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Existence of infinitely many radial and non-radial solutions for quasilinear Schrödinger equations with general nonlinearity

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Abstract. In this paper, we prove the existence of multiple solutions for the following quasilinear Schrödinger equation

$$-\Delta u - u\Delta(|u|^2) + V(|x|)u = f(|x|, u), \qquad x \in \mathbb{R}^N.$$

Under some generalized assumptions on f, we obtain infinitely many radial solutions for $N \geq 2$, many non-radial solutions for $N = 4$ and $N \geq 6$, and a non radial solution for $N = 5$. Our results generalize and extend some existing results.

Keywords: quasilinear Schrödinger equation, radial solutions, non-radial solutions, symmetric mountain pass theorem.

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1 Introduction and preliminaries

This article deals mainly with the following quasilinear Schrödinger equation

$$-\Delta u - u\Delta(|u|^2) + V(|x|)u = f(|x|, u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 2$, $V : [0, \infty) \to \mathbb{R}$ and $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$.

It is well known that Schrödinger equation has already found a great deal of interest in recently years because not only it is very important for other fields to study the Schrödinger equation but also it provides a good model for developing mathematical methods. By virtue of variational methods, Schrödinger equation has been widely studied for multiplicity of non-trivial solutions over the past several years. See, e.g., [4,6,10,12,13,24,29,34] and the references and quoted in them. However, quasilinear Schrödinger equation is taken as a generalisation

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of the Schrödinger equation. Some authors studied the multiplicity of solutions for quasilinear problem. See, e.g., [1, 20, 23, 31] and the references and quoted in them. In the most of the aforementioned references, there are rarely papers to study the radial and non-radial solutions for quasiliner and semilinear Schrödinger equation which has the properties of radial symmetry except for the papers [3, 5, 7, 8, 18, 19, 27, 28] and the references. Especially, in [14], Kristály et al. proved the existence of sequences of non-radial, sign-changing solutions for semilinear Schrödinger equation when $s_N = [\frac{N-1}{2}] + (-1)^N$, $N \geq 4$, where the elements in different sequences cannot be compared from symmetrical point of view. The idea comes from the solution of the Rubik cube, and it has been extended to Heisenberg groups by Kristály and Balogh [15]. Based on this fact, recently, Yang et al. [30] first studied infinitely many radial and non-radial solutions for the problem (1.1) under the following assumptions on V and f:

(V) $V \in C([0,\infty),\mathbb{R}) \cap L^\infty([0,\infty),\mathbb{R})$ and $0 < V_0 := \inf_{r\geq 0} V(r) \leq V(r)$ for all $r \geq 0$.

(f_1) $f \in C([0,\infty)\times\mathbb{R},\mathbb{R})$, and there exist $c > 0$ and $4 < p < 22^*$ such that

$$|f(r,u)| \leq c(|u| + |u|^{p-1}) \quad \text{for any } r \geq 0 \text{ and } u \in \mathbb{R},$$

where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 2$.

(f_2) $f(r,u) = o(|u|)$ as $|u| \to 0$ uniformly in r.

(f_3) There exists $R > 0$ such that

$$C_0 = \inf_{x\in\mathbb{R}^N, |u|\geq R} F(|x|,u) > 0,$$

where $F(r,u) = \int_0^u f(r,s)ds$.

(f_4) There exists $\alpha > 4$ such that

$$\alpha F(r,u) \leq uf(r,u) \quad \text{for any } r \geq 0 \text{ and } u \in \mathbb{R}.$$

(f_5) $f(r,-u) = -f(r,u)$ for any $r \geq 0$ and $u \in \mathbb{R}$.

Moreover, the authors gave the following theorems in [30]. (Note that ℓ is defined in (2.1) in the rest paper.)

Theorem 1.1 ([30]). *Assume that $N \geq 2$, (V), (f_1)–(f_5) hold. Then problem (1.1) has a sequence of radial solutions $\{u_n\}$ such that $\ell(u_n) \to \infty$ as $n \to \infty$.*

Theorem 1.2 ([30]). *Assume that $N = 4$ or $N \geq 6$, (V), (f_1)–(f_5) hold. Then problem (1.1) has a sequence of non-radial solutions $\{u_n\}$ such that $\ell(u_n) \to \infty$ as $n \to \infty$.*

Theorem 1.3. *[30] Assume that (V) and (f_1)–(f_4) hold. If $N = 5$ and*

(f_6) for all $z = (x,y) \in \mathbb{R}\times\mathbb{R}^4$ and for all $g \in O(\mathbb{R}^4)$

$$f(|(x+1,y)|,u) = f(|(x,g(y))|,u) \quad \textit{and} \quad V(|(x+1,y)|) = V(|(x,g(y))|),$$

where $O(\mathbb{R}^4)$ is the orthogonal transform group in $\mathbb{R}^4$. Then problem (1.1) has a nontrivial non-radial solution.

In 2013, Tang [29] gave some much weaker conditions and studied the existence of infinitely many solutions for Schrödinger equation via symmetric mountain pass theorem with sign-changing potential. Using Tang's conditions, some authors studied the existence of infinitely many solutions for different equations. See, e.g., [9, 16, 25, 32, 33, 35, 36] and the references quoted in them. These results generalized and extended some existing results. Especially, Zhang et al. [37] proved many radial and non-radial solutions for a fractional Schrödinger equation by using Tang's conditions and methods which are more weaker than (AR)-condition and super-quadratic conditions.

Inspired by the above references, we consider problem (1.1) with the following more general super-quartic conditions, and establish the existence of infinitely many radial and non-radial solutions by symmetric mountain pass theorem in [2, 26]. To state our results, we give the following much weaker conditions:

(V') $V \in C([0,\infty))$ is bounded from below by a positive constant V_0;

(f_3') $\lim\limits_{|u|\to\infty} \frac{|F(r,u)|}{|u|^4} = \infty$, uniformly in $r \in [0,+\infty)$ and there exists $r_0 \geq 0$ such that

$$F(r,u) \geq 0, \ \ \forall\ u \in \mathbb{R},\ |u| \geq r_0;$$

(f_4') $\mathcal{F}(r,u) := \frac{1}{4}uf(r,u) - F(r,u) \geq 0$, and there exist $c_0 > 0$ and $\kappa > \max\{1, \frac{2N}{N+2}\}$ such that

$$|F(r,u)|^\kappa \leq c_0|u|^{2\kappa}\mathcal{F}(r,u),\ \ \forall\ u \in \mathbb{R},\ |u| \geq r_0.$$

Next, we are ready to state the main results of this paper. (Note that ℓ is defined later in (2.1).)

Theorem 1.4. *Suppose that $N \geq 2$, (V'), (f_1), (f_2), (f_3'), (f_4') and (f_5) hold. Then problem* (1.1) *has a sequence of radial solutions $\{u_n\}$ such that $\ell(u_n) \to \infty$ as $n \to \infty$.*

Theorem 1.5. *Suppose that $N = 4$ or $N \geq 6$, (V'), (f_1), (f_2), (f_3'), (f_4') and (f_5) hold. Then problem* (1.1) *has a sequence of non-radial solutions $\{u_n\}$ such that $\ell(u_n) \to \infty$ as $n \to \infty$.*

Theorem 1.6. *Suppose that $N = 5$, (V'), (f_1), (f_2), (f_3'), (f_4') and (f_6) hold. Then problem* (1.1) *has a nontrivial non-radial solution.*

Remark 1.7. On the one hand, note that the condition (V') is weaker than (V). In (V), $V \in L^\infty([0,+\infty))$, it is very important for ℓ to prove the boundedness of $(C)_c$-sequence $\{v_n\}$. But in (V'), there is no need to assume that $V \in L^\infty([0,+\infty))$, and we give a different approach to prove the boundedness of $(C)_c$-sequence $\{v_n\}$, which is different from Yang's methods (see [30]). On the other hand, note that condition (f_4') is somewhat weaker than the condition (f_4). As for the specific examples, we can see the reference [29].

Remark 1.8. By conditions (f_3') and (f_4'), we can get

$$\mathcal{F}(r,u) \geq \frac{1}{c_0}\left(\frac{|F(r,u)|}{|u|^2}\right)^\kappa \to \infty$$

uniformly in r as $|u| \to \infty$.

2 Variational framework and some lemmas

Before stating this section, we first recall the following important notions.

As usual, for $1 \leq s < +\infty$, let

$$\|u\|_s = \left(\int_{\mathbb{R}^N} |u|^s\right)^{\frac{1}{s}}, \quad u \in L^s(\mathbb{R}^N).$$

Let

$$H^1(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\right\}$$

with the norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)dx\right)^{\frac{1}{2}}.$$

Let S be the best Sobolev constant

$$S\|u\|_{2^*}^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx$$

for any $u \in H^1(\mathbb{R}^N)$.

Our working spaces is defined by

$$H := \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(|x|)u^2 dx < \infty\right\}$$

with the inner product

$$(u, v)_H = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(|x|)uv)dx$$

and the norm

$$\|u\|_H = (u, u)_H^{\frac{1}{2}}.$$

To this end, we define the functional by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} [(1 + 2u^2)|\nabla u|^2 + V(|x|)u^2]dx - \int_{\mathbb{R}^N} F(|x|, u)dx$$

and define the derivative of J at u in the direction of $\phi \in C_0^\infty(\mathbb{R}^N)$ as follows:

$$\langle J(u), \phi\rangle = \int_{\mathbb{R}^N} [(1 + 2u^2)\nabla u \nabla \phi + |\nabla u|^2 u\phi + V(|x|)u\phi]dx - \int_{\mathbb{R}^N} f(|x|, u)\phi dx.$$

In order to prove Theorem 1.4, we denote by E the space of radial functions of H, namely,

$$E := \{u \in H : u(x) = u(|x|)\}.$$

For the proof of Theorem 1.5, following [5], choose an integer $2 \leq m \leq N/2$ with $2m \neq N - 1$, and write the elements of $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{N-2m}$ as $x = (x_1, x_2, x_3)$ with $x_1, x_2 \in \mathbb{R}^m$ and $x_3 \in \mathbb{R}^{N-2m}$. Now, consider the action of

$$G_m := O(m) \times O(m) \times O(N - 2m)$$

on H and define by

$$lu(x) = u(l^{-1}x).$$

Let $\varsigma \in O(N)$ be involution given by $\varsigma(x_1, x_2, x_3) = (x_2, x_1, x_3)$. The action of $G := \{id, \varsigma\}$ on

$$\mathrm{Fix}(G_m) := \{u \in H : lu = u, \forall\, l \in G_m\}$$

is defined by

$$(lu)(x) = \begin{cases} u(x), & \text{if } l = id, \\ -u(l^{-1}x), & \text{if } l = \varsigma. \end{cases}$$

Let

$$E := \mathrm{Fix}(G) = \{u \in \mathrm{Fix}(G_m) : hu = u, \forall\, h \in G\}.$$

Note that 0 is the only radially symmetric function in E for this case.

In both cases, E is a closed subspace of H, and the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ are continuous for $s \in [2, 2^*]$ and the embeddings $E \hookrightarrow L^s(\mathbb{R}^N)$ are compact for $s \in (2, 2^*)$ (see [26, Lemma 2]). It follows from the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ for $s \in [2, 2^*]$ that

$$\|u\|_s \le \gamma_s \|u\|_E = \gamma_s \|u\|_H, \quad \forall\, u \in E,\ s \in [2, 2^*].$$

We know that J is not well defined in general in E. To overcome this difficulty, we apply an argument developed by Liu et al. [17] and Colin and Jeanjean [11]. We make the change of variables by $v = g^{-1}(u)$, where g is defined by

$$g'(t) = \frac{1}{(1 + 2g^2(t))^{\frac{1}{2}}} \text{ on } [0, \infty) \text{ and } g(t) = -g(-t) \text{ on } (-\infty, 0].$$

Let us recall some properties of the change of variables $g : \mathbb{R} \to \mathbb{R}$ which are proved in [11, 17, 21] as follows.

Lemma 2.1. *The function $g(t)$ and its derivative satisfy the following properties:*

(1) g is uniquely defined, C^∞ and invertible;

(2) $|g'(t)| \le 1$ for all $t \in \mathbb{R}$;

(3) $|g(t)| \le |t|$ for all $t \in \mathbb{R}$;

(4) $g(t)/t \to 1$ as $t \to 0$;

(5) $g(t)/\sqrt{t} \to 2^{\frac{1}{4}}$ as $t \to +\infty$;

(6) $g(t)/2 \le tg'(t) \le g(t)$ for all $t > 0$;

(7) $g^2(t)/2 \le tg(t)g'(t) \le g^2(t)$ for all $t \in \mathbb{R}$;

(8) $|g(t)| \le 2^{1/4}|t|^{1/2}$ for all $t \in \mathbb{R}$;

(9) there exists a positive constant C such that

$$|g(t)| \ge \begin{cases} C|t|, & \text{if } |t| \le 1, \\ C|t|^{\frac{1}{2}}, & \text{if } |t| \ge 1; \end{cases}$$

(10) for each $\alpha > 0$, there exists a positive constant $C(\alpha)$ such that

$$|g(\alpha t)|^2 \le C(\alpha)|g(t)|^2;$$

(11) $|g(t)||g'(t)| \le \frac{1}{\sqrt{2}}$.

Hence, by making the change of variables, from $J(u)$ we obtain the following functional

$$\ell(v) = \frac{1}{2}\int_{\mathbb{R}^N}[|\nabla v|^2 + V(|x|)g^2(v)]dx - \int_{\mathbb{R}^N} F(|x|, g(v))dx, \tag{2.1}$$

which is well defined on the space E. Similar to the proof of [30, 37], it is easy to see that $\ell \in C^1(E, \mathbb{R})$, and

$$\langle \ell'(v), \omega\rangle = \int_{\mathbb{R}^N}[\nabla v \nabla \omega + V(|x|)g(v)g'(v)\omega]dx - \int_{\mathbb{R}^N} f(|x|, g(v))g'(v)\omega dx, \tag{2.2}$$

for any $\omega \in E$. Moreover, the critical points of ℓ are the weak solutions of the following equation

$$-\Delta v = \frac{1}{\sqrt{1+2|g(v)|^2}}\left(f(|x|, g(v)) - V(|x|)g(v)\right) \quad \text{in } \mathbb{R}^N.$$

We also know that if v is a critical point of the functional ℓ, then $u = g(v)$ is a critical point of the functional J, i.e. $u = g(v)$ is a solution of problem (1.1).

To prove our results, we need the principle of symmetric criticality theorem (see [22, Theorem 1.28]) as follows.

Lemma 2.2 ([22])**.** *Assume that the action of the topological group G on the Hilbert space X is isometric. If $\Phi \in C^1(X, \mathbb{R})$ is invariant and if u is a critical point of Φ restricted to* $\mathrm{Fix}(G)$*, then u is a critical point of Φ.*

Therefore, from the above lemma, if v is a critical point of $\Phi := \ell|_E$, then v is a critical point of ℓ, i.e. $u = g(v)$ is a solution of (1.1).

A sequence $\{v_n\} \subset E$ is said to be a $(C)_c$-sequence if $\ell(v) \to c$ and $\|\ell'(v)\|(1 + \|v_n\|) \to 0$. ℓ is said to satisfy the $(C)_c$-condition if any $(C)_c$-sequence has a convergent subsequence.

Lemma 2.3. *Suppose that (V'), (f_1), (f_2), (f_3') and (f_4') are satisfied. Then any $(C)_c$-sequence $\{v_n\}$ of ℓ is bounded.*

Proof. Let $\{v_n\}$ be a $(C)_c$-sequence, then we have

$$\ell(v_n) \to c \quad \text{and} \quad \langle \ell'(v_n), v_n\rangle \to 0. \tag{2.3}$$

Hence, by (6) in Lemma 2.1, there is a constant $C_1 > 0$ such that

$$C_1 \geq \ell(v_n) - \frac{1}{2}\langle \ell'(v_n), v_n\rangle \geq \int_{\mathbb{R}^N} \mathcal{F}(|x|, g(v_n))dx. \tag{2.4}$$

Firstly, let $S_n^2 = \int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(|x|)g^2(v_n)\right)dx$. Next, we prove that there exists a constant $C_2 > 0$ such that

$$S_n^2 = \int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(|x|)g^2(v_n)\right)dx \leq C_2.$$

Suppose to the contrary that

$$S_n^2 = \int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(|x|)g^2(v_n)\right)dx \to \infty, \quad \text{as } n \to \infty.$$

On the one hand, setting $\widetilde{g}(v_n) := \frac{g(v_n)}{S_n}$, by (2) in Lemma 2.1, then $\|\widetilde{g}(v_n)\|_E \leq 1$. Passing to a subsequence, we may assume that $\widetilde{g}(v_n) \rightharpoonup \sigma$ in E, $\widetilde{g}(v_n) \to \sigma$ in $L^s(\mathbb{R}^N)$, $2 < s < 2^*$, and $\widetilde{g}(v_n) \to \sigma$ a.e. on $\mathbb{R}^N$.

By (2.1) and (2.3), we can get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|F(|x|,g(v_n))|}{S_n^2}dx=\frac{1}{2}. \tag{2.5}$$

On the other hand, let $\psi_n=\frac{g(v_n)}{g'(v_n)}$, by (6) in Lemma 2.1, then there is a constant $C_3>0$ such that $\|\psi_n\|\le C_3\|v_n\|_E$. Moreover, by (2.4), we know that there exists a constant $C_4>0$ such that

$$\begin{aligned}
C_4 &\ge \ell(v_n)-\frac{1}{4}\langle \ell'(v_n),\psi_n\rangle\\
&=\frac{1}{4}\int_{\mathbb{R}^N}(g'(v_n))^2|\nabla v_n|^2dx+\frac{1}{4}\int_{\mathbb{R}^N}V(|x|)g^2(v_n)dx\\
&\quad+\int_{\mathbb{R}^N}\left(\frac{1}{4}f(|x|,g(v_n))g(v_n)-F(|x|,g(v_n))\right)dx\\
&=\frac{1}{4}\int_{\mathbb{R}^N}(g'(v_n))^2|\nabla v_n|^2dx+\frac{1}{4}\int_{\mathbb{R}^N}V(|x|)g^2(v_n)dx+\int_{\mathbb{R}^N}\mathcal{F}(|x|,g(v_n))dx,
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^N}\mathcal{F}(|x|,g(v_n))dx\le C_4. \tag{2.6}$$

Let

$$\eta(r):=\inf\left\{\mathcal{F}(x,g(v_n))\mid x\in\mathbb{R}^N \text{ with } |g(v_n)|\ge r\right\},$$

for $r>0$. By Remark 1.8, $\eta(r)\to\infty$ as $r\to\infty$. For $0\le a<b$, let

$$\Omega_n(a,b)=\left\{x\in\mathbb{R}^N : a\le|g(v_n)|<b\right\}$$

and

$$C_a^b:=\inf\left\{\frac{\mathcal{F}(|x|,g(v))}{|g(v)|^2} : x\in\mathbb{R}^N \text{ and } v\in\mathbb{R} \text{ with } a\le|g(v)|<b\right\}.$$

Since $\mathcal{F}(|x|,u)>0$ if $u\ne0$, we have $C_a^b>0$ and

$$\mathcal{F}(|x|,g(v_n))\ge C_a^b|g(v_n)|^2.$$

Hence, from (2.6) and the above inequality, we can get

$$\begin{aligned}
C_4 &\ge \int_{\mathbb{R}^N}\mathcal{F}(|x|,g(v_n))dx\\
&\ge \int_{\Omega_n(0,r)}\mathcal{F}(|x|,g(v_n))dx+\int_{\Omega_n(r,+\infty)}\mathcal{F}(|x|,g(v_n))dx\\
&\ge \int_{\Omega_n(0,r)}\mathcal{F}(|x|,g(v_n))dx+\eta(r)\,\mathrm{meas}(\Omega_n(r,+\infty)),
\end{aligned}$$

which shows that $\mathrm{meas}(\Omega_n(r,+\infty))\to0$ as $r\to\infty$ uniformly in n. Thus, for any $s\in[2,22^*)$, by (11) in Lemma 2.1, Hölder's inequality and Sobolev's embedding, we get

$$\begin{aligned}\int_{\Omega_n(r,+\infty)} \widetilde{g}^s(v_n)dx &\le \left(\int_{\Omega_n(r,+\infty)} \widetilde{g}^{22^*}(v_n)dx\right)^{\frac{s}{22^*}} (\text{meas}(\Omega_n(r,+\infty)))^{\frac{22^*-s}{22^*}} \\ &= \frac{1}{S_n^s}\left(\int_{\Omega_n(r,+\infty)} g^{22^*}(v_n)dx\right)^{\frac{s}{22^*}} (\text{meas}(\Omega_n(r,+\infty)))^{\frac{22^*-s}{22^*}} \\ &\le \frac{C_4}{S_n^s}\left(\int_{\Omega_n(r,+\infty)} |\nabla g^2(v_n)|^2dx\right)^{\frac{s}{4}} (\text{meas}(\Omega_n(r,+\infty)))^{\frac{22^*-s}{22^*}} \\ &\le \frac{C_5}{S_n^s}\left(\int_{\Omega_n(r,+\infty)} |\nabla v_n|^2dx\right)^{\frac{s}{4}} (\text{meas}(\Omega_n(r,+\infty)))^{\frac{22^*-s}{22^*}} \\ &\le \frac{C_5}{S_n^{\frac{s}{2}}} (\text{meas}(\Omega_n(r,+\infty)))^{\frac{22^*-s}{22^*}} \to 0\end{aligned} \tag{2.7}$$

as $r \to \infty$ uniformly in n.

If $\sigma = 0$, then $\widetilde{g}(v_n) \to 0$ in $L^s(\mathbb{R}^N)$ for all $s \in (2, 2^*)$, and $\widetilde{g}(v_n) \to 0$ a.e. in $\mathbb{R}^N$. By virtue of (f_2), we can find some number $r_1 > 0$ such that $r_0 > r_1$ and

$$|f(|x|, u)| < \varepsilon|u|, \quad \text{for } |u| \le r_1,$$

where r_0 is given in (f_3'). Then

$$\begin{aligned}\int_{\Omega_n(0,r_1)} \frac{|F(|x|, g(v_n))|}{|g(v_n)|^2}|\widetilde{g}(v_n)|^2dx &\le \int_{\Omega_n(0,r_1)} \left(\frac{\frac{\varepsilon}{2}|g(v_n)|^2}{|g(v_n)|^2}\right)|\widetilde{g}(v_n)|^2dx \\ &\le \frac{\varepsilon}{2}\|\widetilde{g}(v_n)\|_2^2.\end{aligned} \tag{2.8}$$

It follows from (2.4) that

$$\begin{aligned}C_1 &\ge \int_{\Omega_n(0,r_1)} \mathcal{F}(|x|, g(v_n))dx + \int_{\Omega_n(r_1,r_0)} \mathcal{F}(|x|, g(v_n))dx + \int_{\Omega_n(r_0,+\infty)} \mathcal{F}(|x|, g(v_n))dx. \\ &\ge \int_{\Omega_n(0,r_1)} \mathcal{F}(|x|, g(v_n))dx + C_{r_1}^{r_0}\int_{\Omega_n(r_1,r_0)} |g(v_n)|^2dx + \int_{\Omega_n(r_0,+\infty)} \mathcal{F}(|x|, g(v_n))dx,\end{aligned}$$

thus we have

$$\int_{\Omega_n(r_1,r_0)} |g(v_n)|^2dx \le \frac{C_1}{C_{r_1}^{r_0}}. \tag{2.9}$$

By (f_1) and (f_2), for any $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ such that

$$|f(r,u)| \le \left(\varepsilon|u| + C_\varepsilon|u|^{p-1}\right) \quad \text{and} \quad |F(r,u)| \le \left(\frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{p}|u|^p\right), \tag{2.10}$$

and then

$$\begin{aligned}\int_{\Omega_n(r_1,r_0)} \frac{|F(|x|, g(v_n))|}{|g(v_n)|^2}|\widetilde{g}(v_n)|^2dx &\le \int_{\Omega_n(r_1,r_0)} \left(\frac{\frac{\varepsilon}{2}|g(v_n)|^2 + \frac{C_\varepsilon}{p}|g(v_n)|^p}{|g(v_n)|^2}\right)|\widetilde{g}(v_n)|^2dx \\ &\le \left(\frac{\varepsilon}{2} + \frac{C_\varepsilon}{p}r_0^{p-2}\right)\int_{\Omega_n(r_1,r_0)} |\widetilde{g}(v_n)|^2dx.\end{aligned} \tag{2.11}$$

By using (2.9), we have

$$\int_{\Omega_n(r_1,r_0)} |\widetilde{g}(v_n)|^2dx \le \frac{1}{S_n^2}\int_{\Omega_n(r_1,r_0)} |g(v_n)|^2dx \le \frac{1}{S_n^2}\frac{C_1}{C_{r_1}^{r_0}}. \tag{2.12}$$

Therefore, it follows from (2.11) and (2.12) that

$$\int_{\Omega_n(r_1,r_0)} \frac{|F(|x|,g(v_n))|}{|g(v_n)|^2}|\widetilde{g}(v_n)|^2 dx \leq \left(\frac{\varepsilon}{2}+\frac{C_\varepsilon}{p}r_0^{p-2}\right)\frac{1}{S_n^2}\frac{C_1}{C_{r_1}^{r_0}} \to 0, \quad \text{as } n\to\infty. \tag{2.13}$$

Let $\kappa' = \kappa/(\kappa-1)$. Since $\kappa > \max\{1, \frac{2N}{N+2}\}$, we obtain $2\kappa' \in (2, 22^*)$. Hence from (f_4'), (2.4) and (2.7), one has

$$\begin{aligned} &\int_{\Omega_n(r_0,\infty)} \frac{|F(|x|,g(v_n))|}{|g(v_n)|^2}|\widetilde{g}(v_n)|^2 dx \\ &\leq \left[\int_{\Omega_n(r_0,\infty)} \left(\frac{|F(|x|,g(v_n))|}{|g(v_n)|^2}\right)^\kappa dx\right]^{\frac{1}{\kappa}} \left[\int_{\Omega_n(r_0,\infty)} |\widetilde{g}(v_n)|^{2\kappa'} dx\right]^{\frac{1}{\kappa'}} \\ &\leq c_0^{\frac{1}{\kappa}} \left[\int_{\Omega_n(r_0,\infty)} \mathcal{F}(|x|,g(v_n))dx\right]^{\frac{1}{\kappa}} \left(\int_{\Omega_n(r_0,\infty)} |\widetilde{g}(v_n)|^{2\kappa'} dx\right)^{\frac{1}{\kappa'}} \\ &\leq [C_1 c_0]^{\frac{1}{\kappa}} \left(\int_{\Omega_n(r_0,\infty)} |\widetilde{g}(v_n)|^{2\kappa'} dx\right)^{\frac{1}{\kappa'}} \\ &\to 0. \end{aligned} \tag{2.14}$$

Thus it follows from (2.8), (2.13) and (2.14) that

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|F(|x|,g(v_n))|}{S_n^2} dx = &\int_{\Omega_n(0,r_1)} \frac{|F(|x|,g(v_n))|}{|g(v_n)|^2}|\widetilde{g}(v_n)|^2 dx \\ &+ \int_{\Omega_n(r_1,r_0)} \frac{|F(|x|,g(v_n))|}{|g(v_n)|^2}|\widetilde{g}(v_n)|^2 dx \\ &+ \int_{\Omega_n(r_0,\infty)} \frac{|F(|x|,g(v_n))|}{|g(v_n)|^2}|\widetilde{g}(v_n)|^2 dx \\ \to 0, &\quad \text{as } n\to\infty, \end{aligned} \tag{2.15}$$

which contradicts (2.5).

Now, we consider the case $\sigma \neq 0$. Set $A := \{x \in \mathbb{R}^N : \sigma(x) \neq 0\}$. Thus $\operatorname{meas}(A) > 0$. For a.e. $x \in A$, we have $\lim_{n\to\infty} |g(v_n(x))| = \infty$. Hence $A \subset \Omega_n(r_0,\infty)$ for large $n \in \mathbb{N}$, where r_0 is given in (f_3'). By (f_3'), we can get

$$\lim_{n\to\infty} \frac{F(|x|,g(v_n))}{|g(v_n)|^4} = +\infty.$$

It follows from Fatou's Lemma that

$$\lim_{n\to\infty} \int_A \frac{F(|x|,g(v_n))}{|g(v_n)|^4} dx = +\infty. \tag{2.16}$$

Hence, from (2.3), (2.10) and (2.16), we can get

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c+o(1)}{S_n^2} = \lim_{n\to\infty} \frac{\ell(v_n)}{S_n^2} \\
&= \lim_{n\to\infty} \frac{1}{S_n^2}\left(\frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla v_n|^2 + V(|x|)g^2(v_n)\right]dx - \int_{\mathbb{R}^N} F(|x|,g(v_n))dx\right) \\
&= \lim_{n\to\infty}\left[\frac{1}{2} - \int_{\Omega_n(0,r_0)} \frac{F(|x|,g(v_n))}{g^2(v_n)}|\widetilde{g}(v_n)|^2 dx - \int_{\Omega_n(r_0,\infty)} \frac{F(|x|,g(v_n))}{g^2(v_n)}|\widetilde{g}(v_n)|^2 dx\right] \\
&\le \frac{1}{2} + (\epsilon + C_\varepsilon r_0^{p-2})|\widetilde{g}(v_n)|_2^2 - \liminf_{n\to\infty}\int_A \frac{F(|x|,g(v_n))}{g^4(v_n)}|g(v_n)\widetilde{g}(v_n)|^2 dx \\
&\le \frac{1}{2} + (\varepsilon + C_\varepsilon r_0^{p-2})\gamma_2^2\|\widetilde{g}(v_n)\|_E^2 - \liminf_{n\to\infty}\int_A \frac{F(|x|,g(v_n))}{g^4(v_n)}|g(v_n)\widetilde{g}(v_n)|^2 dx \\
&\le \frac{1}{2} + (\varepsilon + C_\varepsilon r_0^{p-2})\gamma_2^2 - \liminf_{n\to\infty}\int_A \frac{F(|x|,g(v_n))}{g^4(v_n)}|g(v_n)\widetilde{g}(v_n)|^2 dx \\
&\to -\infty,
\end{aligned}
\tag{2.17}
$$

which is a contradiction. Thus there exists $C_2 > 0$ such that

$$S_n^2 = \int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(|x|)g^2(v_n)\right)dx \le C_2.$$

Next, we prove $\{v_n\}$ is bounded in E, i.e. we only need to prove that there exists $C_7 > 0$ such that

$$S_n^2 = \int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(|x|)g^2(v_n)\right)dx \ge C_7\|v_n\|_E^2. \tag{2.18}$$

Now, we may assume that $v_n \ne 0$ (otherwise, the conclusion is trivial). If this conclusion is not true, passing to a subsequence, we have $\frac{S_n^2}{\|v_n\|_E^2} \to 0$. Let $\omega_n = \frac{v_n}{\|v_n\|_E}$ and $h_n = \frac{g^2(v_n)}{\|v_n\|_E^2}$. Then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla\omega_n|^2 + V(|x|)h_n(x)\right)dx = 0.$$

Thus

$$\int_{\mathbb{R}^N}|\nabla\omega_n|^2 dx \to 0, \qquad \int_{\mathbb{R}^N} V(|x|)h_n(x)dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} V(|x|)\omega_n^2(x)dx \to 1.$$

Similar to the idea of [23], we assert that for each $\varepsilon > 0$, there exists $C_8 > 0$ independent of n such that $\operatorname{meas}(\Theta_n) < \varepsilon$, where $\Theta_n := \{x \in \mathbb{R}^N : |v_n(x)| \ge C_8\}$. Otherwise, there is an $\varepsilon_0 > 0$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that for any positive integer k,

$$\operatorname{meas}\left(\left\{x \in \mathbb{R}^N : |v_n(x)| \ge k\right\}\right) \ge \varepsilon_0 > 0.$$

Set $\Theta_{n_k} := \{x \in \mathbb{R}^N : |v_{n_k}(x)| \ge k\}$. By (9) in Lemma 2.1, we have

$$S_{n_k}^2 \ge \int_{\mathbb{R}^N} V(|x|)g^2(v_{n_k})dx \ge \int_{\Theta_{n_k}} V(|x|)g^2(v_{n_k})dx \ge C_9 k\varepsilon_0 \to +\infty,$$

as $k \to \infty$, which is a contradiction. Hence the assertion is true. Notice that as $|v_n(x)| \le C_8$, by (9) and (10) in Lemma 2.1, we have

$$C_{10}v_n^2 \le g^2\left(\frac{1}{C_8}v_n\right) \le C_{11}g^2(v_n).$$

Thus

$$\int_{\mathbb{R}^N\setminus\Theta_n} V(|x|)\omega_n^2 dx \le C_{12}\int_{\mathbb{R}^N\setminus\Theta_n} V(|x|)\frac{g^2(v_n)}{\|v_n\|_E^2}dx \le C_{12}\int_{\mathbb{R}^N} V(|x|)h_n(x)dx \to 0. \tag{2.19}$$

At last, by virtue of the integral absolutely continuity, there exists $\varepsilon > 0$ such that whenever $A' \subset \mathbb{R}^N$ and $\operatorname{meas}(A') < \varepsilon$,

$$\int_{A'} V(|x|)\omega_n^2 dx \le \frac{1}{2}. \tag{2.20}$$

It follows from (2.19) and (2.20) that

$$\int_{\mathbb{R}^N} V(|x|)\omega_n^2 dx = \int_{\mathbb{R}^N\setminus\Theta_n} V(|x|)\omega_n^2 dx + \int_{\Theta_n} V(|x|)\omega_n^2 dx \le \frac{1}{2} + o_n(1).$$

This yields that $1 \le \frac{1}{2}$, which is a contradiction. This implies that (2.18) holds. Hence $\{v_n\}$ is bounded in E. □

Lemma 2.4. *Suppose that (V'), (f_1), (f_2), (f_3') and (f_4') are satisfied. Then ℓ satisfies $(C)_c$-condition.*

Proof. By Lemma 2.3, it can conclude that $\{v_n\}$ is bounded in E. Going if necessary to a subsequence, we can assume that $v_n \rightharpoonup v$ in E. By the embedding, we have $v_n \to v$ in $L^s(\mathbb{R}^3)$ for all $2 < s < 2^*$.

Firstly, we prove that there exists $C_{13} > 0$ such that

$$\int_{\mathbb{R}^N} \left(|\nabla(v_n - v)|^2 + V(|x|)(g(v_n)g'(v_n) - g(v)g'(v))(v_n - v)\right) dx \ge C_{13}\|v_n - v\|_E^2. \tag{2.21}$$

Indeed, we may assume $v_n \neq v$ (otherwise the conclusion is trivial). Set

$$\omega_n = \frac{v_n - v}{\|v_n - v\|_E} \quad \text{and} \quad h_n = \frac{g(v_n)g'(v_n) - g(v)g'(v)}{v_n - v}.$$

We argue by contradiction and assume that

$$\int_{\mathbb{R}^N} |\nabla\omega_n|^2 + V(|x|)h_n(x)\omega_n^2 dx \to 0.$$

By

$$\frac{d}{dt}(g(t)g'(t)) = g(t)g''(t) + (g'(t))^2 = \frac{1}{(1+2g^2(t))^2} > 0,$$

then $g(t)g'(t)$ is strictly increasing and for each $C_{14} > 0$ there is $\delta > 0$ such that

$$\frac{d}{dt}(g(t)g'(t)) \ge \delta$$

as $|t| \le C_{14}$. Thus we can see that $h_n(x)$ is positive. Hence

$$\int_{\mathbb{R}^N} |\nabla\omega_n|^2 dx \to 0, \qquad \int_{\mathbb{R}^N} V(|x|)h_n(x)\omega_n^2 dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} V(|x|)\omega_n^2(x)dx \to 1.$$

By a similar fashion as (2.19) and (2.20), we can get a contradiction. This implies that (2.21) holds.

Secondly, by (2), (3), (8), (11) in Lemma 2.1 and (2.10), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \left(f(|x|,g(v_n))g'(v_n) - f(|x|,g(v))g'(v)\right)(v_n - v)dx\right| \\
&\quad\leq \int_{\mathbb{R}^N} \Big(\varepsilon|g(v_n)||g'(v_n)| + C_\varepsilon|g(v_n)|^{p-1}|g'(v_n)| \\
&\qquad\qquad +\varepsilon|g(v)||g(v)| + C_\varepsilon|g(v)|^{p-1}|g'(v)|\Big)|v_n - v|dx \\
&\quad\leq \varepsilon C_{15} + C_\varepsilon\left(\|v_n\|_{\frac{p}{2}}^{\frac{p-2}{2}} + \|v\|_{\frac{p}{2}}^{\frac{p-2}{2}}\right)\|v_n - v\|_{\frac{p}{2}} = o_n(1).
\end{aligned} \tag{2.22}$$

Hence together with (2.21) and (2.22), we get

$$\begin{aligned}
o_n(1) &= \langle \ell(u_n) - \ell(u), u_n - u\rangle \\
&= \int_{\mathbb{R}^3} |\nabla(u_n - u)|^2 dx + \int_{\mathbb{R}^N} V(|x|)\left(g(v_n)g'(v_n) - g(v)g'(v)\right)(v_n - v)dx \\
&\quad - \int_{\mathbb{R}^N} \left(f(|x|,g(v_n))g'(v_n) - f(|x|,g(v))g'(v)\right)(v_n - v)dx \\
&\geq C_{13}\|v_n - v\|_E^2 + o_n(1).
\end{aligned}$$

This implies $v_n \to v$ in E and this completes the proof. □

3 Proof of Theorem 1.4 and Theorem 1.5

To prove our results, we state the following symmetric mountain pass theorem.

Lemma 3.1 ([2, 26])**.** *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional. If $\ell \in C^1(X,\mathbb{R})$ satisfies $(C)_c$-condition for all $c > 0$, and*

(ℓ_1) $\ell(0) = 0$, $\ell(-u) = \ell(u)$ for all $u \in X$;

(ℓ_2) there exist constants $\rho, \alpha > 0$ such that $\ell|_{\partial B_\rho \cap Z} \geq \alpha$;

(ℓ_3) for any finite dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{X}) > 0$ such that $\ell(u) \leq 0$ on $\widetilde{X} \backslash B_R$;

then ℓ possesses an unbounded sequence of critical values.

Let $\{e_j\}$ is a total orthonormal basis of E and define $X_j = \mathbb{R}e_j$,

$$Y_k = \bigoplus_{j=1}^{k} X_j, \ Z_k = \bigoplus_{j=k+1}^{\infty} X_j, \ \forall\, k \in \mathbb{Z}.$$

Then $E = Y_n \oplus Z_n$, Y_n is a finite dimensional space.

Lemma 3.2. *Suppose that (V'), (f_1), (f_2), (f_3') and (f_4') are satisfied. Then there exist constant $\rho, \alpha > 0$ such that*

$$\ell|_{S_\rho \cap Z_m} \geq \alpha.$$

Proof. From (2.1), (3) and (8) in Lemma 2.1, for $u \in Z_m$ and $p \in (4, 22^*)$, we can choose ε small enough such that

$$\begin{aligned}
\ell(v) &= \frac{1}{2}\int_{\mathbb{R}^N}[|\nabla v|^2 + V(|x|)g^2(v)]dx - \int_{\mathbb{R}^N} F(|x|, g(v))dx \\
&\geq \frac{C_{14}}{2}\|g(v)\|_E^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^N}|g(v)|^2dx - \frac{C_\varepsilon}{p}\int_{\mathbb{R}^N}|g(v)|^p dx \\
&\geq \frac{C_{14}}{2}\|g(v)\|_E^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^N}|g(v)|^2dx - \frac{C'_\varepsilon}{p}\int_{\mathbb{R}^N}|g(v)|^{\frac{p}{2}} dx \\
&\geq C_{15}\left(\frac{1}{2}\|g(v)\|_E^2 - \frac{1}{4}\|g(v)\|_E^2 - \frac{1}{4}\|g(v)\|_E^{\frac{p}{2}}\right) \\
&\geq \frac{C_{15}}{4}\|g(v)\|_E^2\left(1 - \|g(v)\|_E^{\frac{p-4}{2}}\right) > 0.
\end{aligned}$$

This completes the proof. □

Lemma 3.3. *Suppose that (V'), (f_1), (f_2), (f'_3) and (f'_4) are satisfied. Then for any finite dimensional subspace $\widetilde{E} \subset E$, there exists constant $R = R(\widetilde{E}) > 0$ such that*

$$\ell(v) \to -\infty, \quad \|v\|_E \to \infty, \quad v \in \widetilde{E}.$$

Proof. Arguing indirectly, assume that for some sequence $\{v_n\} \subset \widetilde{E}$ with $\|v_n\|_E \to \infty$, there is $M > 0$ such that $\ell(v_n) \geq -M$ for all $n \in \mathbb{N}$. On the one hand, let $\omega_n = \frac{v_n}{\|v_n\|_E}$, then $\|\omega_n\|_E = 1$. Since $\widetilde{E}$ is finite dimensional, passing to a subsequence, then we assume that

$$\begin{aligned}
&\omega_n \rightharpoonup \omega \quad \text{in } E, \\
&\omega_n \to \omega \quad \text{in } L^s(\mathbb{R}^N) \text{ for } 2 < s < 2^*, \\
&\omega_n \to \omega \quad \text{a.e. } \mathbb{R}^N,
\end{aligned}$$

and so $\|\omega\|_E = 1$, which implies that $\omega \neq 0$. Let $\Lambda = \{x \in \mathbb{R}^N : \omega(x) \neq 0\}$, then $\text{meas}(\Lambda) > 0$. Since $|v_n| = |\omega_n|\|v_n\|_E$, by $\|v_n\|_E \to \infty$ and (4) in Lemma 2.1, then we have $|g(v_n)| \to \infty$. Therefore, by (2) in Lemma 2.1, we have

$$\begin{aligned}
S_n^2 &= \int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(|x|)g^2(v_n)\right)dx \\
&\geq \int_\Lambda \left(|\nabla g(v_n)|^2 + V(|x|)g^2(v_n)\right)dx \\
&\to \infty.
\end{aligned}$$

On the other hand, set $\widetilde{g}(v_n) = \frac{g(v_n)}{S_n}$, then $\|\widetilde{g}(v_n)\|_E \leq 1$. Passing to a subsequence, we may assume that $\widetilde{g}(v_n) \rightharpoonup \sigma$ in E, $\widetilde{g}(v_n) \to \sigma$ in $L^s(\mathbb{R}^N)$ for all $2 < s < 2^*$, $\widetilde{g}(v_n) \to \sigma$ a.e. on $\mathbb{R}^N$, and so $\|\sigma\|_E \leq 1$. Hence, we can conclude a contradiction by a similar fashion as (2.15) and (2.17). This completes the proof. □

Corollary 3.4. *Suppose that (V'), (f_1), (f_2), (f'_3) and (f'_4) are satisfied. Then for any $\widetilde{E} \subset E$, there exists $R = R(\widetilde{E}) > 0$, such that*

$$\ell(v) \leq 0, \quad \|v\|_E \geq R, \quad \forall\, v \in \widetilde{E}.$$

Proof of Theorem 1.4. Let $X = E$, $Y = Y_m$ and $Z = Z_m$. By Lemmas 2.3, 2.4, 3.2 and Corollary 3.4, all conditions of Lemma 3.1 are satisfied. Thus, problem (1.1) possesses has a sequence of radial solutions $\{v_n\}$ such that $\ell(v_n) \to \infty$ as $n \to \infty$, where $u_n = g(v_n)$. This completes the proof. □

Proof of Theorem 1.5. Using a similar way as Theorem 1.4, we can complete the proof of Theorem 1.5. □ □

4 Proof of Theorem 1.6

In this section, we want to prove Theorem 1.6. Before proving our results, we need the following mountain pass theorem without compactness (see [22], Theorem 1.15)

Lemma 4.1 ([22]). *Let X be an Hilbert space, $\ell \in C^1(X, \mathbb{R})$, $e \in X$, and $r > 0$ such that $\|e\| > r$ and $\inf_{\|v\|=r} \ell(v) > \ell(0) \geq \ell(e)$. Then there exists a sequence $\{v_n\}$ such that $\ell(v_n) \to c$ and $\ell'(v_n) \to 0$, where*

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \ell(\gamma(t)) > 0,$$

and $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0,\ \gamma(1) = e\}$.

The following lemma, has been proved in [30], which is very useful for the proof of Theorem 1.6.

Lemma 4.2 ([30]). *Let $\{\Omega_j\}_{j\in\mathbb{N}}$ be a sequence of open subsets of $\mathbb{R}$ such that*

(1) $\mathbb{R} = \cup_{j\in\mathbb{N}}\bar{\Omega}_j$ and $\Omega_i \cap \Omega_j = \emptyset$, if $i \neq j$.

(2) There exists a constant $c_0 > 0$ such that for $j \in \mathbb{N}$

$$\|v\|_{L^{\frac{10}{3}}(\Omega_j\times\mathbb{R}^4)} \leq c_0\|v\|_{H^1(\Omega_j\times\mathbb{R}^4)}, \quad \forall\ v \in H^1(\Omega_j \times \mathbb{R}^4).$$

Let $\{v_n\}$ be a bounded sequence of $H^1(\mathbb{R}^5)$. If

$$\sup_{j\in\mathbb{N}} \int_{\Omega_j\times\mathbb{R}^4} |v_n|^q \to 0, \ \ when\ n \to \infty,\ for\ q \in \left[2, \frac{10}{3}\right),$$

then $v_n \to 0$ in $L^s(\mathbb{R}^5)$, for all $2 < s < \frac{10}{3}$.

For the proof of Theorem 1.6, following [19], let G be a subgroup of $O(\mathbb{R}^4)$. It is obvious that $\mathbb{R}^4$ is compatible with G if for some $r > 0$,

$$m(y, r, G) = \lim_{|y|\to\infty} m(y, r, G) = +\infty,$$

where

$$m(y, r, G) := \sup_{n\in\mathbb{N}} \{n \in \mathbb{N} : \exists\, g_1, g_2, \ldots, g_n \in G \text{ such that } i \neq j \Rightarrow B(g_i(y)) \cap B(g_j(y)) = \emptyset\}.$$

Note that $\mathbb{R}^4$ is compatible with $O(\mathbb{R}^4)$ and $O(\mathbb{R}^2) \times O(\mathbb{R}^2)$ (see [22]). For simplicity of notation, we denote $G := O(\mathbb{R}^2) \times O(\mathbb{R}^2)$. We consider the action of G on $H^1(\mathbb{R}^5)$, defined by

$$(lu)(x, y) = u(x, l^{-1}y), \quad \text{where } (x, y) \in \mathbb{R} \times \mathbb{R}^4,\ \ l \in O(\mathbb{R}^4).$$

Let

$$H^1_G(\mathbb{R}^5) := \Big\{u \in H^1(\mathbb{R}^5) : lu = u, \ \ \forall\, l \in G\Big\}$$

and $\varsigma \in O(N)$ be the involution in $\mathbb{R}^5 = \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$ given by $\varsigma(x_1, x_2, x_3) = (x_2, x_1, x_3)$. We define an action of the group $G_1 := \{id, \varsigma\}$ on $H^1(\mathbb{R}^5)$ by

$$hu(x) = \begin{cases} u(x), & \text{if } h = id, \\ -u(h^{-1}x), & \text{if } h = \varsigma. \end{cases}$$

Let

$$H^1_{G_1}(\mathbb{R}^5) := \Big\{u \in H^1(\mathbb{R}^5) : lu = u, \ \forall\, l \in G_1\Big\}.$$

Set $E := H^1_G(\mathbb{R}^5) \cap H^1_{G_1}(\mathbb{R}^5)$. It is clear that $u = 0$ is only radial function in E, which is a Hilbert space with the inner product of $H^1(\mathbb{R}^5)$.

The following compactness result is due to [19].

Lemma 4.3 ([19])**.** *The imbedding $H^1_G(\mathbb{R}^5) \hookrightarrow L^s(\Omega \times \mathbb{R}^4)$ is compact, where Ω is a bounded subset of $\mathbb{R}$, $s \in (2, \frac{10}{3})$.*

Proof of Theorem 1.6. Similar to Lemma 3.2 and 3.3, it is easy to verify that ℓ satisfies the conditions of Lemma 4.1. Then, there exists a $(C)_c$-sequence $\{v_n\} \subset E$ i.e., $\{v_n\}$ satisfies

$$\ell(v_n) \to c \ \text{ and } (1 + \|v_n\|)\ell'(v_n) \to 0,$$

where c is the Mountain Pass value given in Lemma 4.1. Similarly as in [30,37], by Lemma 2.1, we have

$$\begin{aligned} o_n(1) &= \|\ell'(v_n)\|(1 + \|v_n\|) \\ &\geq \langle \ell'(v_n), v_n\rangle \\ &= \int_{\mathbb{R}^5} |\nabla v_n|^2 + \int_{\mathbb{R}^5} V(|x|)g(v_n)g'(v_n)v_n - \int_{\mathbb{R}^5} f(|x|, g(v_n))g'(v_n)v_n \\ &\geq C\|v_n\|^2 - C\int_{\mathbb{R}^5} |v_n|^{\frac{p}{2}}. \end{aligned} \tag{4.1}$$

Hence, without loss of generality, we can choose $\delta > 0$ such that for each n

$$\int_{\mathbb{R}^5} |v_n|^{\frac{p}{2}} \geq \|v_n\|^2 \geq \delta. \tag{4.2}$$

Otherwise, (4.1) implies that $v_n \to 0$ in E and hence $c = 0$, which leads to a contradiction. Let $\Omega_j = (j, j+1)$, then $\mathbb{R} = \cup_{j\in\mathbb{N}}\bar{\Omega}_j$. We may claim that there exists $\varrho > 0$ such that

$$\sup_{j\in\mathbb{N}} \int_{\Omega_j \times \mathbb{R}^4} v_n(x, y)dxdy \geq 2\varrho > 0.$$

Otherwise, by Lemma 4.2, we have $v_n \to 0$ in in $L^s(\mathbb{R}^5)$, where $2 < s < \frac{10}{3}$, which contradicts (4.2) since $2 < \frac{p}{2} < 2^*$. Hence, for every n, there exists j_n such that

$$\int_{\Omega_{j_n} \times \mathbb{R}^4} v_n(x, y)dxdy \geq \varrho > 0.$$

Making the change of variable $x = x' + j_n$, one has

$$\int_{\Omega \times \mathbb{R}^4} v_n(x' + j_n, y)dx'dy \geq \varrho > 0.$$

where $\Omega = (0,1)$. Let $w_n(x', y) = v_n(x' + j_n, y)$, then

$$\int_{\Omega\times\mathbb{R}^4} w_n(x', y)dx'dy \geq \varrho > 0. \tag{4.3}$$

Note that $\{v_n\}$ is also a $(C)_c$ sequence of ℓ. Hence

$$w_n \rightharpoonup w \ \text{ in } E,$$

$$w_n \to w \ \text{ in } L^s_{loc}(\mathbb{R}^5), \ \text{ where } 2 < s < \frac{10}{3},$$

$$w_n \rightharpoonup w \ \text{ a.e. on } \mathbb{R}^5.$$

It is standard to prove that w is a critical point of ℓ. Moreover, (4.3) implies $w \neq 0$, i.e., the problem (1.1) has a nonradial solution $v = g(w)$. This completes the proof. □

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