$, we have

$$\int_\theta^\eta G(0,s)ds + \int_\eta^1 G(0,s)ds = \frac{1}{24}\left(-\theta^4 + 4\theta^3 - 6\theta^2 + 4\eta - 1\right) = \frac{1}{24}h(\theta) \geq \frac{1}{24}h(x_0) = 0. \quad □$$

3 Main results

For convenience, we denote

$$\tilde{\theta} = 1 - \frac{\theta}{\eta}, \qquad P = \int_0^\eta G(0,s)ds \quad \text{and} \quad Q = \int_0^\theta G(0,s)ds.$$

Obviously, $0 < \tilde{\theta} < 1$ and $0 < Q < P$.

Theorem 3.1. *Suppose that $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and satisfies the following conditions:*

(A_1) *for any $x \in [0,+\infty)$, the mapping $t \mapsto f(t,x)$ is decreasing;*

(A_2) *for any $t \in [0,1]$, the mapping $x \mapsto f(t,x)$ is increasing;*

(A_3) *there exist three positive constants r_i, $i = 1,2,3$ with $r_1 < r_2 < r_3$ such that either*

(a)

$$f(0,r_i) < \frac{r_i}{P}, \qquad i=1,3$$

and

$$f(\theta,\tilde{\theta}r_2) > \frac{r_2}{Q}, \quad or$$

(b)

$$f(\theta,\tilde{\theta}r_i) > \frac{r_i}{Q}, \qquad i=1,3$$

and

$$f(0,r_2) < \frac{r_2}{P}.$$

Then the BVP (1.1) has at least two decreasing positive solutions u_1 and u_2 satisfying

$$r_1 < \|u_1\| < r_2 < \|u_2\| < r_3.$$

Proof. Let

$$K = \left\{ u \in X : u(t) \text{ is decreasing and nonnegative on } [0,1], \text{ and } \min_{t\in[0,\theta]} u(t) \geq \tilde{\theta}\,\|u\| \right\}.$$

Then it is easy to verify that K is a cone in X.

Now, we define an operator T on K by

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s))ds, \qquad u \in K,\ t \in [0,1].$$

Obviously, if u is a fixed point of the operator T, then u is a decreasing and nonnegative solution of the BVP (1.1).

First, we assert that $T : K \to K$. To see this, suppose $u \in K$. Then by (A_1), (A_2), Lemma 2.2 and (1) of Lemma 2.3, we get

$$\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,s) f(s,u(s))ds \\
&= \int_0^\eta G(t,s) f(s,u(s))ds + \int_\eta^1 G(t,s) f(s,u(s))ds \\
&\geq f(\eta,u(\eta)) \left[\int_0^\eta G(t,s)ds + \int_\eta^1 G(t,s)ds \right] \\
&\geq 0, \qquad t \in [0,1].
\end{aligned}$$

For $t \in [0,\eta]$, it is obvious that

$$(Tu)'(t) = \frac{1}{2}\left[\int_0^t \left(s^2 - 2ts\right) f(s,u(s))ds - \int_t^\eta t^2 f(s,u(s))ds \right] \leq 0,$$

and for $t \in (\eta,1]$, it follows from $u \in K$, (A_1), (A_2) and $\eta \in \left[\frac{1}{3},1\right)$ that

$$\begin{aligned}
(Tu)'(t) &= \frac{1}{2}\left[\int_0^\eta \left(s^2 - 2ts\right) f(s,u(s))ds + \int_\eta^t (t-s)^2 f(s,u(s))ds \right] \\
&\leq \frac{f(\eta,u(\eta))}{2} \left[\int_0^\eta \left(s^2 - 2ts\right) ds + \int_\eta^t (t-s)^2 ds \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{f(\eta, u(\eta))}{6} t^2 (t - 3\eta) \\
&\le \frac{f(\eta, u(\eta))}{6} t^2 (1 - 3\eta) \\
&\le 0.
\end{aligned}
$$

Thus, $(Tu)'(t) \le 0$ for all $t \in [0,1]$, which shows that $(Tu)(t)$ is decreasing on $[0,1]$. Since

$$
(Tu)''(t) = -\left[\int_0^t s f(s, u(s)) ds + \int_t^\eta t f(s, u(s)) ds\right] \le 0, \qquad t \in [0, \eta],
$$

we know that $(Tu)(t)$ is concave on $[0,\eta]$, which together with $0 < \theta \le x_0 < \eta$ and the fact that $(Tu)(t)$ is decreasing and nonnegative on $[0,1]$ indicates that

$$
\begin{aligned}
\min_{t\in[0,\theta]} (Tu)(t) &= (Tu)(\theta) \\
&= (Tu)\left[\left(1 - \frac{\theta}{\eta}\right) \cdot 0 + \frac{\theta}{\eta} \cdot \eta\right] \\
&\ge \left(1 - \frac{\theta}{\eta}\right) (Tu)(0) + \frac{\theta}{\eta} (Tu)(\eta) \\
&\ge \left(1 - \frac{\theta}{\eta}\right) (Tu)(0) \\
&= \tilde{\theta} \, \|Tu\| .
\end{aligned}
$$

This proves that $T : K \to K$.

Next, it follows from known textbook results, for example see Proposition 3.1 [11, p. 164], that $T : K \to K$ is compact.

Since the proof of the case when (b) of (A_3) is satisfied is similar, we only consider the case when (a) of (A_3) is fulfilled. Let

$$
\Omega_{r_i} = \{u \in K : \|u\| < r_i\}, \qquad i = 1, 2, 3.
$$

On the one hand, for any $u \in \partial\Omega_{r_i}$, $i = 1, 3$, we have

$$
0 \le u(s) \le \|u\| = r_i, \qquad s \in [0,1],
$$

which together with (1) of Lemma 2.3, (A_1), (A_2), (a) of (A_3) and the fact $T : K \to K$ implies that

$$
\begin{aligned}
\|Tu\| &= (Tu)(0) \\
&= \int_0^1 G(0,s) f(s, u(s)) ds \\
&= \int_0^\eta G(0,s) f(s, u(s)) ds + \int_\eta^1 G(0,s) f(s, u(s)) ds \\
&\le \int_0^\eta G(0,s) f(s, u(s)) ds \\
&\le f(0, r_i) \int_0^\eta G(0,s) ds \\
&< \frac{r_i}{P} \int_0^\eta G(0,s) ds \\
&= r_i = \|u\| .
\end{aligned}
$$

This indicates that $\|Tu\| < \|u\|$ for any $u \in \partial\Omega_{r_i}$, $i = 1,3$. Hence, by (2) of Theorem 1.1, we get

$$i_K(T, \Omega_{r_i}) = 1, \qquad i = 1,3. \tag{3.1}$$

On the other hand, for any $u \in \partial\Omega_{r_2}$, we have

$$u(\theta) = \min_{s\in[0,\theta]} u(s) \geq \tilde{\theta}\,\|u\| = \tilde{\theta} r_2. \tag{3.2}$$

Let $e(t) \equiv 1$ for $t \in [0,1]$. Then it is obvious that $e \in K \setminus \{0\}$. Now, we prove that $u \neq Tu + \lambda e$ for all $u \in \partial\Omega_{r_2}$ and all $\lambda \geq 0$. Suppose on the contrary that there exist $u^* \in \partial\Omega_{r_2}$ and $\lambda^* \geq 0$ such that $u^* = Tu^* + \lambda^* e$. Then it follows from $u^* \in K$, Lemma 2.3, (3.2), (A_1), (A_2) and (a) of (A_3) that

$$\begin{aligned}
r_2 &= \|u^*\| \\
&= u^*(0) \\
&= (Tu^*)(0) + \lambda^* \\
&= \int_0^1 G(0,s) f\big(s, u^*(s)\big) ds + \lambda^* \\
&= \int_0^\theta G(0,s) f\big(s, u^*(s)\big) ds + \int_\theta^\eta G(0,s) f\big(s, u^*(s)\big) ds + \int_\eta^1 G(0,s) f\big(s, u^*(s)\big) ds + \lambda^* \\
&\geq f\big(\theta, u^*(\theta)\big) \int_0^\theta G(0,s) ds + f\big(\eta, u^*(\eta)\big) \left[\int_\theta^\eta G(0,s) ds + \int_\eta^1 G(0,s) ds\right] + \lambda^* \\
&\geq f\big(\theta, u^*(\theta)\big) \int_0^\theta G(0,s) ds + \lambda^* \\
&\geq f\big(\theta, \tilde{\theta} r_2\big) \int_0^\theta G(0,s) ds + \lambda^* \\
&> \frac{r_2}{Q} \int_0^\theta G(0,s) ds + \lambda^* \\
&= r_2 + \lambda^*,
\end{aligned}$$

which is a contradiction. This shows that $u \neq Tu + \lambda e$ for all $u \in \partial\Omega_{r_2}$ and all $\lambda \geq 0$. Hence, an application of (1) of Theorem 1.1 yields that

$$i_K(T, \Omega_{r_2}) = 0. \tag{3.3}$$

Therefore, it follows from (3.1), (3.3) and (3) of Theorem 1.1 that T has fixed points u_1 and u_2 in K with $r_1 < \|u_1\| < r_2 < \|u_2\| < r_3$, which are two desired decreasing positive solutions of the BVP (1.1). □

Similarly, we can obtain the following more general result.

Corollary 3.2. *If (A_3) in Theorem 3.1 is replaced by the condition*

(A_3') *there exist n (≥ 2) positive constants r_i, $i = 1, 2, \ldots, n$ with $r_1 < r_2 < \cdots < r_n$ such that either*

(a')

$$f(0, r_{2i-1}) < \frac{r_{2i-1}}{P}, \qquad i = 1, \ldots, \left[\frac{n+1}{2}\right]$$

and

$$f(\theta,\tilde{\theta}r_{2i}) > \frac{r_{2i}}{Q}, \qquad i = 1,\dots,\left[\frac{n}{2}\right], \quad \textit{or}$$

(b')

$$f(\theta,\tilde{\theta}r_{2i-1}) > \frac{r_{2i-1}}{Q}, \qquad i = 1,\dots,\left[\frac{n+1}{2}\right]$$

and

$$f(0,r_{2i}) < \frac{r_{2i}}{P}, \qquad i = 1,\dots,\left[\frac{n}{2}\right].$$

Then the BVP (1.1) has at least $n-1$ decreasing positive solutions u_i satisfying

$$r_i < \|u_i\| < r_{i+1}, \qquad i = 1,\dots,n-1.$$

Example 3.3. Consider the following BVP

$$\begin{cases} u^{(4)}(t) = 10(1-t) + 2u^2(t), \ t \in [0,1], \\ u'(0) = u''(0) = u'''\left(\frac{1}{3}\right) = u(1) = 0. \end{cases} \tag{3.4}$$

Let $f(t,u) = 10(1-t) + 2u^2$, $(t,u) \in [0,1] \times [0,+\infty)$. In what follows, we verify that all the conditions of Theorem 3.1 are satisfied.

First, it is obvious that $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and (A_1) and (A_2) are fulfilled.

Next, we show that (b) of (A_3) holds, that is, there exist three positive constants r_i, $i = 1,2,3$ with $r_1 < r_2 < r_3$ such that

$$f(0,r_2) < \frac{r_2}{P} \tag{3.5}$$

and

$$f(\theta,\widetilde{\theta}r_i) > \frac{r_i}{Q}, \qquad i = 1,3. \tag{3.6}$$

On the one hand, since $\eta = \frac{1}{3}$, we get $P = \frac{43}{1944}$, which together with (3.5) implies that

$$r_2 \in (0.2234, 22.3812). \tag{3.7}$$

(Here and in the remainder of this paper, constants have been rounded to four decimal places unless they are exact.)

On the other hand, in view of $\widetilde{\theta} = 1-3\theta$, $Q = \frac{\theta^4-4\theta^3+6\theta^2}{24}$ and (3.6), we know that r_1 and r_3 are dependent on θ. In fact, a direct calculation shows that

$$r_1 \in \left(0, \frac{6-\sqrt{36-5(\theta^4-4\theta^3+6\theta^2)^2(1-3\theta)^2(1-\theta)}}{(\theta^4-4\theta^3+6\theta^2)(1-3\theta)^2}\right)$$

and

$$r_3 \in \left(\frac{6+\sqrt{36-5(\theta^4-4\theta^3+6\theta^2)^2(1-3\theta)^2(1-\theta)}}{(\theta^4-4\theta^3+6\theta^2)(1-3\theta)^2}, +\infty\right).$$

For some examples, one can see the following table:

θ	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$
r_1	$(0, 0.0989)$	$(0, 0.0699)$	$(0, 0.0517)$	$(0, 0.0397)$	$(0, 0.0314)$
r_3	$(606.7159, +\infty)$	$(357.7545, +\infty)$	$(322.2695, +\infty)$	$(330.4350, +\infty)$	$(356.4250, +\infty)$

This indicates that we could find three positive constants r_i, $i = 1, 2, 3$ such that (b) of (A_3) is satisfied.

Therefore, it follows from Theorem 3.1 that the BVP (3.4) has at least two decreasing positive solutions u_1 and u_2. Furthermore, in order to obtain better location of the two solutions, we select $\theta = \frac{1}{6}$. In view of the above table and (3.7), we can obtain that

$$0.05 < \|u_1\| < 0.23 \quad \text{and} \quad 22 < \|u_2\| < 323.$$

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