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Ground states for a class of asymptotically periodic Schrödinger–Poisson systems with critical growth

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Abstract. The purpose of this paper is to study the existence of ground state solution for the Schrödinger–Poisson systems:

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = Q(x)|u|^4 u + f(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $V(x)$, $K(x)$, $Q(x)$ and $f(x,u)$ are asymptotically periodic functions in x.

Keywords: Schrödinger–Poisson systems, ground state solution, variational methods.

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1 Introduction

For past decades, much attention has been paid to the nonlinear Schrödinger–Poisson system

$$\begin{cases} i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + U(x)\Psi + \phi(x)\Psi - |\Psi|^{q-1}\Psi, \ x \in \mathbb{R}^3, \ t \in \mathbb{R} \\ -\Delta\phi = |\Psi|^2, \ x \in \mathbb{R}^3 \end{cases} \tag{1.1}$$

where $\hbar$ is the Planck constant. Equation (1.1) derived from quantum mechanics. For this equation, the existence of stationary wave solutions is often sought, that is, the following form of solution

$$\Psi(x,t) = e^{it}u(x), \qquad x \in \mathbb{R}^3, \ t \in \mathbb{R}.$$

Therefore, the existence of the standing wave solution of the equation (1.1) is equivalent to finding the solution of the following system ($m = \frac{1}{2}, \hbar = 1$ and $V(x) = U(x) + 1$)

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{q-1}u, \ x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, \ x \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

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To the best of our knowledge, the first result on Schrödinger–Poisson system was obtained in [5]. Thereafter, using the variational method, there is a series of work to discuss the existence, non existence, radially symmetric solutions, non-radially symmetric solutions and ground state to Schrödinger–Poisson system (1.2) and similar problems [1,3–5,8–17,20,28,32, 34,37–39,42,44–47].

As far as we know, in [4], Azzollini and Pomponio firstly obtained the ground state solution to the Schrödinger–Poisson system (1.2). They obtained that system (1.2) has a ground state solution when V is a positive constant and $2 < q < 5$, or V is non-constant, possibly unbounded below and $3 < q < 5$. Since it's great physical interests, many scholars pay attention to study ground state solutions to the Schrödinger–Poisson system (1.2) and similar problems [1,8,11,12,14,15,20,37,38,45,46].

In [1], Alves, Souto and Soares studied Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u), \ x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, \ x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $f \in C(\mathbb{R}^+, \mathbb{R})$ and V is bounded, local Hölder continuous and satisfies:

(1) $V(x) \geq \alpha > 0, x \in \mathbb{R}^3$,

(2) $V(x) = V(x+y), \ \forall x \in \mathbb{R}^3, \ \forall y \in \mathbb{Z}^3$,

(3) $\lim_{|x|\to\infty} |V(x) - V_0(x)| = 0$,

(4) $V(x) \leq V_0(x), \ \forall x \in \mathbb{R}^3$, and there exists $\Omega \subset \mathbb{R}^3$ with $m(\Omega) > 0$ such that

$$V(x) < V_0(x), \qquad \forall x \in \Omega,$$

where V_0 satisfies (2). Alves et al. studied the ground state solutions to system (1.3) in case the periodic condition under (1)–(2) and in case the asymptotically periodic condition under (1), (3) and (4) respectively.

In [45], Zhang, Xu and Zhang considered existence of positive ground state solution for the following non-autonomous Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u), \ x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, \ x \in \mathbb{R}^3. \end{cases} \tag{1.4}$$

In some weaken asymptotically periodic sense compare with that of in [1], they obtained the positive ground state solution to system (1.4) when V, K and f are all asymptotically periodic in x.

More recently, Zhang, Xu, Zhang and Du [46] completed the results obtained in [45] to Schrödinger–Poisson system with critical growth

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = Q(x)|u|^4 u + f(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.5}$$

In [46], V, K, Q satisfy: $V, K, Q \in L^\infty(\mathbb{R}^3)$, $\inf_{\mathbb{R}^3} V > 0$, $\inf_{\mathbb{R}^3} K > 0$, $\inf_{\mathbb{R}^3} Q > 0$ and $V - V_p, K - K_p, Q - Q_p \in \mathcal{F}$, where V_p, K_p and Q_p are 1-periodic in $x_i, 1 \leq i \leq 3$, and $\mathcal{F} = \{g \in L^\infty(\mathbb{R}^3) : \forall \varepsilon > 0, \text{ the set } \{x \in \mathbb{R}^3 : |g(x)| \geq \varepsilon\} \text{ has finite Lebesgue measure}\}$.

On the other hand, when $K = 0$ the Schrödinger–Poisson system (1.4) becomes the standard Schrödinger equation (replace $\mathbb{R}^3$ with $\mathbb{R}^N$)

$$-\Delta u + V(x)u = f(x,u), \qquad x \in \mathbb{R}^N. \tag{1.6}$$

The Schrödinger equation (1.6) has been widely investigated by many authors in the last decades, see [2, 6, 19, 24, 25, 29–31, 40, 41, 43] and reference therein. Especially, in [19, 24, 25, 29, 40, 41], they studied the nontrivial solution or ground state solution for problem (1.6) with subcritical growth or critical growth in which V, f satisfy the asymptotically periodic condition. Other context about asymptotically periodic condition, we refer the reader to [18, 21, 35, 36] and reference therein.

Motivated by above results, in this paper, we will study ground state solutions to system (1.5) under reformative condition about asymptotically periodic case of V, K, Q and f at infinity.

To state our main results, we assume that:

(V) there exist $V_p : \mathbb{R}^3 \to \mathbb{R}$, 1-periodic in $x_i, 1 \le i \le 3$, such that

$$V_0 := \inf_{x\in\mathbb{R}^3} V_p > 0, \quad 0 \le V(x) \le V_p(x) \in L^\infty(\mathbb{R}^3) \quad \text{and} \quad V(x) - V_p(x) \in A_0,$$

where

$$A_0 := \{k(x) : \text{for any } \varepsilon > 0,\ m\{x \in B_1(y) : |k(x)| \ge \varepsilon\} \to 0 \text{ as } |y| \to \infty\};$$

(K) there exist $K_p : \mathbb{R}^3 \to \mathbb{R}$, 1-periodic in $x_i, 1 \le i \le 3$, such that

$$K_0 := \inf_{x\in\mathbb{R}^3} V_p > 0, \qquad 0 < K(x) \le K_p(x) \in L^\infty(\mathbb{R}^3) \quad \text{and} \quad K(x) - K_p(x) \in A_0;$$

(Q) there exist $Q_p \in C(\mathbb{R}^3, \mathbb{R})$, 1-periodic in $x_i, 1 \le i \le 3$, and point $x_0 \in \mathbb{R}^3$ such that

$$0 < Q_p(x) \le Q(x) \in C(\mathbb{R}^3, \mathbb{R}), \qquad Q(x) - Q_p(x) \in A_0$$

and

$$Q(x) = |Q|_\infty + O(|x - x_0|), \quad \text{as } x \to x_0;$$

and $f \in C(\mathbb{R}^3 \times \mathbb{R}^+, \mathbb{R})$ satisfies

(f_1) $\lim_{s\to 0^+} \frac{f(x,s)}{s} = 0$ uniformly for $x \in \mathbb{R}^3$,

(f_2) $\lim_{s\to +\infty} \frac{f(x,s)}{s^5} = 0$ uniformly for $x \in \mathbb{R}^3$,

(f_3) $s \to \frac{f(x,s)}{s^3}$ is nondecreasing on $(0, +\infty)$,

(f_4) there exists an open bounded set $\Omega \subset \mathbb{R}^3$, containing x_0 given by (Q), satisfies

$$\lim_{s\to+\infty} \frac{F(x,s)}{s^4} = +\infty \quad \text{uniformly for } x \in \Omega,$$

(f_5) there exists $f_p \in C(\mathbb{R}^3 \times \mathbb{R}^+, \mathbb{R}^+)$, 1-periodic in x_i, $1 \le i \le 3$, such that

(i) $f(x,s) \geq f_p(x,s)$ for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}^+$ and $f(x,s) - f_p(x,s) \in A$, where

$$A := \{h(x,s) : \text{for any } \varepsilon > 0,\ m\{x \in B_1(y) : |h(x,s)| \geq \varepsilon\} \to 0 \\ \text{as } |y| \to \infty \text{ uniformly for } |s| \text{ bounded }\},$$

(ii) $s \to \frac{f_p(x,s)}{s^3}$ is nondecreasing on $(0,+\infty)$.

The next theorem is the main result of the present paper.

Theorem 1.1. *Suppose that conditions* (V)*,* (K)*,* (Q) *and* (f_1)–(f_5) *are satisfied. Then the system* (1.5) *has a ground state solution.*

Remark 1.2.

(i) Functional sets A_0 in V, Q, K and A in (f_5) were introduced by [24, 25] in which Liu, Liao and Tang studied positive ground state solution to Schrödinger equation (1.6) with subcritical growth or critical growth.

(ii) Since $\mathcal{F} \subset A_0$, our assumptions on V, Q and K are weaker than [46]. Furthermore, $V(x) \geq 0$ in our paper but in [46] they assumed $V(x) > 0$.

(iii) In [46], to obtained the ground state to system (1.5), they firstly consider the periodic system

$$\begin{cases} -\Delta u + V_p(x)u + K_p(x)\phi u = Q_p(x)|u|^4 u + f_p(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K_p(x)u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.7}$$

Then a solution of system (1.5) was obtained by applying inequality between the energy of periodic system (1.7) and that of system (1.5). In this paper, we do not use methods of [46] and prove Theorem 1.1 directly.

2 The variational framework and preliminaries

To fix some notations, the letter C and C_i will be repeatedly used to denote various positive constants whose exact values are irrelevant. $B_R(z)$ denotes the ball centered at z with radius R. We denote the standard norm of L^p by $|u|_p = (\int_{\mathbb{R}^3} |u|^p dx)^{\frac{1}{p}}$ and $|u|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^3} |u|$. Since we are looking for a nonnegative solution, we may assume that $f(x,s) = f_p(x,s) = 0$ for all $(x,s) \in (\mathbb{R}^3, \mathbb{R}^-)$.

The Sobolev space $H^1(\mathbb{R}^3)$ endowed with the norm

$$\|u\|_H^2 := \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx.$$

The space $D^{1,2}(\mathbb{R}^3)$ endowed with the standard norm

$$\|u\|_{D^{1,2}}^2 := \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

Let $E := \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} V(x)u^2 dx < \infty\}$ be the Sobolev space endowed with the norm

$$\|u\|^2 := \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx.$$

Lemma 2.1 ([24]). *Suppose (V) holds. Then there exists two positive constants C_1 and C_2 such that $C_1\|u\|_H^2 \le \|u\| \le C_2\|u\|_H^2$ for all $u \in E$. Moreover, $E \hookrightarrow L^p(\mathbb{R}^3)$ for any $p \in [2,6]$ is continuous.*

The system (1.5) can be transformed into a Schrödinger equation with a nonlocal term. In fact, for all $u \in E$ (then $u \in H^1(\mathbb{R}^3)$), considering the linear functional L_u defined in $D^{1,2}(\mathbb{R}^3)$ by

$$L_u(v) = \int_{\mathbb{R}^3} K(x)u^2 v dx.$$

By the Hölder inequality, we have

$$|L_u(v)| \le |K|_\infty |u|^2_{\frac{12}{5}} |v|_6 \le C|u|^2_{\frac{12}{5}} \|v\|_{D^{1,2}}. \tag{2.1}$$

Therefor, the Lax–Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla v dx = (\phi_u, v)_{D^{1,2}} = L_u(v) = \int_{\mathbb{R}^3} K(x)u^2 v dx \quad \text{for any } v \in D^{1,2}(\mathbb{R}^3).$$

Namely, ϕ_u is the unique solution of $-\Delta\phi = K(x)u^2$. Moreover, ϕ_u can be expressed as

$$\phi_u = \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|} dy.$$

Substituting ϕ_u into the systems (1.5), we obtain

$$-\Delta u + V(x)u + K(x)\phi_u u = Q(x)|u|^4 u + f(x,u), \qquad x \in \mathbb{R}^3. \tag{2.2}$$

By (2.1), we get

$$\|\phi_u\|_{D^{1,2}} = \|L_u\| \le C|u|^2_{\frac{12}{5}} \le C\|u\|^2.$$

Then, we have

$$\begin{aligned} \left|\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx\right| &\le |K(x)|_\infty |\phi_u|_6 |u|^2_{\frac{12}{5}} \\ &\le C|K(x)|_\infty \|\phi_u\|_{D^{1,2}} |u|^2_{\frac{12}{5}} \\ &\le C|u|^4_{\frac{12}{5}} \\ &\le C_0\|u\|^4. \end{aligned} \tag{2.3}$$

So the energy functional $I : E \to \mathbb{R}$ corresponding to Eq. (2.2) is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} Q(x)(u^+)^6 dx - \int_{\mathbb{R}^3} F(x,u)dx,$$

where $F(x,s) = \int_0^s f(x,t)dt$.

Moreover, under our conditions, I belongs to C^1, so the Fréchet derivative of I is

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3}(\nabla u \cdot \nabla v + V(x)uv)dx + \int_{\mathbb{R}^3} K(x)\phi_u uv dx - \int_{\mathbb{R}^3} Q(x)(u^+)^5 v dx - \int_{\mathbb{R}^3} f(x,u)v dx$$

and $(u,\phi) \in E \times D^{1,2}(\mathbb{R}^3)$ is a solution of system (1.5) if and only if $u \in E$ is a critical point of I and $\phi = \phi_u$.

For all $u \in E$, let $\tilde{\phi}_u \in D^{1,2}(\mathbb{R}^3)$ is unique solution of the following equation

$$-\Delta\phi = K_p(x)u^2.$$

Moreover, $\widetilde{\phi}_u$ can be expressed as

$$\widetilde{\phi}_u = \int_{\mathbb{R}^3} \frac{K_p(y)u^2(y)}{|x-y|}dy.$$

Let

$$\begin{aligned} I_p(u) = &\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V_p(x)u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} K_p(x)\widetilde{\phi}_u u^2 dx \\ &- \frac{1}{6}\int_{\mathbb{R}^3} Q_p(x)(u^+)^6 dx - \int_{\mathbb{R}^3} F_p(x,u)dx, \end{aligned}$$

where $F_p(x,s) = \int_0^s f_p(x,t)dt$. Then I_p is the energy functional corresponding to the following equation

$$-\Delta u + V_p(x)u + K_p(x)\widetilde{\phi}_u u = Q_p(x)|u|^4 u + f_p(x,u), \qquad x \in \mathbb{R}^3. \tag{2.4}$$

It is easy to see that $(u,\phi) \in E \times D^{1,2}(\mathbb{R}^3)$ is a solution of periodic system (1.7) if and only if $u \in E$ is a critical point of I_p and $\phi = \widetilde{\phi}_u$.

Lemma 2.2. *Suppose* (K) *holds. Then,*

$$\int_{\mathbb{R}^3} K_p(x)\widetilde{\phi}_{u(\cdot+z)}u^2(\cdot+z)dx = \int_{\mathbb{R}^3} K_p(x)\widetilde{\phi}_u u^2 dx, \qquad \forall z \in \mathbb{Z}^3,\ u \in E.$$

Lemma 2.3. *Suppose that* (f_1), (f_3) *and* (f_5) *hold. Then*

(i) $\frac{1}{4}f(x,s)s \geq F(x,s) \geq 0$ *for all* $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$,

(ii) $\frac{1}{4}f_p(x,s)s \geq F_p(x,s) \geq 0$ *for all* $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$.

Proof. The proof is similar to that of in [27], so we omitted here. □

Lemma 2.4. *I' is weakly sequentially continuous. Namely if* $u_n \rightharpoonup u$ *in* E, $I'(u_n) \rightharpoonup I'(u)$ *in* $E^{-1}(\mathbb{R}^3)$.

Proof. The proof is similar to that of Lemma 2.3 in [45,46], so we omitted here. □

Lemma 2.5 ([24]). *Suppose that* (f_1), (f_2) *and (i) of* (f_5) *hold. Assume that* $\{u_n\}$ *is bounded in* E *and* $u_n \to 0$ *in* $L^s_{\mathrm{loc}}(\mathbb{R}^3)$, *for any* $s \in [2,6)$. *Then up to a subsequence, one has*

$$\int_{\mathbb{R}^3}(F(x,u_n) - F_p(x,u_n))dx = o(1). \tag{2.5}$$

Lemma 2.6 ([24,25]). *Suppose that* (V), (Q), (f_1), (f_2) *and* (i) *of* (f_5) *hold. Assume that* $\{u_n\}$ *is bounded in* E *and* $|z_n| \to \infty$. *Then up to a subsequence, one has*

$$\int_{\mathbb{R}^3}(V_p(x) - V(x))u_n\varphi(\cdot - z_n)dx = o(1), \tag{2.6}$$

$$\int_{\mathbb{R}^3}(f(x,u_n) - f_p(x,u_n))\varphi(\cdot - z_n)dx = o(1), \tag{2.7}$$

and

$$\int_{\mathbb{R}^3}(Q(x) - Q_p(x))(u_n^+)^5\varphi(\cdot - z_n)dx = o(1), \tag{2.8}$$

where $\varphi \in C_0^\infty(\mathbb{R}^3)$.

Lemma 2.7. *Suppose that (K), (f_1) and (f_2) hold. Assume that $u_n \rightharpoonup 0$ in E. Then up to a subsequence, one has*

$$\int_{\mathbb{R}^3} (K(x)\phi_{u_n} u_n \varphi(\cdot - z_n) - K_p(x)\widetilde{\phi}_{u_n} u_n \varphi(\cdot - z_n))dx = o(1), \tag{2.9}$$

where $|z_n| \to \infty$ and $\varphi \in C_0^\infty(\mathbb{R}^3)$.

Proof. Set $h(x) := K(x) - K_p(x)$. By (K), we have $h(x) \in A_0$. Then for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$m\{x \in B_1(y) : |h(x)| \geq \varepsilon\} < \varepsilon, \quad \text{for any } |y| \geq R_\varepsilon.$$

We cover $\mathbb{R}^3$ by balls $B_1(y_i)$, $i \in \mathbb{N}$. In such a way that each point of $\mathbb{R}^3$ is contained in at most $N+1$ balls. Without any loss of generality, we suppose that $|y_i| < R_\varepsilon$, $i = 1, 2, \ldots, n_\varepsilon$ and $|y_i| \geq R_\varepsilon$, $i = n_\varepsilon + 1, n_\varepsilon + 2, n_\varepsilon + 3, \ldots, +\infty$. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^3} (K(x)\phi_{u_n} u_n \varphi(\cdot - z_n) - K_p(x)\widetilde{\phi}_{u_n} u_n \varphi(\cdot - z_n))dx \\
&\quad = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K_p(y)u_n(y)\varphi(y-z_n)}{|x-y|} dy h(x) u_n^2(x) dx \\
&\qquad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K_p(y)u_n^2(y)}{|x-y|} dy h(x) u_n(x)\varphi(x-z_n) dx \\
&\qquad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{h(y)u_n^2(y)}{|x-y|} dy h(x) u_n(x)\varphi(x-z_n) dx \\
&\quad =: E_1 + E_2 + E_3.
\end{aligned}$$

Like the argument of [45], we define

$$\begin{aligned}
H(x) &:= \int_{\mathbb{R}^3} \frac{K_p(y)u_n(y)\varphi(y-z_n)}{|x-y|} dy \\
&= \int_{\{y:|x-y|\leq 1\}} \frac{K_p(y)u_n(y)\varphi(y-z_n)}{|x-y|} dy + \int_{\{y:|x-y|>1\}} \frac{K_p(y)u_n(y)\varphi(y-z_n)}{|x-y|} dy.
\end{aligned}$$

By the Hölder inequality and the Sobolev embeddings, we have

$$\begin{aligned}
|H(x)| &\leq |K_p|_\infty |u_n|_3 |\varphi|_6 \left(\int_{\{y:|x-y|\leq 1\}} \frac{1}{|x-y|^2} dy\right)^{\frac{1}{2}} + |K_p|_\infty |u_n|_2 |\varphi|_4 \left(\int_{\{y:|x-y|>1\}} \frac{1}{|x-y|^4} dy\right)^{\frac{1}{4}} \\
&\leq C\left(\int_{\{z:|z|\leq 1\}} \frac{1}{|z|^2} dz\right)^{\frac{1}{2}} + C\left(\int_{\{z:|z|>1\}} \frac{1}{|z|^4} dz\right)^{\frac{1}{4}}.
\end{aligned}$$

So, $\sup_{x\in\mathbb{R}^3} |H(x)| < \infty$. Then, we obtain

$$\begin{aligned}
E_1 &= \int_{\mathbb{R}^3} H(x)h(x)u_n^2(x)dx \\
&\leq \int_{\{x:|h(x)|\geq\varepsilon\}} |H(x)h(x)u_n^2(x)|dx + \int_{\{x:|h(x)|<\varepsilon\}} |H(x)h(x)u_n^2(x)|dx \\
&=: Q_1 + Q_2,
\end{aligned}$$

$$\begin{aligned}
Q_1 &= \int_{\{x:|h(x)|\geq\varepsilon\}} |H(x)h(x)u_n^2(x)|dx \\
&\leq \int_{\{x:|h(x)|\geq\varepsilon,|x|>R_\varepsilon+1\}} |H(x)h(x)u_n^2(x)|dx + \int_{\{x:|h(x)|\geq\varepsilon,|x|\leq R_\varepsilon+1\}} |H(x)h(x)u_n^2(x)|dx \\
&\leq \sum_{n_\varepsilon+1}^{\infty} \int_{\{x\in B_1(y_i):|h(x)|\geq\varepsilon,|x|>R_\varepsilon+1\}} |H(x)h(x)u_n^2(x)|dx + 2\sup_{x\in\mathbb{R}^3}|H(x)||K_p|_\infty \int_{B_{R_\varepsilon+1}} |u_n(x)|^2 dx \\
&=: Q_{11} + Q_{12},
\end{aligned}$$

$$\begin{aligned}
Q_{11} &= \sum_{n_\varepsilon+1}^{\infty} \int_{\{x\in B_1(y_i):|h(x)|\geq\varepsilon,|x|>R_\varepsilon+1\}} |H(x)h(x)u_n^2(x)|dx \\
&\leq 2\sup_{x\in\mathbb{R}^3}|H(x)||K_p|_\infty \sum_{n_\varepsilon+1}^{\infty} \int_{\{x\in B_1(y_i):|h(x)|\geq\varepsilon,|x|>R_\varepsilon+1\}} |u_n^2(x)|dx \\
&\leq C\sum_{n_\varepsilon+1}^{\infty} (m\{x\in B_1(y):|h(x)|\geq\varepsilon\})^{\frac{2}{3}} \left(\int_{\{x\in B_1(y_i):|h(x)|\geq\varepsilon,|x|>R_\varepsilon+1\}} |u_n^6(x)|dx\right)^{\frac{1}{3}} \\
&\leq C_1\varepsilon^{\frac{2}{3}} \sum_{n_\varepsilon+1}^{\infty} \int_{\{x\in B_1(y_i):|h(x)|\geq\varepsilon,|x|>R_\varepsilon+1\}} (|\nabla u_n|^2 + u_n^2)dx \\
&\leq C_1(N+1)\varepsilon^{\frac{2}{3}} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2)dx \\
&\leq C_2\varepsilon^{\frac{2}{3}}.
\end{aligned}$$

Let $\varepsilon \to 0$, we obtain $Q_{11} \to 0$. By the condition $u_n \rightharpoonup 0$, one has $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^3)$. Therefore $Q_{12} \to 0$. So $Q_1 \to 0$.

$$\begin{aligned}
Q_2 &= \int_{\{x:|h(x)|<\varepsilon\}} |H(x)h(x)u_n^2(x)|dx \\
&\leq \varepsilon \sup_{x\in\mathbb{R}^3}|H(x)| \int_{\mathbb{R}^3} |u_n^2(x)|dx \\
&\leq C\varepsilon.
\end{aligned}$$

Let $\varepsilon \to 0$, we have $Q_2 \to 0$. Then, we get $E_1 \to 0$. In the same way, we can prove $E_2 \to 0$ and $E_3 \to 0$. □

Let $F = \{u \in E : u^+ \neq 0\}$, define

$$\mathcal{N} := \{u \in E \setminus \{0\} : \langle I'(u), u\rangle = 0\} = \{u \in F : \langle I'(u), u\rangle = 0\}.$$

Then $\mathcal{N}$ is a Nehari type associate to I, and set $c := \inf_{u\in\mathcal{N}} I$.

Lemma 2.8. *Suppose that (V), (K), (Q) and (f_1)–(f_3) hold. For any $u \in F$, there is a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$. Moreover, the maximum of $I(tu)$ for $t \geq 0$ is achieved.*

Proof. Fix $u \in F$, define $g(t) := I(tu)$, $t > 0$. Using (f_1), (f_2), and (f_3), we can prove that $g(0) = 0$, $g(t) > 0$ for t small and $g(t) < 0$ for t large.

In fact, by (f_1) and (f_2), $\forall\delta > 0$ there exists a $C_\delta > 0$ such that

$$|f(x,s)| \leq \delta|s| + C_\delta|s|^5, \qquad |F(x,s)| \leq \frac{\delta}{2}|s|^2 + \frac{C_\delta}{6}|s|^6 \quad \text{for any } (x,s) \in (\mathbb{R}^3, \mathbb{R}).$$

So, we get that

$$
\begin{aligned}
g(t) &= \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \frac{t^6}{6}\int_{\mathbb{R}^3} Q(x)(u^+)^6 dx - \int_{\mathbb{R}^3} F(x,tu)dx \\
&\geq \frac{t^2}{2}\|u\|^2 - \frac{\delta t^2}{2}\int_{\mathbb{R}^3} |u|^2 dx - \frac{C_\delta t^6}{6}\int_{\mathbb{R}^3} |u|^6 dx - Ct^6\int_{\mathbb{R}^3} |u|^6 dx \\
&\geq \frac{t^2}{2}\|u\|^2 - C\delta t^2\|u\|^2 - CC_\delta t^6\|u\|^6.
\end{aligned}
$$

Hence, $g(t) > 0$ for t small.

On the other hand, let $\Theta = \{x \in \mathbb{R}^3 : u(x) > 0\}$, we have that

$$
\begin{aligned}
g(t) &= \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \frac{t^6}{6}\int_{\mathbb{R}^3} Q(x)(u^+)^6 dx - \int_{\mathbb{R}^3} F(x,tu)dx \\
&\leq \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \frac{t^6}{6}\int_{\Theta} Q(x)(u^+)^6 dx.
\end{aligned}
$$

Hence, it is easy to see that $g(t) \to -\infty$ as $t \to +\infty$.

Therefore, there exists a t_u such that $I(t_u u) = \max_{t>0} I(tu)$ and $t_u u \in \mathcal{N}$. Suppose that there exist $t_1 > t_2 > 0$ such that $t_1 u,\ t_2 u \in \mathcal{N}$. Then, we have that

$$
\frac{1}{t_1^2}\|u\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx = t_1^2\int_\Theta Q(x)(u^+)^6 dx + \int_\Theta \frac{f(x,t_1 u)u}{t_1^3}dx,
$$

$$
\frac{1}{t_2^2}\|u\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx = t_2^2\int_\Theta Q(x)(u^+)^6 dx + \int_\Theta \frac{f(x,t_2 u)u}{t_2^3}dx.
$$

Therefore, one has that

$$
\left(\frac{1}{t_1^2} - \frac{1}{t_2^2}\right)\|u\|^2 = (t_1^2 - t_2^2)\int_\Theta Q(x)(u^+)^6 dx + \int_\Theta \left(\frac{f(x,t_1 u)}{(t_1 u)^3} - \frac{f(x,t_2 u)}{(t_2 u)^3}\right)u^4 dx,
$$

which is absurd according to (f_3) and $t_1 > t_2 > 0$. □

Remark 2.9. As in [31,43], we have

$$
c = \inf_{u\in\mathcal{N}} I(u) = \inf_{u\in F}\max_{t>0} I(tu) = \inf_{\gamma(t)\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) > 0
$$

where

$$
\Gamma := \{\gamma \in C([0,1],E) : \gamma(0) = 0,\ I(\gamma(1)) < 0\}.
$$

Lemma 2.10. *Suppose that (V), (K), (Q) and (f_1)–(f_3) hold. Then there exists a bounded sequence $\{u_n\} \in E$ such that*

$$
I(u_n) \to c \quad \text{and} \quad \|I'(u_n)\|_{E^{-1}} \to 0.
$$

Proof. From the proof of Lemma 2.8, it is easy to see that I satisfies the mountain pass geometry. By [33], there exists an $\{u_n\}$ such that $I(u_n) \to c$ and $(1 + \|u_n\|)\|I'(u_n)\|_{E^{-1}} \to 0$, so we have $\langle I'(u_n), u_n\rangle = o(1)$. By (f_3), we can obtain

$$
\frac{1}{4}f(x,s)s \geq F(x,s) \quad \text{for any } (x,s) \in (\mathbb{R}^3, \mathbb{R}).
$$

Then, we have that

$$\begin{aligned}c &= I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2 + \frac{1}{12}\int_{\mathbb{R}^3} Q(x)(u_n^+)^6 dx + \int_{\mathbb{R}^3}(\frac{1}{4}f(x,u_n)u_n - F(x,u_n))dx \\ &\geq \frac{1}{4}\|u_n\|^2.\end{aligned}$$

Therefor, $\{u_n\}$ is bounded and the proof is finished. □

The proof of next lemma similar to that of [24,26]. For easy reading, we give the proof.

Lemma 2.11. *Suppose that* (V), (K), (Q) *and* (f_1)–(f_3) *hold. If* $u \in \mathcal{N}$ *and* $I(u) = c$, *u is a solution of Eq.* (2.2).

Proof. Suppose by contradiction u is not a solution. Then there exists $\varphi \in E$ such that

$$\langle I'(u), \varphi\rangle < -1.$$

Choose $\varepsilon \in (0,1)$ small enough such that for all $|t-1| \leq 1$ and $|\sigma| \leq \varepsilon$,

$$\langle I'(tu + \sigma\varphi), \varphi\rangle \leq -\frac{1}{2}.$$

We define a smooth cut-off function $\zeta(t) \in [0,1]$, which satisfies $\zeta(t) = 1$ for$|t-1| \leq \frac{\varepsilon}{2}$ and $\zeta(t) = 0$ for $|t-1| \geq \varepsilon$. For $t > 0$ we introduce a curve $\gamma(t) = tu$ for $|t-1| \geq \varepsilon$ and $\gamma(t) = tu + \varepsilon\zeta(t)\varphi$ for $|t-1| < \varepsilon$. Obviously, $\gamma(t)$ is a continuous curve and when ε small enough, $\|\gamma(t)\| > 0$ for $|t-1| < \varepsilon$. Next we prove $I(\gamma(t)) < c$, for $t > 0$. If $|t-1| \geq \varepsilon$, $I(\gamma(t)) = I(tu) < I(u) = c$. If $|t-1| < \varepsilon$, we define $A : \sigma \mapsto I(tu + \sigma\zeta(t)\varphi)$. Obviously, $A \in C^1$. By the mean value therm, there exists $\overline{\sigma} \in (0, \varepsilon)$ such that

$$I(tu + \varepsilon\zeta(t)\varphi) = I(tu) + \langle I'(tu + \overline{\sigma}\zeta(t)\varphi), \varepsilon\zeta(t)\varphi\rangle \leq I(tu) - \frac{\varepsilon}{2}\zeta(t) < c.$$

Define $\nu(u) := \langle I'(u), u\rangle$, then $\nu(\gamma(1-\varepsilon)) = \nu((1-\varepsilon)u) > 0$ and $\nu(\gamma(1+\varepsilon)) = \nu((1+\varepsilon)u) < 0$. By the continuity of $t \to \nu(\gamma(t))$, there exists $t' \in (1-\varepsilon, 1+\varepsilon)$ such that $\nu(\gamma(t')) = 0$. Thus $\gamma(t') \in \mathcal{N}$ and $I(\gamma(t')) < c$, which is a contradiction. □

Define

$$\mathcal{N}_p = \{u \in F : \langle I_p'(u), u\rangle = 0\} \quad \text{and} \quad c_p = \inf_{u \in \mathcal{N}_p} I_p(u).$$

In fact, $c_p = \inf_{u \in F} \max_{t>0} I_p(tu)$.

Remark 2.12. For any $u \in F$, by Lemma 2.8, there exists $t_u > 0$ such that $t_u u \in \mathcal{N}$ and then $I(t_u u) \geq c$. Using $V(x) \leq V_p(x)$, $Q(x) \geq Q_p(x)$ and $F(x,s) \geq F_p(x,s)$, we have $c \leq I(t_u u) \leq I_p(t_u u) \leq \max_{t>0} I_p(tu)$. Then we obtain $c \leq c_p$.

3 Estimates

In this section, we will estimate the least energy c, and the method comes from the celebrated paper [7].

Let

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{|\nabla u|_2^2}{|u|_6^2}.$$

In fact, S is the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$.

Without loss of generality, we assume that $x_0 = 0$. For $\varepsilon > 0$, the function $w_\varepsilon : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$w_\varepsilon(x) = \frac{3^{\frac{1}{4}}\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x|^2)^{\frac{1}{2}}}$$

is a family of functions on which S is attained. Let $\varphi \in C_0^\infty(\mathbb{R}^3, [0,1])$ be a cut-off function satisfying $\varphi = 1$, for $x \in B_{\frac{\rho}{2}}$ and $\varphi = 0$, for $x \in \mathbb{R}^3 \setminus B_\rho$, where $B_\rho \subset \Omega$. Define the test function by

$$v_\varepsilon = \frac{u_\varepsilon}{(\int_{\mathbb{R}^3} Q(x)u_\varepsilon^6 dx)^{\frac{1}{6}}},$$

where $u_\varepsilon = \varphi w_\varepsilon$. Then one has

$$\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx \le |Q|_\infty^{-\frac{1}{3}} S + O(\varepsilon^{\frac{1}{2}}), \quad \text{as } \varepsilon \to 0^+, \tag{3.1}$$

$$\int_{\mathbb{R}^3} |v_\varepsilon|^2 dx = O(\varepsilon^{\frac{1}{2}}), \quad \text{as } \varepsilon \to 0^+, \tag{3.2}$$

$$\int_{\mathbb{R}^3} |u_\varepsilon|^6 dx = K_1 + O(\varepsilon^{\frac{3}{2}}), \quad \text{as } \varepsilon \to 0^+, \text{ where } K_1 \text{ is some positive constant}, \tag{3.3}$$

$$\int_{\mathbb{R}^3} Q(x)v_\varepsilon^6 dx = 1, \tag{3.4}$$

$$\int_{\mathbb{R}^3} |v_\varepsilon|^{\frac{12}{5}} dx = O(\varepsilon^{\frac{3}{5}}), \quad \text{as } \varepsilon \to 0^+. \tag{3.5}$$

Lemma 3.1. *Suppose (V), (K), (Q) and (f_1)–(f_4) are satisfied. Then $c < \frac{1}{3}|Q|_\infty^{-\frac{1}{2}} S^{\frac{3}{2}}$.*

Proof. For $t > 0$, define

$$\begin{aligned} g(t) &:= I(tv_\varepsilon) \\ &= \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx + \frac{t^2}{2}\int_{\mathbb{R}^3} V(x)v_\varepsilon^2 dx + \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{v_\varepsilon} v_\varepsilon^2 dx \\ &\quad - \frac{t^6}{6}\int_{\mathbb{R}^3} Q(x)v_\varepsilon^6 dx - \int_{\mathbb{R}^3} F(x, tv_\varepsilon)dx. \end{aligned}$$

By Lemma 2.8, there exists a unique $t_\varepsilon > 0$ such that $g(t_\varepsilon) = \max_{t>0} g(t)$ and $g'(t_\varepsilon) = 0$. We claim that there exists C_1, C_2 such that $C_1 \le t_\varepsilon \le C_2$ for ε small enough. Indeed, if $t_\varepsilon \to 0$ as $\varepsilon \to 0$, one has $g(t_\varepsilon) \to 0$, which is a contradiction. If $t_\varepsilon \to +\infty$ as $\varepsilon \to +\infty$, one has $g(t_\varepsilon) \to -\infty$, which is a contradiction. Thus the claim holds. For $s > 0$, define

$$\psi(s) := \frac{s^2}{2}\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx - \frac{s^6}{6}.$$

Then there exists $s_\varepsilon := \left(\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx\right)^{\frac{1}{4}}$ such that

$$\psi(s_\varepsilon) = \max_{s>0} \psi(s) = \frac{1}{3}\left(\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx\right)^{\frac{3}{2}}.$$

By (3.1) and the inequality $(a+b)^\alpha \leq a^\alpha + \alpha(a+b)^{\alpha-1}b, a > 0, b > 0, \alpha \geq 1$, we have

$$\psi(s_\varepsilon) \leq \frac{1}{3}|Q|_\infty^{-\frac{1}{2}} S^{\frac{3}{2}} + O(\varepsilon^{\frac{1}{2}}). \tag{3.6}$$

We claim

$$\lim_{\varepsilon\to 0^+} \frac{\int_{\mathbb{R}^3} F(x, t_\varepsilon v_\varepsilon)dx}{O(\varepsilon^{\frac{1}{2}})} = +\infty. \tag{3.7}$$

By (3.3), for ε small enough, one has $|u_\varepsilon|_6 \leq 2K_1$ and then for $|x| < \varepsilon^{\frac{1}{2}} < \frac{\rho}{2}$,

$$t_\varepsilon v_\varepsilon \geq \frac{C_1}{2|Q|_\infty K_1} u_\varepsilon = \frac{C_1}{2|Q|_\infty K_1} w_\varepsilon = \frac{C_1}{2|Q|_\infty K_1} \frac{3^{\frac{1}{4}}\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x|^2)^{\frac{1}{2}}} \geq C\varepsilon^{-\frac{1}{4}}.$$

It follows from (f_4) that for any $R > 0$, there exists $A_R > 0$ such that for all $(x, s) \in \Omega \times [A_R, +\infty)$,

$$F(x, s) \geq Rs^4.$$

Thus for ε small enough, one has

$$\int_{\{x:|x|<\varepsilon^{\frac{1}{2}}\}} F(x, t_\varepsilon v_\varepsilon)dx \geq CR \int_{\{x:|x|<\varepsilon^{\frac{1}{2}}\}} \varepsilon^{-1}dx = CR\varepsilon^{\frac{1}{2}}.$$

Combining with $F(x, s) \geq 0$ and the arbitrariness of R, we can obtain the claim. By (2.3) and (3.5), we get

$$\left|\int_{\mathbb{R}^3} K(x)\phi_{v_\varepsilon} v_\varepsilon^2 dx\right| \leq C_0 |v_\varepsilon|^4_{\frac{12}{5}} \leq C_2\varepsilon.$$

Hence for ε small enough, by (3.2), (3.6) and (3.7), we have

$$\begin{aligned}
c &\leq \max_{t>0} I(tv_\varepsilon) \\
&= \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3} V(x)v_\varepsilon^2 dx \\
&\quad + \frac{t_\varepsilon^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{v_\varepsilon} v_\varepsilon^2 dx - \frac{t_\varepsilon^6}{6}\int_{\mathbb{R}^3} Q(x)v_\varepsilon^6 dx - \int_{\mathbb{R}^3} F(x, t_\varepsilon v_\varepsilon)dx \\
&\leq \frac{1}{3}|Q|_\infty^{-\frac{1}{2}} S^{\frac{3}{2}} + O(\varepsilon) + O(\varepsilon^{\frac{1}{2}}) - \int_{\mathbb{R}^3} F(x, t_\varepsilon v_\varepsilon)dx \\
&\leq \frac{1}{3}|Q|_\infty^{-\frac{1}{2}} S^{\frac{3}{2}} + O(\varepsilon^{\frac{1}{2}}) - \int_{\mathbb{R}^3} F(x, t_\varepsilon v_\varepsilon)dx \\
&< \frac{1}{3}|Q|_\infty^{-\frac{1}{2}} S^{\frac{3}{2}}.
\end{aligned}$$

□

4 The proof of main result

The proof of Theorem 1.1. From Lemma 2.10, there exists a bounded sequence $\{u_n\} \in E$ satisfying $I(u_n) \to c$ and $\|I'(u_n)\|_{E^{-1}} \to 0$. Then there exists $u \in E$ such that, up to a subsequence,

$u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$. By Lemma 2.4, for any $v \in E$, we have

$$0 = \langle I'(u_n), v\rangle + o(1) = \langle I'(u), v\rangle,$$

that is u is a solution of Eq. (2.2). Since

$$0 = \langle I'(u), u^-\rangle = \|u^-\| + \int_{\mathbb{R}^3} K(x)\phi_u |u^-|^2 dx \geq \|u^-\|,$$

then $u \geq 0$.

We next distinguish the following two case to prove Eq. (2.2) has a nonnegative ground state solution.

Case 1. Suppose that $u \neq 0$. Then $I(u) \geq c$. By the Fatou lemma, we obtain

$$\begin{aligned} c &= \liminf_{n\to\infty} \left(I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \right) \\ &= \liminf_{n\to\infty} \left(\frac{1}{4}\|u_n\|^2 + \frac{1}{12}\int_{\mathbb{R}^3} Q(x)(u_n^+)^6 dx + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(x,u_n)u_n - F(x,u_n) \right) dx \right) \\ &\geq \frac{1}{4}\|u\|^2 + \frac{1}{12}\int_{\mathbb{R}^3} Q(x)(u^+)^6 dx + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(x,u)u - F(x,u) \right) dx \\ &= I(u) - \frac{1}{4}\langle I'(u), u\rangle \\ &= I(u). \end{aligned}$$

Therefore, $I(u) = c$ and $I'(u) = 0$.

Case 2. Suppose that $u = 0$. Define

$$\beta := \limsup_{n\to\infty} \sup_{z\in\mathbb{R}^3} \int_{B_1(z)} u_n^2 dx.$$

If $\beta = 0$, by using the Lions lemma [22,23], we have $u_n \to 0$ in $L^q(\mathbb{R}^3)$ for all $q \in (2,6)$. By the condition of (f_1) and (f_2), $\forall \delta > 0$ there exists a $C_\delta > 0$ such that $f(x,u)u \leq \delta(|u|^2 + |u|^6) + C_\delta |u|^\alpha$ and $F(x,u) \leq \frac{\delta}{2}|u|^2 + \frac{\delta}{6}|u|^6 + C_\delta|u|^\alpha$ for any $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$ and $\alpha \in (2,6)$. So

$$\int_{\mathbb{R}^3} f(x,u_n)u_n dx \to 0, \qquad \int_{\mathbb{R}^3} F(x,u_n) dx \to 0.$$

Then

$$c = \frac{1}{2}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} Q(x)(u_n^+)^6 dx + o_n(1), \tag{4.1}$$

$$\|u_n\|^2 + \int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} Q(x)(u_n^+)^6 dx + o_n(1). \tag{4.2}$$

By (4.2), we have

$$\|u_n\|^2 \leq |Q|_\infty |u_n|_6^6 + o_n(1) \leq |Q|_\infty S^{-3}\|u_n\|^6 + o_n(1), \tag{4.3}$$

which deduces that (i) $\|u_n\| \to 0$ or (ii) $\|u_n\| \geq |Q|_\infty^{-\frac{1}{4}} S^{\frac{3}{4}} + o_n(1)$.

If (i) holds, by (2.3), one has $\int_{\mathbb{R}^3} K(x)\phi_{u_n} u_n^2 dx \to 0$. It follows from (4.1) and (4.2) that $c = 0$, which is a contradiction with $c > 0$.

If (ii) holds, by (4.2) we have

$$\int_{\mathbb{R}^3} Q(x)(u_n^+)^6 dx \geq |Q|_\infty^{-\frac{1}{2}} S^{\frac{3}{2}} + o_n(1). \tag{4.4}$$

From (4.1) and (4.2) we easily conclude that

$$c = \frac{1}{4}\|u_n\|^2 + \frac{1}{12}\int_{\mathbb{R}^3} Q(x)(u_n^+)^6 dx + o_n(1).$$

Then from (4.4) it follows that $c \geq \frac{1}{3}|Q|_\infty^{-\frac{1}{2}} S^{\frac{3}{2}}$, contradicting the fact that $c < \frac{1}{3}|Q|_\infty^{-\frac{1}{2}} S^{\frac{3}{2}}$. Thus $\beta > 0$. Up to a subsequence, there exist $R > 0$ and $\{z_n\} \subset \mathbb{Z}^3$ such that

$$\int_{B_R} u_n(x+z_n)^2 dx = \int_{B_R(z_n)} u_n^2 dx > \frac{\beta}{2}.$$

Define $w_n := u_n(x+z_n)$. Thus there exists $w \in E$ satisfying, up to a subsequence, $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^2_{\text{loc}}(\mathbb{R}^3)$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^3$. Obviously, $w \neq 0$. If $\{z_n\}$ is bounded, there exists R' such that

$$\int_{B_{R'}} u_n^2 dx \geq \int_{B_R(z_n)} u_n^2 dx \geq \frac{\beta}{2},$$

which contradicts with $u_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^3)$. Thus $\{z_n\}$ is unbounded. Up to a subsequence, we have $z_n \to \infty$. By Lemma 2.6 and Lemma 2.7, then

$$\begin{aligned}
0 &= \langle I'(u_n, \varphi(x-z_n))\rangle + o(1) \\
&= \int_{\mathbb{R}^3} (\nabla u_n \cdot \nabla\varphi(x-z_n) + V(x)u_n\varphi(x-z_n))dx + \int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n\varphi(x-z_n)dx \\
&\quad - \int_{\mathbb{R}^3} Q(x)(u_n^+)^5\varphi(x-z_n)dx - \int_{\mathbb{R}^3} f(x,u_n)\varphi(x-z_n)dx + o(1) \\
&= \int_{\mathbb{R}^3} (\nabla u_n \cdot \nabla\varphi(x-z_n) + V_p(x)u_n\varphi(x-z_n))dx + \int_{\mathbb{R}^3} K_p(x)\widetilde{\phi}_{u_n}u_n\varphi(x-z_n)dx \\
&\quad - \int_{\mathbb{R}^3} Q_p(x)(u_n^+)^5\varphi(x-z_n)dx - \int_{\mathbb{R}^3} f_p(x,u_n)\varphi(x-z_n)dx + o(1) \\
&= \int_{\mathbb{R}^3} (\nabla w_n \cdot \nabla\varphi + V_p(x)w_n\varphi)dx + \int_{\mathbb{R}^3} K_p(x)\widetilde{\phi}_{w_n}w_n\varphi dx \\
&\quad - \int_{\mathbb{R}^3} Q_p(x)(w_n^+)^5\varphi dx - \int_{\mathbb{R}^3} f_p(x,w_n)\varphi dx + o(1) \\
&= \langle I_p'(w), \varphi\rangle,
\end{aligned}$$

that is w is a solution of Eq. (2.2). Obviously, $w \geq 0$. By Lemma 2.5, (f_5) and Fatou lemma, we have

$$\begin{aligned}
c &= I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle + o(1) \\
&= \frac{1}{4}\|u_n\|^2 + \frac{1}{12}\int_{\mathbb{R}^3} Q(x)(u_n^+)^6 dx + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(x,u_n)u_n - F(x,u_n)\right)dx + o(1) \\
&\geq \frac{1}{4}\|u_n\|^2 + \frac{1}{12}\int_{\mathbb{R}^3} Q_p(x)(u_n^+)^6 dx + \int_{\mathbb{R}^3}\left(\frac{1}{4}f_p(x,u_n)u_n - F_p(x,u_n)\right)dx + o(1) \\
&= \frac{1}{4}\|w_n\|^2 + \frac{1}{12}\int_{\mathbb{R}^3} Q_p(x)(w_n^+)^6 dx + \int_{\mathbb{R}^3}\left(\frac{1}{4}f_p(x,w_n)w_n - F_p(x,w_n)\right)dx + o(1) \\
&\geq \frac{1}{4}\|w\|^2 + \frac{1}{12}\int_{\mathbb{R}^3} Q_p(x)(w^+)^6 dx + \int_{\mathbb{R}^3}\left(\frac{1}{4}f_p(x,w)w - F_p(x,w)\right)dx + o(1) \\
&= I_p(w) - \frac{1}{4}\langle I_p'(w), w\rangle \\
&= I_p(w) \\
&\geq c_p.
\end{aligned}$$

Using Remark 2.12, $I_p(w) = c_p = c$. By the properties of c and $\mathcal{N}$, there exits $t_w > 0$ such that $t_w w \in \mathcal{N}$. Thus, we obtain $c \le I(t_w w) \le I_p(t_w w) \le I_p(w) = c$. So c is achieved by $t_w w$. By Lemma 2.11, we have $I'(t_w w) = 0$.

In a word, we obtain that Eq. (2.2) has a nonnegative ground state solution $u \in E$. $\square$

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