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Multiple small solutions for $p(x)$-Schrödinger equations with local sublinear nonlinearities via genus theory

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Abstract. In this paper, we deal with the following $p(x)$-Schrödinger problem:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + V(x)\,|u|^{p(x)-2}\,u = f(x,u) \quad \text{in } \mathbb{R}^N\,; \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases}$$

where the nonlinearity is sublinear. We present the existence of infinitely many solutions for the problem. The main tool used here is a variational method and Krasnoselskii's genus theory combined with the theory of variable exponent Sobolev spaces. We also establish a Bartsch–Wang type compact embedding theorem for the variable exponent spaces.

Keywords: $p(x)$-Laplace operator, Schrödinger equation, variable exponent Lebesgue–Sobolev spaces, Krasnoselskii's genus.

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1 Introduction

In this paper, we consider the following $p(x)$-Schrödinger equation in $\mathbb{R}^N$

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + V(x)\,|u|^{p(x)-2}\,u = f(x,u), \quad \text{in } \mathbb{R}^N\,; \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{P}$$

where $N \geq 2$, $p : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous, $1 < p^- := \inf_{x\in\mathbb{R}^N} p(x) \leq p^+ := \sup_{x\in\mathbb{R}^N} p(x) < N$ and $V \in C(\mathbb{R}^N, \mathbb{R})$ is the new potential function, f obeys some conditions which will be stated later and $W^{1,p(x)}(\mathbb{R}^N)$ is the variable exponent Sobolev space.

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These interests are stimulated mainly by the development of the studies of problems in elasticity, electrorheological fluids, flow in porous media, calculus of variations, differential equations with $p(x)$-growth (see [1, 4, 10, 20, 27, 28]). We refer to the $p(x)$-Laplace operator $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, where p is a continuous non-constant function. This differential operator is a natural generalization of the p-Laplace operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a real constant. However, the $p(x)$-Laplace operator possesses more complicated nonlinearity than p-Laplace operator, due to the fact that $\Delta_{p(x)}$ is not homogeneous. This fact implies some difficulties; for example, we cannot use the Lagrange multiplier theorem in many problems involving this operator. Among these problems, the study involving $p(x)$-Laplacian problems via variational methods is an interesting topic. Many researchers have devoted their work to this area (see [2, 3, 13, 15, 17, 26]).

When $V(x)$ is radial (for example $V(x) \equiv 1$), Dai studied the following problem in [9]:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = f(x,u), & \text{in } \mathbb{R}^N; \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases}$$

by means of a direct variational approach and the theory of variable exponent Sobolev spaces, sufficient conditions ensuring the existence of infinitely many distinct homoclinic radially symmetric solutions is proved.

The case of p, when V are radially symmetric on $\mathbb{R}^N$ with $1 < p^- \le p^+ < N$ and $V^- > 1$ was discussed by Ge, Zhou and Xue in [19]. The existence of at least two nontrivial solutions has been established. In [30], Zhou and Wang studied the existence of infinitely many solutions for a class of (P) when the potential function does not satisfy the coercive condition.

For $p(x) = p$, problem (P) reduces to

$$\begin{cases} -\Delta_p u + V(x)\,|u|^{p-2}u = f(x,u), & \text{in } \mathbb{R}^N; \\ u \in W^{1,p}\left(\mathbb{R}^N\right). \end{cases} \tag{P_0}$$

Liu [24] studied that the existence of ground states of problem (P_0) with a potential which is periodic or has a bounded potential. Liu, Wang [23] discussed the problem (P_0) with sign-changing potential and subcritical p-superlinear nonlinearity, by using the cohomological linking method for cones, and obtained an existence result of nontrivial solution. Recently, Alves, Liu [3] established the existence of ground state solution for problem (P) via modern variational methods on the potential function V under following hypothesis

$$(V_0) \qquad V \in C(\mathbb{R}^N, \mathbb{R}),\ \inf_{x\in\mathbb{R}^N} V(x) > 0$$

and for each $M > 0$, $\mu\{x \in \mathbb{R}^N : V^{-1}(-\infty, M]\} < +\infty$, where $\mu(.)$ denotes the Lebesgue measure in $\mathbb{R}^N$. By using a variational method combined with the theory of variable exponent Sobolev spaces, Duan, Huang [13] and Wang, Yao, Liu [29] studied the existence of infinitely many solutions for a class of (P) equations in $\mathbb{R}^N$, which the potential V satisfies hypothesis (V_0) and $f(x,u)$ is sublinear at infinity in u. Moreover, the authors proposed new assumptions on the nonlinear term to yield bounded Palais–Smale sequences and then proved that the special sequences converge to critical points respectively. The main arguments are based on the geometry supplied by the fountain theorem. Consequently, they showed that the problem (P) under investigation admits a sequence of weak solutions with high energies.

Now, $p(x)$ and $V(x)$ satisfy the following assumptions.

(p_1) The function $p : \mathbb{R}^N \to \mathbb{R}$ is ipschitz continuous and $1 < p^- \le p^+ < N$;

(V_1) $V \in C\left(\mathbb{R}^N, \mathbb{R}\right)$, $\inf_{x\in\mathbb{R}^N} V(x) > 0$ and there are constants $r > 0$, $\alpha > N$ such that for any $b > 0$,

$$\lim_{|y|\to+\infty} \mu\left(\left\{x \in B_r(y) : \frac{V(x)}{|x|^\alpha} \le b\right\}\right) = 0,$$

where $B_r(y) = \left\{x \in \mathbb{R}^N : |x - y| < r\right\}$ and $\mu(\cdot)$ denotes the Lebesgue measure.

Remark 1.1. Condition (V_1) which is weaker than (V_0), is originally introduced by Bartsch, Wang, Willem [6] to guarantee the compact embedding of the working space. There are functions V satisfying (V_1) and not satisfying (V_0).

In the present paper, we concern with the existence of infinitely many solutions for (P) in $\mathbb{R}^N$ without any growth conditions imposed on $f(x, u)$ at infinity with respect to u and to the best of our knowledge, no results on this case have been obtained up to now. The main tool used here is a variational method and Krasnoselskii's genus theory combined with the theory of variable exponent Sobolev spaces. We also prove a Bartsch–Wang type compact embedding theorem for variable exponent spaces. We emphasize that in our approach, no coerciveness hypothesis (V_1) and not necessarily radially symmetric will be required on the potential V . Based on the above fact and motivated by techniques used in [24, 29, 30], the main purpose of this paper is devoted to investigate the existence of infinitely many solutions for problem (P) when the nonlinearity is sublinear in u at infinity.

Assume that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory and the following conditions.

(f_1)
$$|f(x,t)| \le m(x)g(x)\,|t|^{m(x)-1}, \qquad \forall\,(x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $g : \mathbb{R}^N \to \mathbb{R}^+$ is a positive continuous function such that $g \in L^{\frac{q(x)}{q(x)-m(x)}}\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right)$, $m \in C\left(\mathbb{R}^N\right)$, $1 < m^- \le m^+ < p^-$, $p(x) < q(x) \ll p^*(x) = \frac{Np(x)}{N-p(x)}$, and $q(x) \ll p^*(x)$ means that $\operatorname{ess\,inf}_{x\in\mathbb{R}^N}\left(p^*(x) - q(x)\right) > 0$.

(f_2) There exist an $x_0 \in \mathbb{R}^N$ and a constant $r > 0$ such that

$$\liminf_{t\to 0}\left(\inf_{x\in B_r(x_0)} \frac{\int_0^t f(x,s)ds}{|t|^{p(x)}}\right) > -\infty,$$
$$\limsup_{t\to 0}\left(\inf_{x\in B_r(x_0)} \frac{\int_0^t f(x,s)ds}{|t|^{p^-}}\right) = +\infty.$$

(f_3) f is an odd function according to t, that is,

$$f(x,t) = -f(x,-t)$$

for all $t \in \mathbb{R}$ and for all $x \in \mathbb{R}^N$.

2 Preliminaries

In this section, we recall some results on variable exponent Lebesgue and Sobolev spaces. Over the last decade, the variable exponent Lebesgue spaces $L^{p(x)}$ and the corresponding Sobolev space $W^{1,p(x)}$ have been a subject of active research area (we refer to [11,14,16,18,21,25] for the fundamental properties of these spaces). Write

$$C_+(\mathbb{R}^N) = \left\{ p \in C(\mathbb{R}^N) : p(x) > 1 \text{ for any } x \in \mathbb{R}^N \right\},$$

$$p^- := \inf_{x \in \mathbb{R}^N} p(x), p^+ := \sup_{x \in \mathbb{R}^N} p(x) \ \text{ for any } p \in C_+(\mathbb{R}^N).$$

The set of all measurable real-valued functions defined on $\mathbb{R}^N$ will be denote by $\Re(\mathbb{R}^N)$. Note that two measurable functions in $\Re(\mathbb{R}^N)$ are considered as the same element of $\Re(\mathbb{R}^N)$ when they are equal almost everywhere.

Let $p \in C_+(\mathbb{R}^N)$. The variable exponent Lebesgue space $L^{p(x)}(\mathbb{R}^N)$ is defined by

$$L^{p(x)}(\mathbb{R}^N) = \left\{ u \in \Re(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u(x)|^{p(x)}\, dx < \infty \right\},$$

which is equipped with the norm, so-called Luxemburg norm

$$|u|_{L^{p(x)}(\mathbb{R}^N)} := |u|_{p(x)} = \inf \left\{ \mu > 0 : \int_{\mathbb{R}^N} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\}.$$

Moreover, we define the variable exponent Sobolev space by

$$W^{1,p(x)}(\mathbb{R}^N) = \left\{ u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N) \right\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} := \|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

for all $u \in W^{1,p(x)}(\mathbb{R}^N)$.The spaces $L^{p(x)}(\mathbb{R}^N)$, $W^{1,p(x)}(\mathbb{R}^N)$ are separable and reflexive Banach spaces [11,18,21]. Now, let us introduce the modular of the space $L^{p(x)}\left(\mathbb{R}^N\right)$ as the functional $\sigma_{p(x)} : L^{p(x)}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\sigma_{p(x)}(u) = \int_{\mathbb{R}^N} |u(x)|^{p(x)}\, dx$$

for all $u \in L^{p(x)}(\mathbb{R}^N)$. The relation between modular and Luxemburg norm is clarified by the following propositions.

Proposition 2.1 ([11,18,21]). *Let $u, u_n \in L^{p(x)}(\mathbb{R}^N) \quad (n = 1, 2, \dots)$:*

(i) $|u|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \sigma_{p(x)}(u) < 1 (= 1; > 1)$;

(ii) $|u|_{p(x)} > 1 \implies |u|_{p(x)}^{p^-} \le \sigma_{p(x)}(u) \le |u|_{p(x)}^{p^+}$;

$|u|_{p(x)} < 1 \implies |u|_{p(x)}^{p^+} \le \sigma_{p(x)}(u) \le |u|_{p(x)}^{p^-}$;

(iii) $|u|_{p(x)} \le \sigma_{p(x)}(u)^{\frac{1}{p^-}} + \sigma_{p(x)}(u)^{\frac{1}{p^+}}$;

(iv) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \sigma_{p(x)}(u_n - u) = 0$;

$\lim_{n\to\infty} |u_n|_{p(x)} = \infty \Leftrightarrow \sigma_{p(x)}(u_n) = \infty$.

Proposition 2.2 ([11,21], Hölder-type inequality). *The conjugate space of $L^{p(x)}(\mathbb{R}^N)$ is $L^{p'(x)}(\mathbb{R}^N)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\mathbb{R}^N)$ and $v \in L^{p'(x)}(\mathbb{R}^N)$, we have*

$$\left| \int_{\mathbb{R}^N} uv dx \right| \le 2 |u|_{p(x)} |v|_{p'(x)}.$$

Proposition 2.3 ([2,11,16]). *Let $k, h \in C(\mathbb{R}^N)$ with $1 < k(x) \le h(x)$ for all $x \in \mathbb{R}^N$ and $u \in L^{h(x)}(\mathbb{R}^N)$. Then, $|u|^{k(x)} \in L^{\frac{h(x)}{k(x)}}(\mathbb{R}^N)$ via*

$$\left| |u|^{k(x)} \right|_{\frac{h(x)}{k(x)}} \le |u|_{h(x)}^{k^-} + |u|_{h(x)}^{k^+}$$

or there exists a number $\widehat{k} \in [k^-, k^+]$ such that

$$\left| |u|^{k(x)} \right|_{\frac{h(x)}{k(x)}} = |u|_{h(x)}^{\widehat{k}}.$$

Proposition 2.4 ([14,16,21]). *Let $p : \mathbb{R}^N \to \mathbb{R}$ be Lipschitz continuous and satisfy $p^+ < N$, let $q : \mathbb{R}^N \to \mathbb{R}$ be a measurable function. If $p(x) \le q(x) \le p^*(x) = \frac{Np(x)}{N-p(x)}$, then there is a continuous embedding $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N)$.*

Proposition 2.5 ([10,15,19]). *Let Ω be a bounded domain in $\mathbb{R}^N$. Assume that the boundary $\partial\Omega$ possesses cone property and $q(x) \in C(\overline{\Omega}, \mathbb{R})$ with $1 \le q(x) \ll p^*(x) = \frac{Np(x)}{N-p(x)}$ for $N > p(x)$ and $p^*(x) = +\infty$ for $N \le p(x)$, then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$.*

Proposition 2.6 ([18,21]). *Let Ω be an open subset of $\mathbb{R}^N$ and $G : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, and*

$$|G(x,t)| \le a(x) + b\,|t|^{p_1(x)/p_2(x)}, \qquad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $a \in L^{p_2(x)}(\Omega)$, b is a positive constant, $p_1, p_2 \in L^\infty(\Omega)$. Denoted by N_G the Nemytsky operator is defined by G, i.e.

$$(N_G(u))(x) = G(x, u(x)),$$

then $N_F : L^{p_1(x)}(\Omega) \to L^{p_2(x)}(\Omega)$ is a continuous and bounded map. When $p(x) < N$, write $p^*(x) = \frac{Np(x)}{N-p(x)}$.

Next, we consider the case that V satisfies (V_1). On the linear subspace

$$E = \left\{ u \in W^{1,p(x)}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + V(x)\,|u|^{p(x)} \right) dx < +\infty \right\},$$

we equip with the norm

$$\|u\|_E = \inf\left\{ \eta > 0 : \int_{\mathbb{R}^N} \left(\left|\frac{\nabla u}{\eta}\right|^{p(x)} + V(x) \left|\frac{u}{\eta}\right|^{p(x)} \right) dx \le 1 \right\}.$$

Then $(E, \|\cdot\|_E)$ is continuously embedded into $W^{1,p(x)}\left(\mathbb{R}^N\right)$ as a closed subspace. Therefore, $(E, \|\cdot\|_E)$ is also a separable reflexive Banach space.

In addition, we define the modular $\Lambda_{p(x),V} : E \to \mathbb{R}$ associated with E as follows:

$$\Lambda_{p(x),V}(u) = \int_{\mathbb{R}^N} \left(|\nabla u(x)|^{p(x)} + V(x)\,|u(x)|^{p(x)} \right) dx$$

for all $u \in E$, in a similar way to Proposition 2.1. The following proposition holds.

Proposition 2.7 ([11,18,21]). *Let $u, u_n \in E$:*

(i) $\|u\|_E < 1 (= 1; > 1) \Leftrightarrow \Lambda_{p(x),V}(u) < 1 (= 1; > 1)$;

(ii) $\|u\|_E > 1 \implies \|u\|_E^{p^-} \leq \Lambda_{p(x),V}(u) \leq \|u\|_E^{p^+}$;

$\|u\|_E < 1 \implies \|u\|_E^{p^+} \leq \Lambda_{p(x),V}(u) \leq \|u\|_E^{p^-}$;

(iii) $\lim_{n\to\infty} \|u_n\|_E = 0 \Leftrightarrow \lim_{n\to\infty} \Lambda_{p(x),V}(u_n) = 0$;

$\lim_{n\to\infty} \|u_n\|_E = \infty \Leftrightarrow \lim_{n\to\infty} \Lambda_{p(x),V}(u_n) = \infty$.

Proposition 2.8 ([17]). *$J \in C^1(E, \mathbb{R})$ and*

$$\langle J'(u), v \rangle = \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + V(x) |u|^{p(x)-2} uv \right) dx, \qquad \forall v, u \in E,$$

J is a convex functional, $J' : E \to E^*$ is strictly monotone, bounded homeomorphism, and is of (S_+) type, namely $u_n \rightharpoonup u$ (weakly) and

$$\overline{\lim}_{n\to\infty} \langle J'(u_n) - J'(u), (u_n - u) \rangle \leq 0 \quad \textit{implies} \quad u_n \to u \textit{ (strongly) in } E.$$

Definition 2.9. We say that the functional J satisfies the Palais–Smale condition ((PS) for short) if every sequence $\{u_n\} \in E$ such that

$$|I(u_n)| \leq c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to \infty$$

contains a convergent subsequence in the norm of E.

3 Proof of main result

In order to discuss the problem (P), we need to consider the energy functional $I : E \to \mathbb{R}$ defined by

$$I(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + V(x) |u|^{p(x)} \right) dx - \int_{\mathbb{R}^N} F(x,u) dx, \forall u \in E, \tag{3.1}$$

where $F(x,t) = \int_0^t f(x,s) ds$, $\forall (x,t) \in \mathbb{R}^N \times \mathbb{R}$. Set

$$J(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + V(x) |u|^{p(x)} \right) dx,$$
$$\Psi(u) = \int_{\mathbb{R}^N} F(x,u) dx,$$

then

$$I(u) = J(u) - \Psi(u).$$

Under our conditions, it follows from Hölder-type inequality and Sobolev embedding theorem that the energy functional I is well-defined. It is well known that $I \in C^1(E, \mathbb{R})$ and its derivative is given by

$$\langle I'(u), \varphi \rangle = \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2} \nabla u \nabla \varphi + V(x) |u|^{p(x)-2} u\varphi \right) dx - \int_{\mathbb{R}^N} f(x,u) \varphi dx \tag{3.2}$$

for all $u, \varphi \in E$.

Definition 3.1. We call that $u \in E \setminus \{0\}$ is a weak solution of (P), if

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2} \nabla u \nabla \varphi + V(x) |u|^{p(x)-2} u\varphi \right) dx = \int_{\mathbb{R}^N} f(x,u)\varphi dx,$$

where $\varphi \in E$.

We are now in a position to state our main results.

Theorem 3.2. *Assume that conditions* (p_1), (V_1) *and* (f_1) *hold.*

(1) *Problem* (P) *has a solution.*

(2) *Furthermore, if f admits the conditions (f_2) and (f_3), then problem (P) has a sequence of solution $\{\pm u_k : k = 1, 2, \ldots, \}$ such that $I(\pm u_k) < 0$ and $I(\pm u_k) \to 0$ as $k \to \infty$.*

For more existence and multiplicity results on $p(x)$-Laplacian equation in $\mathbb{R}^N$, we refer to Fan and Han [15]. In [15, Theorem 3.2], the similar results are obtained for the case $V(x) \equiv 1$.

For the proof of Theorem 3.2 we need some preliminary lemmas. The following Bartsch–Wang type compact embedding will play a crucial role in our subsequent arguments.

Lemma 3.3. *If V satisfies (V_1) and $p : \mathbb{R}^N \to \mathbb{R}$ is continuous and $1 < p^- \le p^+ < N$, then*

(i) we have a compact embedding $E \hookrightarrow\hookrightarrow L^{\frac{p(x)}{p^-}}(\mathbb{R}^N)$.

(ii) for any measurable function $q : \mathbb{R}^N \to \mathbb{R}$ *with* $\frac{p(x)}{p^-} < q(x) \ll p^*(x)$, *we have a compact embedding* $E \hookrightarrow\hookrightarrow L^{q(x)}(\mathbb{R}^N)$.

Proof. (i). Let $\{u_n\} \subset E$ such that $\|u_n\|_E \le C$. We assume up to a subsequence $u_n \rightharpoonup u$ in E. Then, we have $\|\omega_n\|_E \le C$ and $\omega_n \rightharpoonup 0$ in E, where $\omega_n = u_n - u$. We need to show $\omega_n \to 0$ in $L^{\frac{p(x)}{p^-}}(\mathbb{R}^N)$ to complete the proof.

Set $A_b(y) = \left\{x \in B_r(y) : \frac{V(x)}{|x|^\alpha} \le b\right\} \cap B_r(y_i)$ and $D_b(y) = \left\{x \in B_r(y) : \frac{V(x)}{|x|^\alpha} > b\right\} \cap B_r(y_i)$. By the Sobolev compact imbedding theorem in bounded domains (see Proposition 2.5), it implies that $\omega_n \to 0$ strongly in $L^{\frac{p(x)}{p^-}}(B_R)$ for any $R > 0$, where $B_R = \left\{x \in \mathbb{R}^N : |x| \le R\right\}$. To estimate $\int_{B_R^c} |\omega_n|^{\frac{p(x)}{p^-}} dx$, let $\{y_i\}_{i \in N}$ be a sequence of points in $\mathbb{R}^N$ satisfying such that $\mathbb{R}^N \subset \cup_{i=1}^\infty B_r(y_i)$ and each point x is contained in at most 2^N such balls $B_r(y_i)$. So for all $R > 2r$, we have

$$\begin{aligned}\int_{B_R^c} |\omega_n|^{\frac{p(x)}{p^-}} dx &\le \sum_{|y_i| \ge R-r} \int_{B_r(y_i)} |\omega_n|^{\frac{p(x)}{p^-}} dx \\ &= \sum_{|y_i| \ge R-r} \int_{A_b(y_i)} |\omega_n|^{\frac{p(x)}{p^-}} dx + \sum_{|y_i| \ge R-r} \int_{D_b(y_i)} |\omega_n|^{\frac{p(x)}{p^-}} dx.\end{aligned}$$

On the other hand, choose a number $\tau \in \left(1, \frac{N}{N-p^-}\right)$ arbitrarily, then $p(x) < \tau p(x) < p^*(x)$. By Proposition 2.4, we have that

$$E \hookrightarrow W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{\tau p(x)}(\mathbb{R}^N)$$

is continuous, and there are constants $C_0, C_1, C > 0$ such that

$$|\omega_n|_{\tau p(x)} \le C_0 \|\omega_n\|_{1,p(x)} \le C_1 \|\omega_n\|_E \le C_1 C, \qquad \forall \omega_n \in E. \tag{3.3}$$

Applying the Hölder-type inequality, we get

$$\begin{aligned}
\sum_{|y_i|\geq R-r}\int_{A_b(y_i)}|\omega_n|^{\frac{p(x)}{p^-}}dx &\leq \sum_{|y_i|\geq R-r}\left||\omega_n|^{\frac{p(x)}{p^-}}\right|_{\tau p^-,A_b(y_i)}|1|_{\frac{\tau p^-}{\tau p^- -1},A_b(y_i)} \\
&\leq \sum_{|y_i|\geq R-r}\left(|\omega_n|_{\tau p(x),A_b(y_i)}+|\omega_n|^{\frac{p^+}{p^-}}_{\tau p(x),A_b(y_i)}\right)\sup_{|y_i|\geq R-r}\left[\mu(A_b(y_i))\right]^{\frac{\tau p^- -1}{\tau p^-}} \\
&\leq \sum_{|y_i|\geq R-r}\left(|\omega_n|_{\tau p(x),B_r(y_i)}+|\omega_n|^{\frac{p^+}{p^-}}_{\tau p(x),B_r(y_i)}\right)\sup_{|y_i|\geq R-r}\left[\mu\left(A_b\left(y_i\right)\right)\right]^{\frac{\tau p^- -1}{\tau p^-}} \\
&\leq 2^N\left(|\omega_n|_{\tau p(x),B^c_{R-2r}}+|\omega_n|^{\frac{p^+}{p^-}}_{\tau p(x),B^c_{R-2r}}\right)\sup_{|y_i|\geq R-r}\left[\mu\left(A_b\left(y_i\right)\right)\right]^{\frac{\tau p^- -1}{\tau p^-}} \\
&\leq 2^N\left(|\omega_n|_{\tau p(x)}+|\omega_n|^{\frac{p^+}{p^-}}_{\tau p(x)}\right)\sup_{|y_i|\geq R-r}\left[\mu\left(A_b\left(y_i\right)\right)\right]^{\frac{\tau p^- -1}{\tau p^-}} \\
&\leq 2^N C_1^{\frac{\widehat{p}}{p^-}}\|\omega_n\|_E^{\frac{\widehat{p}}{p^-}}\sup_{|y_i|\geq R-r}\left[\mu\left(A_b\left(y_i\right)\right)\right]^{\frac{\tau p^- -1}{\tau p^-}} \\
&\leq 2^N\left(C_1 C\right)^{\frac{\widehat{p}}{p^-}}\sup_{|y_i|\geq R-r}\left[\mu\left(A_b\left(y_i\right)\right)\right]^{\frac{\tau p^- -1}{\tau p^-}},
\end{aligned}$$

where $\frac{\widehat{p}}{p^-}\in\left[1,\frac{p^+}{p^-}\right]$ and $|u|_{s(x),\Omega}=\inf\left\{\nu>0:\int_\Omega\left|\frac{u(x)}{\nu}\right|^{s(x)}dx\leq 1\right\}$.

Setting $\Omega_n:=\{x\in\mathbb{R}^N:|x|^\alpha|\omega_n|^{\left(\frac{p^- -1}{p^-}\right)p(x)}\leq 1\}$, we conclude that

$$\begin{aligned}
&\sum_{|y_i|\geq R-r}\int_{D_b(y_i)}|\omega_n|^{\frac{p(x)}{p^-}}dx \\
&=\sum_{|y_i|\geq R-r}\int_{D_b(y_i)\cap\Omega_n^c}|\omega_n|^{\frac{p(x)}{p^-}}dx+\sum_{|y_i|\geq R-r}\int_{D_b(y_i)\cap\Omega_n}|\omega_n|^{\frac{p(x)}{p^-}}dx \\
&\leq \sum_{|y_i|\geq R-r}\int_{D_b(y_i)\cap\Omega_n^c}|x|^\alpha|\omega_n|^{\frac{p(x)}{p^-}}|x|^{-\alpha}dx+\sum_{|y_i|\geq R-r}\int_{D_b(y_i)\cap\Omega_n}|x|^{\frac{-\alpha}{p^- -1}}dx \\
&\leq \sum_{|y_i|\geq R-r}\int_{D_b(y_i)}|x|^\alpha|\omega_n|^{p(x)}dx+\sum_{|y_i|\geq R-r}\int_{D_b(y_i)}|x|^{\frac{-\alpha}{p^- -1}}dx \\
&\leq \sum_{|y_i|\geq R-r}\int_{D_b(y_i)}|x|^\alpha|\omega_n|^{p(x)}dx+\sum_{|y_i|\geq R-r}\int_{D_b(y_i)}|x|^{\frac{-\alpha}{p^- -1}}dx \\
&\leq \frac{1}{b}\sum_{|y_i|\geq R-r}\int_{D_b(y_i)}V(x)|\omega_n|^{p(x)}dx+\sum_{|y_i|\geq R-r}\int_{D_b(y_i)}|x|^{\frac{-\alpha}{p^- -1}}dx \\
&\leq \frac{2^N}{b}\int_{B^c_{R-2r}}V(x)|\omega_n|^{p(x)}dx+\int_{B^c_{R-2r}}|x|^{\frac{-\alpha}{p^- -1}}dx \\
&\leq \frac{2^N}{b}\int_{\mathbb{R}^N}(|\nabla\omega_n|^{p(x)}+V(x)|\omega_n|^{p(x)})dx+\frac{2^N}{(R-2r)^{\frac{\alpha}{p^- -1}-N}} \\
&\leq \frac{2^N}{b}\|\omega_n\|_E^{\widehat{p}}+\frac{2^N}{(R-2r)^{\frac{\alpha}{p^- -1}-N}}\leq\frac{2^N C^{\widehat{p}}}{b}+\frac{2^N}{(R-2r)^{\frac{\alpha}{p^- -1}-N}}
\end{aligned}$$

with $\alpha>\frac{N}{p^- -1}$. Therefore, we get

$$\int_{B_R^c} |\omega_n|^{\frac{p(x)}{p^-}}\,dx \le 2^N (C_1 C)^{\frac{\widehat{p}}{p^-}} \sup_{|y_i|\ge R-r} [\mu(A_b(y_i))]^{\frac{\tau p^- -1}{\tau p^-}} + \frac{2^N C^{\widehat{p}}}{b} + \frac{2^N}{(R-2r)^{\frac{\alpha}{p^- -1}-N}}. \tag{3.4}$$

Now, for any $\varepsilon > 0$ and choose $b > 0$, we can find a positive constant $R > 0$ big enough such that

$$2^N (C_1 C)^{\frac{\widehat{p}}{p^-}} \sup_{|y_i|\ge R-r} [\mu(A_b(y_i))]^{\frac{\tau p^- -1}{\tau p^-}} < \frac{\varepsilon}{6}, \tag{3.5}$$

$$\frac{2^N C^{\widehat{p}}}{b} < \frac{\varepsilon}{6}, \tag{3.6}$$

and

$$\frac{2^N}{(R-2r)^{\frac{\alpha}{p^- -1}-N}} < \frac{\varepsilon}{6}. \tag{3.7}$$

It follows from (3.4)–(3.7) that

$$\int_{B_R^c} |\omega_n|^{\frac{p(x)}{p^-}}\,dx \le \frac{\varepsilon}{2}.$$

This implies $\omega_n \to 0$ in $L^{\frac{p(x)}{p^-}}(\mathbb{R}^N)$ and the proof of Lemma 3.3 (i) is completed.

(ii). Since $\{\omega_n\}$ is bounded in E and by Proposition 2.4, we get that $\{\omega_n\}$ is bounded in $L^{p^*(x)}(\mathbb{R}^N)$. It is taken into account $\frac{p(x)}{p^-} < q(x) \ll p^*(x)$, we can use the interpolation inequality [21, Corollary 2.2] and obtain

$$|\omega_n|_{q(x)} \le c\,|\omega_n|^{\eta}_{\frac{p(x)}{p^-}}\,|\omega_n|^{\sigma}_{p^*(x)}, \qquad \forall \omega_n \in L^{\frac{p(x)}{p^-}}(\mathbb{R}^N) \cap L^{p^*(x)}(\mathbb{R}^N), \tag{3.8}$$

where $c > 0$ is a constant and

$$\eta = \begin{cases} \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} \dfrac{p(x)}{q(x)} \dfrac{p^*(x)-q(x)}{p^- p^*(x)-p(x)} & \text{if } |\omega_n|_{\frac{p(x)}{p^-}} > 1, \\ \operatorname*{ess\,inf}_{x\in\mathbb{R}^N} \dfrac{p(x)}{q(x)} \dfrac{p^*(x)-q(x)}{p^- p^*(x)-p(x)} & \text{if } |\omega_n|_{\frac{p(x)}{p^-}} \le 1, \end{cases}$$

and

$$\sigma = \begin{cases} \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} \dfrac{p^*(x)}{q(x)} \dfrac{p^- q(x)-p(x)}{p^- p^*(x)-p(x)} & \text{if } |\omega_n|_{p^*(x)} > 1, \\ \operatorname{ess\,inf} \dfrac{p^*(x)}{q(x)} \dfrac{p^- q(x)-p(x)}{p^- p^*(x)-p(x)} & \text{if } |\omega_n|_{p^*(x)} \le 1. \end{cases}$$

Moreover, it follows from Lemma 3.3 (i) that $|\omega_n|_{\frac{p(x)}{p^-}} \to 0$ and (3.8) implies $\omega_n \to 0$ in $L^{q(x)}(\mathbb{R}^N)$. In the proof, some ideas in [5, 6, 12] have been followed. Proof of Lemma 3.3 is completed. □

Lemma 3.4. *Suppose (f_1) holds. Then functional I is coercive and bounded from below.*

Proof. Let $M = |g|_{\frac{q(x)}{q(x)-m(x)}}$. From (f_1), we have $|F(x,t)| \leq g(x)\,|u|^{m(x)}$. By using Propositions 2.1, 2.2, 2.3, 2.4, 2.7, we get

$$\begin{aligned} I(u) &\geq \frac{1}{p^+}\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)} + V(x)\,|u|^{p(x)}\right)dx - \int_{\mathbb{R}^N} g(x)\,|u|^{m(x)}\,dx \\ &\geq \frac{1}{p^+}\,\|u\|_E^{p^-} - 2\,|g|_{\frac{q(x)}{q(x)-m(x)}}\,|u|_{q(x)}^{\widehat{m}} \\ &\geq \frac{1}{p^+}\,\|u\|_E^{p^-} - 2C_1^{\widehat{m}}M\,\|u\|_E^{\widehat{m}}, \qquad \forall u \in E, \end{aligned} \tag{3.9}$$

for $\|u\|_E$ large enough. Therefore, $\widehat{m} < p^-$ gives the coercivity of I and I is bounded from below. Proof of Lemma 3.4 is completed. □

Lemma 3.5. *Suppose (f_1) holds. Then I satisfies the (PS) condition.*

Proof. Let us assume that there exists a sequence $\{u_n\}$ in E such that

$$|I(u_n)| \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.10}$$

From (3.10), we have $|I(u_n)| \leq c_0$. Combining this fact with (3.9) implies that

$$c_0 \geq I(u_n) \geq \frac{1}{p^+}\,\|u_n\|_E^{p^-} - 2C_1^{\widehat{m}}\,|g|_{\frac{q(x)}{q(x)-m(x)}}\,\|u_n\|_E^{\widehat{m}} \geq c_1,$$

for $\|u\|_E$ large enough. Since $\widehat{m} < p^-$, we obtain that $\{u_n\}$ is bounded in E. Finally, we show that there is a strongly convergent subsequence of $\{u_n\}$ in E. Indeed, in view of the boundedness of $\{u_n\}$, passing to a subsequence if necessary, still denoted by $\{u_n\}$, we may assume that

$$u_n \rightharpoonup u \quad \text{in } E.$$

By Lemma 3.3, we obtain the following results:

$$u_n \to u \quad \text{in } L^{\frac{p(x)}{p^-}}(\mathbb{R}^N), \qquad \frac{p(x)}{p^-} \leq q(x) \ll p^*(x),$$

that is

$$|u_n - u|_{\frac{p(x)}{p^-}} \to 0 \quad \text{as } n \to \infty,$$

and

$$|u_n - u|_{q(x)} \to 0 \quad \text{as } n \to \infty.$$

In view of the definition of weak convergence, we have $\langle I'(u_n) - I'(u), u_n - u\rangle \to 0$. Hence

$$\begin{aligned} \langle J'(u_n) - J'(u), u_n - u\rangle &= \langle I'(u_n) - I'(u), u_n - u\rangle + \int_{\mathbb{R}^N}(f(x,u_n) - f(x,u))\,(u_n - u)\,dx \\ &= \int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\,\nabla u_n\,(\nabla u_n - \nabla u)\,dx \\ &\quad + \int_{\mathbb{R}^N}V(x)\,|u_n|^{p(x)-2}\,u_n\,(u_n - u)\,dx \\ &\quad + \int_{\mathbb{R}^N}(f(x,u_n) - f(x,u))(u_n - u)dx \to 0. \end{aligned} \tag{3.11}$$

It is clear that

$$\langle I'(u_n) - I'(u), u_n - u\rangle = \langle I'(u_n), u_n - u\rangle + \langle I'(u), u_n - u\rangle \to 0. \tag{3.12}$$

By (f_1), Lemma 3.3 and Propositions 2.1, 2.2, 2.4, 2.7 it follows,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))\,(u_n - u)\,dx \right| \\
&\quad \le m^+ \left| \int_{\mathbb{R}^N} g(x) \left(|u_n|^{m(x)-1} + |u|^{m(x)-1} \right) (u_n - u)\,dx \right| \\
&\quad \le m^+ \left| \int_{\mathbb{R}^N} g(x)\,|u_n|^{m(x)-1}\,(u_n - u)\,dx \right| + m^+ \left| \int_{\mathbb{R}^N} g(x)\,|u|^{m(x)-1}\,(u_n - u)\,dx \right| \\
&\quad \le 3m^+ \,|g|_{\frac{q(x)}{q(x)-m(x)}} \left| |u_n|^{m(x)-1} \right|_{\frac{q(x)}{m(x)-1}} |u_n - u|_{q(x)} \\
&\qquad + 3m^+ \,|g|_{\frac{q(x)}{q(x)-m(x)}} \left| |u|^{m(x)-1} \right|_{\frac{p(x)}{m(x)-1}} |u_n - u|_{q(x)} \\
&\quad \le 3m^+ M \left[\max\left\{ |u_n|_{q(x)}^{m^- -1}, |u_n|_{q(x)}^{m^+ -1} \right\} + \max\left\{ |u|_{q(x)}^{m^- -1}, |u|_{q(x)}^{m^+ -1} \right\} \right] |u_n - u|_{q(x)} \\
&\quad \le C_3 \left(\|u_n\|_E^{m^- -1} + \|u_n\|_E^{m^+ -1} + \|u\|_E^{m^- -1} + \|u\|_E^{m^+ -1} \right) |u_n - u|_{q(x)} \\
&\quad \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.13}$$

Together (3.12) with (3.13), one deduces from (3.11) that

$$\langle J'(u_n) - J'(u), u_n - u \rangle \to 0 \quad \text{as } n \to \infty.$$

Since J is of (S_+) type (see Proposition 2.8), we obtain $u_n \to u$ in E. Proof of Lemma 3.5 is completed. □

Let X be a separable and reflexive Banach space, then there exist $(e_n) \subset X$ and $(e_n^*) \subset X^*$ such that

$$\langle e_n^*, e_n \rangle = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \ne m, \end{cases}$$

where $X = \overline{\text{span}}\,\{e_n : n = 1, 2, \ldots,\}$ and $X^* = \overline{\text{span}}\,\{e_n^* : n = 1, 2, \ldots,\}$. For each $k \in \mathbb{N}$ we consider

$$X_k = \text{span}\,\{e_1, e_2, \ldots, e_k\},$$

then the subspace of E spanned by the vectors $e_1, e_2, \ldots, e_k$ and

$$Y_k = \mathop{\oplus}_{j=1}^{k} X_j, \; Z_k = \overline{\mathop{\oplus}_{j=k}^{\infty} X_j}.$$

Lemma 3.6 ([14]). *Assume that $\Psi : X \to \mathbb{R}$ is weakly-strongly continuous and $\Psi(0) = 0$, $\gamma > 0$ is a given positive number. Set*

$$\beta_k = \sup_{u \in Z_k, \|u\|_X \le \gamma} |\Psi(u)|,$$

then $\beta_k \to 0$ as $k \to \infty$.

We denote by $\gamma(A)$ the genus of A (see [7, 22]). Set

$$\Re = \Big\{ A \subset X \setminus \{0\} : A \text{ is compact and } A = -A \Big\}.$$

$$\Re_k = \{A \subset \Re : \gamma(A) \geq k\}, \qquad k = 1, 2, \ldots, \qquad c_k = \inf_{A \in \Re_k} \sup_{u \in A} I(u),$$

we have

$$-\infty < c_1 \leq c_2 \leq \cdots \leq c_k \leq c_{k+1} \leq \cdots$$

The following result obtained by Clarke in [8] is the main idea, which we use in the proof of Theorem 3.2.

Theorem A ([8]). *Let $I \in C^1(X, \mathbb{R})$ be functional satisfying the (PS) condition. Furthermore, let us suppose that*

(i) I is bounded from below and even;

(ii) There is a compact set $K \in \Re$ such that $\gamma(A) = k$ and $\sup_{x \in K} I(x) < I(0)$.

Then I possesses at least k pairs of distinct critical points, and their corresponding critical values $c_k < 0$ satisfying $c_k \to 0$ as $k \to \infty$ are less than $I(0)$.

Proof of Theorem 3.2. (1). Proposition 2.6 and Lemma 3.5 conclude that I is weakly lower semi-continuous. From Lemma 3.4, I is coercive on E, i.e., $I(u) \to +\infty$ as $\|u\|_E \to \infty$. Hence, I can attain its minimum on E, this provides a solution of problem (P).

(2). As I is coercive, by Lemma 3.3 we know that satisfies (PS) condition. By (f_3), I is an even functional. Now, we will show that $c_k < 0$ for every $k \in \mathbb{N}$. Because E is a reflexive and separable Banach space for any $k \in \mathbb{N}$, we can choose a k-dimensional linear subspace X_k of E such that $X_k \subset C_0^\infty(\mathbb{R}^N)$. As the norms $W^{1,p(x)}(\mathbb{R}^N)$ and E on X_k are equivalent, there exists $r_k \in (0, 1)$ and $\delta > 0$ such that $u \in X_k$ with $\|u\|_E \leq r_k$ implies $|u|_{L^\infty(\mathbb{R}^N)} \leq \delta$.

Set $S_{r_k}^{(k)} = \{u \in X_k : \|u\|_E = r_k\}$. From condition (f_2), there exist $L > 0, \delta > 0$ such that

$$\operatorname*{ess\,inf}_{x \in B_r(x_0)} \int_0^t f(x,s)ds > -L\,|t|^{p(x)} > -L\,|t|^{p^-} \tag{3.14}$$

for every $0 < |t| \leq \delta$.

Let us consider a compact set $B_d(x_0) \subset B_r(x_0)$, $0 < d < r$ with $|B_d(x_0)| = (L+1)\,|B_r(x_0) \setminus B_d(x_0)|$ and a nonzero nonnegative function $v \in C^\infty(\mathbb{R}^N)$ such that

$$v(x) \equiv 1 \quad \text{if } x \in B_d(x_0), \tag{3.15}$$

$$0 < v(x) \leq 1 \quad \text{if } x \in B_r(x_0) \setminus B_d(x_0) \tag{3.16}$$

and

$$v(x) \equiv 0 \quad \text{if } x \in \mathbb{R}^N \setminus B_r(x_0). \tag{3.17}$$

Then we have $|v(x)| \leq 1$. By condition (f_2), there exists $t' \in \mathbb{R}$ and $c(k) > 1$ such that

$$\operatorname*{ess\,inf}_{x \in B_r(x_0)} \int_0^{t'} f(x,s)ds \geq \max\left(\frac{c(k)\int_{\mathbb{R}^N}\left(|\nabla v|^{p(x)} + V(x)\,|v|^{p(x)}\right)dx}{p^+\,|B_r(x_0) \setminus B_d(x_0)|}, 1\right)|t'|^{p^-} \tag{3.18}$$

for all $v \in S_{r_k}^{(k)}$ and $0 < |t'| \leq \delta$. We take a $t' > 0$ such that $t' \sup_{x \in B_r(x_0)} v(x) \leq \delta$. By (3.14) and (3.18), we get

$$\int_0^{t'v} f(x,s)ds > -L \operatorname*{ess\,inf}_{x \in B_r(x_0)} \int_0^{t'} f(x,s)ds \tag{3.19}$$

for every $0 < |t'| \leq \delta$. Combining (3.14)–(3.19), for $v \in S_{r_k}^{(k)}$ and $t' \in (0,1)$, we have

$$\begin{aligned}
I(t'v) &= \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla (t'v)|^{p(x)} + V(x) |t'v|^{p(x)} \right) dx - \int_{\mathbb{R}^N} \left(\int_0^{t'v} f(x,s) ds \right) dx \\
&\leq \frac{(t')^{p^-}}{p^+} \int_{\mathbb{R}^N} \left(|\nabla v|^{p(x)} + V(x) |v|^{p(x)} \right) dx - \int_{B_d(x_0)} \left(\int_0^{t'} f(x,s) ds \right) dx \\
&\quad - \int_{B_r(x_0) \setminus B_d(x_0)} \left(\int_0^{t'v} f(x,s) ds \right) dx \\
&\leq \frac{(t')^{p^-}}{p^+} \int_{\mathbb{R}^N} \left(|\nabla v|^{p(x)} + V(x) |v|^{p(x)} \right) dx - |B_d(x_0)| \operatorname*{ess\,inf}_{x \in B_r(x_0)} \int_0^{t'} f(x,s) ds \\
&\quad - L |B_r(x_0) \setminus B_d(x_0)| \operatorname*{ess\,inf}_{x \in B_r(x_0)} \int_0^{t'v} f(x,s) ds \\
&\leq \frac{(t')^{p^-}}{p^+} \int_{\mathbb{R}^N} \left(|\nabla v|^{p(x)} + V(x) |v|^{p(x)} \right) dx - (L+1) |B_r(x_0) \setminus B_d(x_0)| \operatorname*{ess\,inf}_{x \in B_r(x_0)} \int_0^{t'} f(x,s) ds \\
&\quad + L |B_r(x_0) \setminus B_d(x_0)| \operatorname*{ess\,inf}_{x \in B_r(x_0)} \int_0^{t'} f(x,s) ds \\
&= \frac{(t')^{p^-}}{p^+} \int_{\mathbb{R}^N} \left(|\nabla v|^{p(x)} + V(x) |v|^{p(x)} \right) dx - |B_r(x_0) \setminus B_d(x_0)| \operatorname*{ess\,inf}_{x \in B_r(x_0)} \int_0^{t'} f(x,s) ds \\
&\leq \frac{(t')^{p^-}}{p^+} \int_{\mathbb{R}^N} \left(|\nabla v|^{p(x)} + V(x) |v|^{p(x)} \right) dx - \frac{(t')^{p^-} c(k)}{p^+} \int_{\mathbb{R}^N} (|\nabla v|^{p(x)} + V(x) |v|^{p(x)}) dx \\
&= \frac{(t')^{p^-} (1 - c(k))}{p^+} r_k^{p^-} < 0,
\end{aligned}$$

we can find $t'_k \in (0,1)$ and $\epsilon_k > 0$ such that $I(t'_k v) \leq -\epsilon_k$ for all $u \in S_{r_k}^{(k)}$, that is, $I(u) \leq -\epsilon_k$, for all $u \in S_{t_k r_k}^{(k)}$.

It is clear that $\gamma(S_{t_k r_k}^{(k)}) = k$ so $c_k \leq -\epsilon_k < 0$. Moreover, from (f_3), I is even. Finally, by Lemma 3.4, Lemma 3.5 and above results, we can apply Theorem A to obtain that the functional I admits at least k pairs of distinct critical points, and since k is arbitrary, we obtain infinitely many critical points of I.

It remains to prove $c_k \to 0$ as $k \to \infty$. By Lemma 3.4 there exists a constant $\gamma > 0$ such that $I(u) > 0$ when $\|u\|_E \geq \gamma$. Taking arbitrarily $A \in \Re_k$, then $\gamma(A) \geq k$, $k = 1, 2, \ldots$ According to the properties of genus we know that $A \cap Z_k \neq \varnothing$. By Lemma 3.4 we have $\beta_k \to 0$ as $k \to \infty$. When $u \in Z_k$ and $\|u\|_E \leq \gamma$, we have $I(u) = J(u) - \Psi(u) \geq -\Psi(u) \geq -\beta_k$, hence $\sup_{u \in A} I(u) \geq -\beta_k$ and then $c_k \geq -\beta_k$, this concludes $c_k \to 0$ as $k \to \infty$. Proof of Theorem 3.2 is completed. □

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