# Periodic Solutions of Semilinear Equations at Resonance with a $2n$-Dimensional Kernel

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Abstract. In this paper, we obtain some sufficient conditions for the existence of $2\pi$-periodic solutions of some semilinear equations at resonance where the kernel of the linear part has dimension $2n (n \geq 1)$. Our technique is essentially based on the Brouwer degree theory and Mawhin's coincidence degree theory.

Key words: Semilinear equation, resonance, periodic solution, kernel, dimension, Brouwer degree, coincidence degree.

1991 AMS Classification: 34K15

## 1. Introduction

For a long time, many authors have payed much attention to the existence problem of periodic solutions for the perturbed systems of ordinary as well as functional differential equations. In recent years, we see an increasing interest in the more difficult problem "a*t resonance*" in the sense that the associated linear homogenous system has a nontrivial periodic solution. In this side, some useful techniques, say the averaging method, have been developed and many significant results have been obtained for the existence of periodic solutions to some nonlinear systems of first order differential equations at resonance that involve a small parameter (see [1,2] and references therein).

Much research has also been devoted to the study of existence results for some nonlinear systems whose nonlinearities satisfy so-called Landesman-Lazer conditions. Several of these results are mentioned in [3]. However, less is known when the linear part has a two-dimensional kernel. Some work has been done by Lazer & Leach [4], Cesari[5], Iannacci & Nkashama[6], Nagle & Sinkala[7,8] and Ma, Wang & Yu[9]. To the best of our knowledge, few authors have considered the case when the linear part has dimension greater than two. In this direction, an example with

This project was supported by NNSF of China

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a three-dimensional kernel and a fourth order ordinary differential equation are considered in [8] and [10] respectively. In a recent paper[12], the results in [8] have been improved and unified by Ma, Wang & Yu.

This paper is concerned with the existence of $2\pi$-periodic solutions for the nonlinear system of first order functional differential equations of mixed type

$$\dot{x}_j(t) = B_j x_j(t) + F_j(t, x(t+\cdot)) + p_j(t), \quad j = 1, 2, \cdots, n \tag{1.1}$$

where $x_j(t) \in R^2, x(t+\cdot) \in BC(R, R^{2n})$ is defined by $x(t+s) = (x_1(t+s), x_2(t+s), \cdots, x_n(t+s))$, $p_j \in C(R, R^2)$ is $2\pi$-periodic, and $F_j : R \times BC(R, R^{2n}) \to R^2$ is continuous, bounded and $2\pi$-periodic in its first variable $t$. The constant matrix $B_j$ has a pair of purely imaginary eigenvalues $\pm im_j$ with $m_j$ some positive integer. Without loss of generality, we assume

$$B_j = \begin{pmatrix} 0 & m_j \\ -m_j & 0 \end{pmatrix}, \quad j = 1, 2, \cdots, n.$$

In this paper, we also need the following hypothesis

(F) There exists a permutation $k_1, k_2, \cdots, k_n$ consisting of $1, 2, \cdots, n$ and for any positive integer $j$ with $1 \le j \le n$, there exist $\tau_j \in R, H_j \in BC(R^2, R^2)$ with the asymptotic limits $H_j(\pm, \pm) = \lim_{r,s\to\pm\infty} H_j(r, s)$ and $G_j : R \times BC(R, R^{2n}) \to R^2$, which is continuous, bounded and $2\pi$-periodic with respect to its first variable $t$, such that for any $t \in R$ and $\varphi \in BC(R, R^{2n})$,

$$F_j(t, \varphi) = H_j(\varphi_{2k_j-1}(-\tau_j), \varphi_{2k_j}(-\tau_j)) + G_j(t, \varphi).$$

## 2. Main Results

In order to state our main results, we need some notations. For any positive integer $N$, we will denote by $|\cdot|$ the Euclidean norm in $R^N$. We always denote by $A$ the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $m$ and $l$ be some positive integers. If $p \in C(R, R^2)$ is $2\pi$-periodic, we set

$$p(m) := \frac{1}{2\pi} \int_0^{2\pi} e^{A^T(ms)} p(s) ds. \tag{2.1}$$

where "$T$" denotes the transpose and $e^{\cdot}$ denotes the exponential of an operater.

For $H \in C(R^2, R^2)$, whenever the asymptotic limits

$$H(\pm, \pm) = \lim_{r,s\to\pm\infty} H(r, s)$$

exist, we set

$$W^H := \frac{1}{2\pi}\left[\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} H(+,+) + \begin{pmatrix}-1 & -1\\ 1 & -1\end{pmatrix} H(+,-) + \begin{pmatrix}-1 & 1\\ -1 & -1\end{pmatrix} H(-,-) + \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} H(-,+)\right]; \tag{2.2}$$

$$W^H(m,l) := \frac{1}{2\pi}\left[\int_0^{\pi/2} L(m,l)(s)dsH(+,+) + \int_{\pi/2}^{\pi} L(m,l)(s)dsH(+,-) + \int_{\pi}^{3\pi/2} L(m,l)(s)dsH(-,-) + \int_{3\pi/2}^{2\pi} L(m,l)(s)dsH(-,+)\right], \tag{2.3}$$

where the matrix value mapping $L(m,l): R \to R^{2\times 2}$ is defined by

$$L(m,l)(s) = \frac{1}{l}\sum_{k=0}^{l-1} e^{A^T(\frac{m}{l}(s+2k\pi))}. \tag{2.4}$$

It is easy to verify that if $m = l$, then $W^H(m,l) = W^H$. Finally, let $X$ be a normed space, if $G : X \to R^N$ is continuous and bounded, we denote by $M_G$ the supremum of $G$, i.e.,

$$M_G := \sup_{x\in X} |G(x)|. \tag{2.5}$$

**Theorem 2.1.** *If, in addition to (F), we assume that for any* $1 \le j \le n$,

$$|m_j - m_{k_j}| < \frac{1}{2} m_{k_j} \tag{2.6}$$

*and*

$$|W^{H_j}(m_j, m_{k_j})| > \frac{1}{2}(M_{G_j} + |p_j(m_j)|) + \frac{1}{2}\left(\sum_{i=1}^{n}(M_{G_i} + |p_i(m_i)|)^2\right)^{1/2} \tag{2.7}$$

*hold, then Eq.(1.1) has at least one* $2\pi$*-periodic solution.*

The following is a direct corollary of Theorem 2.1.

**Corollary 2.1.** *If, in addition to (F), we assume that for any* $1 \le j \le n$, *(2.6) and*

$$|W^{H_j}(m_j, m_{k_j})| > \frac{1}{2} M_{G_j} + \frac{1}{2}\left(\sum_{i=1}^{n} M_{G_i}^2\right)^{1/2} + \left(\sum_{j=1}^{n} |p_j(m_j)|^2\right)^{1/2} \tag{2.8}$$

*hold, then Eq.(1.1) has at least one* $2\pi$*-periodic solution.*

## 3. Proof of Main Results

Let $X$ and $Z$ be real normed spaces with respective norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, and $L : \mathrm{d}omL \subset X \to Z$ be a linear Fredholm mapping of index zero. Let $P$ be a continuous projection in $X$ onto $\ker L, I-Q$ be a continuous projection in $Z$ onto $\mathrm{I}mL$, and $K_P : \mathrm{I}mL \to \mathrm{d}omL \cap \ker P$ be the (unique) pseudo-inverse of $L$ associated to $P$ in the sense that $LK_Pz = z$ for all $z \in \mathrm{I}mL$ and $PK_P = 0$. Let $J : \mathrm{I}mQ \to \ker L$ be an isomorphism. In addition, we assume that $N : X \to Z$ is $L$-completely continuous and that $\langle \cdot, \cdot \rangle$ is an inner product on $\ker L$.

The folowing useful lemma is proved in [11].

**Lemma 3.1 [11].** *Assume that* $\dim \ker L \geq 2$ *and there exist* $M > 0$*, a bounded open subset* $\Omega_0 \subset \ker L$ *with* $0 \in \Omega_0$ *and* $\partial\Omega_0$ *a connected subset in* $\ker L$*, such that the following conditions hold:*

*(I)* $\|K_P(I-Q)Nx\|_X \leq M, \quad$ *for all* $x \in X$;

*(II) For any* $x \in X$ *with* $Px \in \partial\Omega_0$ *and* $\|(I-P)x\|_X < M$,

$$\langle JQNx, JQNx \rangle > 0; \tag{3.1}$$

*(III) There exist a continuous mapping* $\eta : \bar{\Omega}_0 \to \ker L$ *and a family of continuous mappings* $\eta_i : \ker L \to \ker L \quad (i = 1, 2, \cdots, N)$ *satisfying*

$$\langle \eta_i(u), \eta_i(u) \rangle \;>\; 0, \quad \langle u, \eta_i(u) \rangle \neq 0 \;\; \text{for} \;\; i = 1, 2, \cdots, N \;\; \text{and} \;\; u \neq 0, \tag{3.2}$$

*such that for any* $u \in \partial\Omega_0$,

$$\langle JQNu - \eta(u), \eta_1\eta_2 \cdots \eta_N(u) \rangle \neq 0; \tag{3.3}$$

$$\langle JQNu, JQNu - \eta(u) \rangle \neq 0. \tag{3.4}$$

*Then* $Lx = Nx$ *has at least one solution* $x$ *satisfying*

$$Px \in \bar{\Omega}_0 \qquad \text{and} \qquad \|(I-P)x\|_X \leq M.$$

Let $N$ be a positive integer. Set $P_{2\pi}^{(N)} = \{x \in C(R, R^N) : x(t + 2\pi) = x(t), \forall t \in R\}$, $\|x\| = \sup_{t\in R} |x(t)| = \sup_{t\in[0,2\pi]} |x(t)|$. Then $P_{2\pi}^{(N)} \subset BC(R, R^N)$ is a Banach space.

Let $D = \mathrm{d}iag(B_1, B_2, \cdots, B_n)$, where

$$B_j = \begin{pmatrix} 0 & m_j \\ -m_j & 0 \end{pmatrix}.$$

Define the operator $L : P_{2\pi}^{(2n)} \to P_{2\pi}^{(2n)}$ by $Lx(t) = \dot{x}(t) - Dx(t)$,

$$\mathrm{d}omL = \{x \in P_{2\pi}^{(2n)} : \dot{x}(t) \ \ exists \ \ and \ \ is \ \ continuous\}.$$

It is not hard to check that $L$ is a Fredholm mapping of index zero. Let $J : \ker L \to \ker L$ be the identical operator and let $P = Q : P_{2\pi}^{(2n)} \to P_{2\pi}^{(2n)}$ be the projections defined by

$$Px(t) = \frac{1}{2\pi} e^{Dt} \int_0^{2\pi} e^{D^T s} x(s) ds. \tag{3.5}$$

Then the (unique) pseudo-inverse of $L$ associated to $P$, denoted by $K : \mathrm{I}mL \to \mathrm{d}omL \cap \ker P$, is a compact operator with $\|K\| \leq 2\pi$ (see [11] for details).

Define the operator $N : P_{2\pi}^{(2n)} \to P_{2\pi}^{(2n)}$ by

$$Nx(t) = (N_1 x(t), N_2 x(t), \cdots, N_n x(t)),$$

$$N_j x(t) = F_j(t, x(t + \cdot)) + p_j(t), \quad j = 1, 2, \cdots, n.$$

Then $N$ is continuous and takes bounded sets into bounded sets, and hence is $L$-completely continuous. Moreover, Eq.(1.1) is equivalent to the operater equation $Lx = Nx$.

It is easy to see that $H : R^{2n} \to \ker L$ defined by

$$H(a) = e^{Dt} a, \quad for \ \ a \in R^{2n}$$

is an isometric isomorphism. In this paper, we identify $a \in R^{2n}$ with its image $H(a) \in \ker L$, i.e., $H(a) = a, a \in R^{2n}$.

For the sake of convenience, we also introduce the following notations. Let $m, l$ be some positive integers and $H \in C(R^2, R^2)$. For any real number $\rho \geq 0$, we set

$$M^H(\rho) := \frac{1}{2\pi} \int_0^{2\pi} e^{A^T s} H((\rho \sin s, \rho \cos s)^T) ds; \tag{3.6}$$

$$M^H(\rho, m, l) := \frac{1}{2l\pi} \int_0^{2l\pi} e^{A^T (\frac{ms}{l})} H((\rho \sin s, \rho \cos s)^T) ds. \tag{3.7}$$

It is easy to know that for any positive integer $m$,

$$M^H(\rho, m, m) = M^H(\rho)$$

In what follows, the following lemmas are needed.

**Lemma 3.2 [11].** *Let $m, l$ be some positive integers and $0 < r_0 < 1$. If $H \in C(R^2, R^2)$ is bounded and the asymptotic limits $H(\pm, \pm) = \lim_{r,s \to \pm\infty} H(r, s)$ exist, then*

$$\lim_{\rho \to \infty} M^H(r\rho, m, l) = W^H(m, l) \tag{3.8}$$

*and*

$$\lim_{\rho \to \infty} M^H(r\rho) = W^H \tag{3.9}$$

*uniformly for $r$ in $[r_0, 1]$.*

**Lemma 3.3.** *For any permutation $k_1, k_2, \cdots, k_n$ consisting of $1, 2, \cdots, n$ there exists a family of continuous mappings $\eta_i : R^{2n} \to R^{2n}$ $(i = 1, 2, \cdots, N_1)$ with*

$$\langle \eta_i(u), \eta_i(u) \rangle > 0, \quad \langle u, \eta_i(u) \rangle > 0, \quad \textit{for} \quad u \in R^{2n} \backslash \{0\}, \tag{3.10}$$

*such that for any $a_j \in R^2$ $(j = 1, 2, \cdots, n)$,*

$$\eta_1 \eta_2 \cdots \eta_{N_1}(a_1, a_2, \cdots, a_n) = (a_{k_1}, a_{k_2}, \cdots, a_{k_n}) \tag{3.11}$$

*holds.*

**Proof.** It suffices to show that there exists a family of continuous mappings $\zeta_i : R^{2n} \to R^{2n}$ $(i = 1, 2, \cdots, n_1)$ with

$$\langle \zeta_i(u), \zeta_i(u) \rangle > 0, \quad \langle u, \zeta_i(u) \rangle > 0, \quad \text{for} \quad u \in R^{2n} \backslash \{0\}, \tag{3.12}$$

such that for any $a_j \in R^2$ $(j = 1, 2, \cdots, n)$ and $1 \leq j_1 < j_2 \leq n$,

$$\zeta_1 \zeta_2 \cdots \zeta_{n_1}(a_1, \cdots, a_{j_1}, \cdots, a_{j_2}, \cdots, a_n) = (a_1, \cdots, a_{j_2}, \cdots, a_{j_1}, \cdots, a_n), \tag{3.13}$$

Define $\zeta_i : R^{2n} \to R^{2n}$ $(i = 1, 2, \cdots, 6)$ by

$$\zeta_i(u) = (\zeta_i^{(1)}, \zeta_i^{(2)}, \cdots, \zeta_i^{(n)}),$$

$$\zeta_i^{(k)} = \begin{cases} u_k, & k \neq j_1, j_2 \\ \frac{\sqrt{2}}{2} u_{j_1} + \frac{\sqrt{2}}{2} u_{j_2}, & k = j_1 \\ -\frac{\sqrt{2}}{2} u_{j_1} + \frac{\sqrt{2}}{2} u_{j_2}, & k = j_2 \end{cases}, \quad i = 1, 2$$

$$\zeta_i^{(k)} = \begin{cases} u_k, & k \neq j_1 \\ ((\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2})u_{j_1}^T, (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})u_{j_1}^T), & k = j_1 \end{cases}, \quad i = 3, 4, 5, 6$$

where $u = (u_1, u_2, \cdots, u_n) \in R^{2n}, u_k \in R^2, k = 1, 2, \cdots, n.$

It is easy to see that $\zeta_i \quad (i = 1, 2, \cdots, 6)$ is continuous, moreover, (3.12) and (3.13) hold with $n_1 = 6$. This completes the proof.

We are now in a position to prove our main result.

**Proof of Theorem 2.1.** Let $M = 4\pi[(\sum_{j=1}^n M_{F_j}^2)^{1/2} + (\sum_{j=1}^n \|p_j\|^2)^{1/2}]$, then for any $x \in P_{2\pi}^{(2n)}$, $\|K(I-Q)Nx\| \leq M$, and hence the condition (I) of Lemma 3.1 holds.

Let $\rho > 0$, take

$$\Omega_0 = \{u \in \ker L : u = (r_1\rho a_1, r_2\rho a_2, \cdots, r_n\rho a_n), a_s \in \partial B_1(0) \subset R^2,$$

$$0 \leq r_s < 1, s = 1, 2, \cdots, n\}.$$

Then $\Omega_0$ is a bounded open set in $\ker L$ and

$$\partial\Omega_0 = \bigcup_{j=1}^n \{u \in \ker L : u = (r_1\rho a_1, r_2\rho a_2, \cdots, r_n\rho a_n), a_s \in \partial B_1(0) \subset R^2,$$
$$0 \leq r_s \leq 1, s = 1, 2, \cdots, n, r_j = 1\}$$

For $x \in P_{2\pi}^{(2n)}$ with $x_j(t) = r_j\rho e^{B_j t}a_j + \bar{x}_j(t), a_j \in \partial B_1(0) = \{a \in R^2 : |a| = 1\} \subset R^2; \bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in \mathrm{Im}L, \bar{x}_j(t) \in R^2$, $\|\bar{x}_j\| \leq M, j = 1, 2, \cdots, n$, it is not hard to verify that

$$JQNx = ((JQNx)_1, (JQNx)_2, \cdots, (JQNx)_n), \tag{3.14}$$

$$(JQNx)_j = e^{B_j^T \tau_j} Y_j(\rho, a_{k_j}, r_{k_j}) + X_j(x) + p_j(m_j), \tag{3.15}$$

where

$$X_j(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{B_j^T s} G_j(s, x(s+\cdot))ds, \tag{3.16}$$

$$Y_j(\rho, a_{k_j}, r_{k_j}) = \frac{1}{2\pi}\int_0^{2\pi} e^{B_j^T s} H_j(r_{k_j}\rho e^{B_{k_j}s}a_{k_j} + \bar{x}_{k_j}(s+\tau))ds. \tag{3.17}$$

By using the fact that $\|\bar{x}_{k_j}\| \le M$ and a similar argument used in the proof of Lemma 3.2, it is not hard to show that

$$\lim_{\rho\to\infty} |Y_j(\rho, a_{k_j}, 1)| = |W^{H_j}(m_j, m_{k_j})| \tag{3.18}$$

uniformly for $a_{k_j}$ in $\partial B_1(0) \subset R_2$ and $\|\bar{x}_{k_j}\| \le M$.

It follows from (2.7) and (3.16) that

$$|W^{H_j}(m_j, m_{k_j})| > M_{G_j} + |p_j(m_j)| \ge |X_j(x)| + |p_j(m_j)| \tag{3.19}$$

If $r_{j_0} = 1$ for some $j_0$, then (3.14), (3.15), (3.18) and (3.19) imply that for $\rho$ sufficiently large,

$$JQN(x) \neq 0.$$

Thus, we have proved that for $\rho$ sufficiently large,

$$JQN(x) \neq 0$$

for any $x \in P_{2\pi}^{(2n)}$ with $Px \in \partial\Omega_0$ and $\|(I-P)x\| \le M$, that is, the condition (II) of Lemma 3.1 also holds.

Define the mapping $\eta : \bar{\Omega}_0 \to \ker L$ by

$$\eta(u) = (\eta^{(1)}(u), \eta^{(2)}(u), \cdots, \eta^{(n)}(u)),$$

$$\eta^{(j)}(u) = \frac{1}{2\pi}\int_0^{2\pi} e^{B_j^T s} G_j(s, r_1\rho e^{B_1(s+\cdot)}a_1, \cdots, r_n\rho e^{B_n(s+\cdot)}a_n)ds + p_j(m_j),$$

$$u = (r_1\rho a_1, \cdots, r_n\rho a_n), a_j \in \partial B_1(0) \subset R^2, 0 \le r_j \le 1, j = 1, 2, \cdots, n.$$

Then it is easy to see that $\eta$ is continuous.

Let $\beta_j$ $(-\pi < \beta_j \le \pi)$ be defined by

$$\sin\beta_j = \frac{W_1^{H_j}(m_j, m_{k_j})}{|W^{H_j}(m_j, m_{k_j})|}, \qquad \cos\beta_j = \frac{W_2^{H_j}(m_j, m_{k_j})}{|W^{H_j}(m_j, m_{k_j})|}$$

here

$$W^{H_j}(m_j, m_{k_j}) = (W_1^{H_j}(m_j, m_{k_j}), W_2^{H_j}(m_j, m_{k_j})).$$

Let $N_2$ be a positive integer satisfying

$$\left|\frac{\tau_j - \beta_j/m_j}{N_2}\right| < \frac{\pi}{2m_j}, j = 1, 2, \cdots, n.$$

Define $\eta_i : \ker L \to \ker L \quad (i = 1, 2, \cdots, N_2)$ by

$$\eta_i(u) = (\eta_i^{(1)}(u), \eta_i^{(2)}(u), \cdots, \eta_i^{(n)}(u)),$$

$$\eta_i^{(j)}(u) = e^{B_j^T \gamma_j} u_j, \ \gamma_j = \frac{\tau_j - \beta_j/m_j}{N_2}, \qquad j = 1, 2, \cdots, n,$$

where $u = (u_1, u_2, \cdots, u_n), u_j \in R^2 \quad (j = 1, 2, \cdots, n)$. Then it is clear that $\eta_i \quad (i = 1, 2, \cdots, N_2)$ is continuous and (3.2) holds.

By Lemma 3.3, we can also define $\eta_i : \ker L \to \ker L \quad (i = N_2 + 1, \cdots, N_2 + N_1)$, which are continuous and satisfy (3.2), such that

$$\eta_{N_2+1}\eta_{N_2+2} \cdots \eta_{N_2+N_1}(a_1, a_2, \cdots, a_n) = (a_{k_1}, a_{k_2}, \cdots, a_{k_n})$$

holds for any $a_j \in R^2 \quad (j = 1, 2, \cdots, n)$.

In the sequel, we assume that $u = (r_1\rho a_1, r_2\rho a_2, \cdots, r_n\rho a_n) \in \partial\Omega_0, \ a_j \in \partial B_1(0) \subset R^2, 0 \leq r_j \leq 1 \quad j = 1, 2, \cdots, n$. Clearly, we may assume, without loss of generality, that $r_{k_{j_0}} = 1$ for some $j_0$.

Let $\alpha_j = \alpha_j(a_j) \quad (-\pi < \alpha_j \leq \pi)$ defined by $\sin\alpha_j = a_j^{(1)}, \cos\alpha_j = a_j^{(2)}$, where $a_j = (a_j^{(1)}, a_j^{(2)})$.

Therefore, we have

$$\bar{\eta}(u) := \eta_1\eta_2 \cdots \eta_{N_2+N_1}(u) = (\bar{\eta}_1(u), \bar{\eta}_2(u), \cdots, \bar{\eta}_n(u)), \tag{3.20}$$

$$\bar{\eta}_j(u) = r_{k_j}\rho e^{B_j^T(\tau_j - \beta_j/m_j)} a_{k_j} = \rho r_{k_j} e^{B_j^T \tau_j} e^{A\alpha_{k_j}} W^{H_j}(m_j, m_{k_j})/|W^{H_j}(m_j, m_{k_j})| \tag{3.21}$$

$$\bar{\xi}(u) := JQNu - \eta(u) = (\bar{\xi}_1(u), \bar{\xi}_2(u), \cdots, \bar{\xi}_n(u)), \tag{3.22}$$

$$\begin{aligned} \bar{\xi}_j(u) &= \frac{1}{2\pi} e^{B_j^T \tau_j} \int_0^{2\pi} e^{B_j^T s} H_j(r_{k_j}\rho e^{B_{k_j} s} a_{k_j}) ds \\ &= e^{B_j^T \tau_j} e^{A(m_j \alpha_{k_j}/m_{k_j})} M^{H_j}(r_{k_j}\rho, m_j, m_{k_j}). \end{aligned} \tag{3.23}$$

It follows from (3.20)-(3.23) that

$$\begin{aligned} &\langle \bar{\xi}(u), \bar{\eta}(u) \rangle \\ &= \rho \sum_{j=1}^{n} \frac{r_{k_j}}{|W^{H_j}(m_j, m_{k_j})|} \langle W^{H_j}(m_j, m_{k_j}), e^{A\varpi(j)\alpha_{k_j}} M^{H_j}(r_{k_j}\rho, m_j, m_{k_j}) \rangle. \end{aligned} \tag{3.24}$$

where $\varpi(j) = \frac{m_j - m_{k_j}}{m_{k_j}}$.

Since $|\alpha_{k_j}| \le \pi$ and $|m_j - m_{k_j}| < \frac{1}{2} m_{k_j}$, we have

$$\left|\varpi(j)\alpha_{k_j}\right| \le |\varpi(j)\pi| < \frac{\pi}{2}, \quad j = 1, 2, \cdots, n.$$

Hence,

$$\langle W^{H_j}(m_j, m_{k_j}), e^{A\varpi(j)\alpha_{k_j}} W^{H_j}(m_j, m_{k_j})\rangle = |W^{H_j}(m_j, m_{k_j})|^2 \cos\left(\varpi(j)\alpha_{k_j}\right)$$

$$\ge |W^{H_j}(m_j, m_{k_j})|^2 \cos(\varpi(j)\pi) > 0, \tag{3.25}$$

For $j \ne j_0$, we set

$$I_j^0 = \left[0, |W^{H_{j_0}}(m_{j_0}, m_{k_{j_0}})| \cos(\varpi(j_0)\pi)/(4nM_{H_j})\right),$$

$$I_j^1 = \left[|W^{H_{j_0}}(m_{j_0}, m_{k_{j_0}})| \cos(\varpi(j_0)\pi)/(4nM_{H_j}), 1\right],$$

then by (3.24), and noting that $r_{k_{j_0}} = 1$, we have

$$\langle \bar{\xi}(u), \bar{\eta}(u)\rangle = \rho\left[Z_0 + Z_1 + Z_2\right] \tag{3.26}$$

where,

(3.27)
$$Z_0 = \frac{1}{|W^{H_{j_0}}(m_{j_0}, m_{k_{j_0}})|}\langle W^{H_{j_0}}(m_{j_0}, m_{k_{j_0}}), e^{A(\varpi(j_0)\alpha_{k_{j_0}})} M^{H_{j_0}}(\rho, m_{j_0}, m_{k_{j_0}})\rangle$$

(3.28)
$$Z_1 = \sum_{j \ne j_0, r_{k_j} \in I_j^0} \frac{r_{k_j}}{|W^{H_j}(m_j, m_{k_j})|}\langle W^{H_j}(m_j, m_{k_j}), e^{A(\varpi(j)\alpha_{k_j})} M^{H_j}(r_{k_j}\rho, m_j, m_{k_j})\rangle$$

(3.29)
$$Z_2 = \sum_{j \ne j_0, r_{k_j} \in I_j^1} \frac{r_{k_j}}{|W^{H_j}(m_j, m_{k_j})|}\langle W^{H_j}(m_j, m_{k_j}), e^{A(\varpi(j)\alpha_{k_j})} M^{H_j}(r_{k_j}\rho, m_j, m_{k_j})\rangle$$

Since $M^{H_j}(r_{k_j}\rho, m_j, m_{k_j}) \to W^{H_j}(m_j, m_{k_j})$ $(\rho \to \infty)$ uniformly for $r_{k_j}$ in $I_j^1$ by Lemma 3.2, we have

$$Z_2 > 0, \tag{3.30}$$

for large $\rho$.

By the Schwartz inequility, we also find

$$|Z_1| \le \sum_{j\neq j_0, r_{k_j}\in I_j^0} r_{k_j} M_{H_j} \le |W^{H_{j_0}}(m_{j_0}, m_{k_{j_0}})| \cos(\varpi(j_0)\pi)/4. \tag{3.31}$$

Therefore, it follows from (3.26)-(3.31) that for $\rho$ suffciently large,

$$\langle \bar{\xi}(u), \bar{\eta}(u)\rangle > 0.$$

Thus, (3.3) holds for any $u \in \partial\Omega_0$.

On the other hand, it is not hard to show that

$$\begin{aligned}
&\langle JQNu, JQNu - \eta(u)\rangle \\
&= \sum_{j=1}^{n} |M^{H_j}(r_{k_j}\rho, m_j, m_{k_j})|^2 \\
&+ \sum_{j=1}^{n} \langle M^{H_j}(r_{k_j}\rho, m_j, m_{k_j}), e^{A^T(m_j\alpha_{k_j}/m_{k_j})} e^{B_j\tau_j} X_j(u)\rangle \\
&+ \sum_{j=1}^{n} \langle M^{H_j}(r_{k_j}\rho, m_j, m_{k_j}), e^{A^T(m_j\alpha_{k_j}/m_{k_j})} e^{B_j\tau_j} p_j(m_j)\rangle,
\end{aligned}$$

where

$$X_j(u) = \frac{1}{2\pi}\int_0^{2\pi} e^{B_j^T s} G_j(s, r_1\rho e^{B_1(s+\cdot)}a_1, \cdots, r_n\rho e^{B_n(s+\cdot)}a_n)ds.$$

By using the Schwartz inequality, it follows that

$$\begin{aligned}
\langle JQNu, JQNu - \eta(u)\rangle &\ge \sum_{j=1}^{n} |M^{H_j}(r_{k_j}\rho, m_j, m_{k_j})|^2 \\
&\quad - \sum_{j=1}^{n} |M^{H_j}(r_{k_j}\rho, m_j, m_{k_j})|[M_{G_j} + |p_j(m_j)|] \\
&= \sum_{j=1}^{n} [|M^{H_j}(r_{k_j}\rho, m_j, m_{k_j})| - \frac{1}{2}(M_{G_j} + |p_j(m_j)|)]^2 \\
&\quad - \frac{1}{4}\sum_{j=1}^{n} (M_{G_j} + |p_j(m_j)|)^2.
\end{aligned}$$

Since $r_{k_{j_0}} = 1$ for some $j_0$, and $M^{H_{j_0}}(\rho, m_{j_0}, m_{k_{j_0}}) \to W^{H_{j_0}}(m_{j_0}, m_{k_{j_0}})$ $(\rho \to \infty)$, it follows from (2.7) that for $\rho$ sufficiently large,

$$\langle JQNu, JQNu - \eta(u)\rangle > 0.$$

Thus, (3.4) also holds for any $u \in \partial\Omega_0$.

By virtue of Lemma 3.1, Eq.(2.1) has at least one $2\pi$-periodic solution and the proof is complete.

Finally, we give an example to illustrate our main results.

**Example** Consider the system

$$
\begin{cases}
x_1' = x_2 + x_3/(1+x_3^2) + \arctan x_1 + p_1(t) \\
x_2' = -x_1 + 3\arctan x_4 + \dfrac{1}{2}\arctan x_2 + p_2(t) \\
x_3' = x_4 + x_5 e^{-x_5^2} + \sqrt{2}\sin x_3 + p_3(t) \\
x_4' = -x_3 - \sqrt{6}\arctan x_6 + \sqrt{2}\cos x_3 + p_4(t) \\
x_5' = x_6 + \arctan x_5 - 2\arctan x_1 + p_5(t) \\
x_6' = -x_5 + \dfrac{1}{2}\arctan x_6 - 2\arctan x_2 + p_6(t)
\end{cases}
\tag{3.32}
$$

where $p_j (j = 1, 2, \cdots, 6)$ are continuous, $2\pi$-periodic functions. By Corollary 2.1, it is easy to check that Eq.(3.32) has at least one $2\pi-$periodic solution provided

$$
\sqrt{|c_1|^2 + |c_2|^2 + |c_3|^2} < \sqrt{6} - \frac{\sqrt{2}}{2} - \frac{1}{4\sqrt{2}}\sqrt{5\pi^2 + 32},
$$

where

$$
c_k = \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} p_{2k-1}(s) \\ p_{2k}(s) \end{pmatrix} ds, \quad k = 1, 2, 3.
$$

## References


1. J. K. Hale, *Ordnary Differential Equations*, Wiley Interscience, New York, 1969.
2. R. K. Nagle, *Nonlinear boundary value problems for ordinary differential equations with a small parameter*, SIAM J. Math. Analysis **9** (1978), 719-729.
3. S. Fucik, *Solvability of Nonlinear Equations and Boundary Value Problems*, D. Reidel Publishing, Dordrecht, Holland, 1980.
4. A. C. Lazer & D. E. Leach, *Bounded perturbations of forced harmonic oscillations at resonance*, Ann. Mat. Pura. Appl., **82** (1969), 49-68.
5. L. Cesari, *Non-linear problems across a point of resonance for non-self-adjoint systems*, Acdemic Press (1978), 43-67.
6. *R. Iannacci & M. N. Nkashma, Unbounded perturbations of forced second order ordinary differential equations at resonance*, J. Diff. Eqns. **69** (1978), 289-309.
7. R. K. Nagle & Z. Sinkala, *Semilinear equations at resonance where the kernel has dimension two*, in Differential Equations:Stability and Control (Edited by S.Elaydi) Marcel Dekker, New York (1991).

8. R. K. Nagle & Z. Sinkala, *Existence of $2\pi$-periodic solutions for nonlinear systems of first-order ordinary differential equations at resonance*, Nonlinear Analysis (TMA) **25** (1995), 1-16.
9. S. W. Ma, Z. C. Wang & J. S. Yu, *Coincidence degree and periodic solutions of Duffing equations*, Nonlinear Analysis (TMA) **34** (1998), 443-460.
10. J. D. Schuur, *Perturbation at resonance for a fourth order ordinary differential equation*, J. Math. Anal. Appl. **65** (1978), 20-25.
11. S. W. Ma, Z. C. Wang & J. S. Yu, An abstract existence theorem at resonance and its applications, J. Diff. Eqns., **145** (1998), 274-294.