Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **117**, 1–13; doi: 10.14232/ejqtde.2016.1.117 http://www.math.u-szeged.hu/ejqtde/

On the representation of solutions of delayed differential equations via Laplace transform

Michal Pospíšil ✉ [1, 2] and **František Jaroš** [2]

[1] Mathematical Institute of Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia
[2] Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia

Received 9 May 2016, appeared 13 December 2016
Communicated by Ivan Kiguradze

Abstract. In this paper, the unilateral Laplace transform is used to derive a closed-form formula for a solution of a system of nonhomogeneous linear differential equations with any finite number of constant delays and linear parts given by pairwise permutable matrices. This unifies the recent results on the representation of such solutions.

Keywords: delay equation, multiple delays, matrix polynomial.

2010 Mathematics Subject Classification: 34A30, 34K06.

1 Introduction

Recently, the method of steps [5] was applied in [6] to obtain representation of solutions of differential equations with one delay. We recall this result.

Theorem 1.1. *Let $\tau > 0$, B be $N \times N$ matrix, and $\varphi \in C^1([-\tau, 0], \mathbb{R}^N)$, $f : [0, \infty) \to \mathbb{R}^N$ be given functions. Then the solution of the Cauchy problem consisting of the equation*

$$\dot{x}(t) = Bx(t-\tau) + f(t), \qquad t \geq 0$$

and initial condition

$$x(t) = \varphi(t), \qquad t \in [-\tau, 0] \tag{1.1}$$

has the form

$$x(t) = \mathrm{e}_\tau^{Bt}\varphi(-\tau) + \int_{-\tau}^{0} \mathrm{e}_\tau^{B(t-\tau-s)}\varphi'(s)ds + \int_0^t \mathrm{e}_\tau^{B(t-\tau-s)} f(s)ds$$

for any $t \geq -\tau$, where e_τ^{Bt} is the delayed matrix exponential defined as

$$\mathrm{e}_\tau^{Bt} = \begin{cases} \Theta, & t < -\tau, \\ \mathbb{I}, & -\tau \leq t < 0, \\ \mathbb{I} + Bt + B^2\frac{(t-\tau)^2}{2} + \dots + B^k\frac{(t-(k-1)\tau)^k}{k!}, & (k-1)\tau \leq t < k\tau,\ k \in \mathbb{N}, \end{cases}$$

Θ and $\mathbb{I}$ are the $N \times N$ zero and identity matrix, respectively.

✉ Corresponding author. Email: Michal.Pospisil@fmph.uniba.sk

This result was generalized in [8] for the case of $n \in \mathbb{N}$ constant delays and C^1 initial function. Nevertheless, we rather recall a simplified result from [9] that was originally published for equations with variable coefficients, time-dependent delays and only continuous function φ.

Theorem 1.2. *Let $n \in \mathbb{N}$, $0 < \tau_1, \dots, \tau_n \in \mathbb{R}$, $\tau := \max\{\tau_1, \tau_2, \dots, \tau_n\}$, $B_1, \dots, B_n$ be pairwise permutable $N \times N$ matrices, i.e. $B_iB_j = B_jB_i$ for each $i, j \in \{1, \dots, n\}$, $\varphi \in C([-\tau, 0], \mathbb{R}^N)$, and $f : [0, \infty) \to \mathbb{R}^N$ be a given function. Then the solution of the Cauchy problem consisting of the equation*

$$\dot{x}(t) = B_1x(t - \tau_1) + B_2x(t - \tau_2) + \dots + B_nx(t - \tau_n) + f(t), \qquad t \geq 0 \tag{1.2}$$

and initial condition (1.1) possesses the form

$$x(t) = \begin{cases} \varphi(t), & -\tau \leq t < 0, \\ X_n(t)\varphi(0) + \int_0^t X_n(t-s) \sum_{m=1}^n B_m \psi(s - \tau_m) ds & \\ \quad + \int_0^t X_n(t-s) f(s) ds, & 0 \leq t \end{cases} \tag{1.3}$$

where

$$\psi(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0), \\ \theta, & t \notin [-\tau, 0), \end{cases} \tag{1.4}$$

θ is the N-dimensional vector of zeros, and $X_n(t) = \mathrm{e}_{\tau_1,\tau_2,\dots,\tau_n}^{B_1,B_2,\dots,B_n(t-\tau_n)}$ is the multi-delayed matrix exponential given by

$$\mathrm{e}_{\tau_1,\dots,\tau_j}^{B_1,\dots,B_jt} = \begin{cases} \Theta, & t < -\tau_j, \\ X_{j-1}(t + \tau_j), & -\tau_j \leq t < 0, \\ X_{j-1}(t + \tau_j) + B_j \int_0^t X_{j-1}(t - s_1) X_{j-1}(s_1) ds_1 + \cdots & \\ \quad \cdots + B_j^k \int_{(k-1)\tau_j}^t \int_{(k-1)\tau_j}^{s_1} \cdots \int_{(k-1)\tau_j}^{s_{k-1}} X_{j-1}(t - s_1) & \\ \quad \times \prod_{i=1}^{k-1} X_{j-1}(s_i - s_{i+1}) X_{j-1}(s_k - (k-1)\tau_j) ds_k \dots ds_1, & \\ & (k-1)\tau_j \leq t < k\tau_j,\ k \in \mathbb{N} \end{cases} \tag{1.5}$$

for each $j = 2, 3, \dots, n$, where $X_{j-1}(t) = \mathrm{e}_{\tau_1,\dots,\tau_{j-1}}^{B_1,\dots,B_{j-1}(t-\tau_{j-1})}$.

Formula (1.3) was used in [8] to derive stability results. So it is usable for theoretical purposes. However, it does not seem to be very suitable for practical calculation of a solution, as the multi-delayed matrix exponential is built up inductively.

In the present paper, we provide another representation of solutions of linear nonhomogeneous differential equations with any finite number of delays in the sense of the following definition.

Definition 1.3. Let $n \in \mathbb{N}$, $0 < \tau_1, \dots, \tau_n \in \mathbb{R}$, $\tau := \max\{\tau_1, \tau_2, \dots, \tau_n\}$, $\varphi \in C([-\tau, 0], \mathbb{R}^N)$, $B_1, \dots, B_n$ be $N \times N$ matrices, and $f : [0, \infty) \to \mathbb{R}^N$ be a given function. The function $x : [-\tau, \infty) \to \mathbb{R}^N$ is a solution of the Cauchy problem (1.2), (1.1), if $x \in C^1([0, \infty), \mathbb{R}^N)$ (at $t = 0$ the derivative in equation (1.2) represents the right-hand derivative), $x(t)$ solves equation (1.2) on $[0, \infty)$ and satisfies the condition (1.1).

To obtain the representation we involve the unilateral Laplace transform. Of course, the idea to apply the Laplace transform to delay differential equations is not a new one. For instance in [2], the Laplace transform of a solution of a linear delayed differential equation is expressed using a Laplace transform of its fundamental solution. Rather than focusing on the Laplace image of a solution, in this paper we make use of properties of the Laplace transform and its inverse [4, 10], in particular of the uniqueness of the inverse on the set of continuous functions. So we obtain a closed-form formula for the solution.

The paper is organized as follows. The next section concludes some known and basic results on the Laplace transform. Section 3 contains our main results on the representation of a solution of (1.2), (1.1) which is another extension of Theorem 1.1 to the case of multiple delays (clearly equivalent to Theorem 1.2). Here we consider also the equation

$$\dot{x}(t) = Ax(t) + B_1x(t-\tau_1) + \cdots + B_nx(t-\tau_n) + f(t), \qquad t \geq 0 \tag{1.6}$$

with the initial condition (1.1), and derive the representation of its solution (see [6] for the case of one delay). This section is enclosed by an example.

In the whole paper we shall denote $|\cdot|$ the norm of a vector without any respect to its dimension. Further, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of all positive and nonnegative integers, respectively. We also assume the property of an empty sum, $\sum_{i\in\emptyset} z(i) = 0$ for any function z.

2 Preliminary results

The main tool we use in our computations is the unilateral Laplace transform defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-pt} f(t)dt$$

for $\operatorname{Re} p > a$ and an exponentially bounded function f such that $|f(t)| \leq c\mathrm{e}^{at}$ for all $t \geq 0$ and some constants $a, c \in \mathbb{R}$. For the case of brevity we sometimes adopt the notation $F(p) = \mathcal{L}\{f(t)\}$. Then $f(t) = \mathcal{L}^{-1}\{F(p)\}$. Note that here the preimage is assumed to vanish on $(-\infty, 0)$, which we emphasize by $\mathcal{L}^{-1}\{F(p)\} = f(t)\sigma(t)$, when needed. Recall σ is the Heaviside step function defined as

$$\sigma(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

Moreover, we apply $\mathcal{L}$ (and $\mathcal{L}^{-1}$) to each coordinate when considering the Laplace transform (or its inverse) of a vector.

The next lemma concludes some of properties of the Laplace transform (see e.g. [4, 10]).

Lemma 2.1. *The following equalities hold true for sufficiently large* $\operatorname{Re} p$ *and appropriate functions* f, g*:*

1. $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$ *for constants* $a, b \in \mathbb{R}$,
2. $\mathcal{L}^{-1}\left\{\frac{\mathrm{e}^{-p\tau}}{p}\right\} = \sigma(t-\tau)$ *for* $\tau \geq 0$,
3. $\mathcal{L}^{-1}\{F(p)G(p)\} = (f * g)(t)$ *for convolution operator* $*$,
4. $\mathcal{L}\{f'(t)\} = p\mathcal{L}\{f(t)\} - f(0)$,
5. $\mathcal{L}^{-1}\{1\} = \delta(t)$ *where* $\delta(t)$ *is Dirac delta distribution.*

Note that due to the arguments preceding the above lemma, point (3) of the lemma can be written as

$$\mathcal{L}^{-1}\{F(p)G(p)\} = ((f\sigma) * (g\sigma))(t) = \int_0^t f(s)g(t-s)ds.$$

The next two lemmas are corollaries of the latter one.

Lemma 2.2. *The following identities hold true for sufficiently large* Re *p:*

1. $\mathcal{L}^{-1}\{F_1(p)F_2(p)\dots F_n(p)\} = (f_1 * f_2 * \dots * f_n)(t)$ *for* $n \in \mathbb{N}$, $n \geq 2$ *and appropriate functions* $f_1, f_2, \dots, f_n$,
2. $\mathcal{L}^{-1}\left\{\left(\frac{e^{-p\tau}}{p}\right)^n\right\} = \frac{(t-n\tau)^{n-1}}{(n-1)!}\sigma(t-n\tau)$ *for* $\tau > 0$, $n \in \mathbb{N}$.

Proof. (1) If $n = 2$, the statement coincides with Lemma 2.1.3. On suppose that the statement holds for $n = k$, using Lemma 2.1, one obtains

$$\begin{aligned}\mathcal{L}^{-1}\{F_1(p)F_2(p)\dots F_{k+1}(p)\} &= (\mathcal{L}^{-1}\{F_1(p)F_2(p)\dots F_k(p)\} * f_{k+1})(t)\\ &= (f_1 * f_2 * \dots * f_{k+1})(t).\end{aligned}$$

(2) If $n = 1$ the statement becomes Lemma 2.1.2. Now, suppose that it holds for $n = k$. Lemma 2.1 yields

$$\begin{aligned}\mathcal{L}^{-1}\left\{\left(\frac{e^{-p\tau}}{p}\right)^{k+1}\right\} &= \left(\mathcal{L}^{-1}\left\{\left(\frac{e^{-p\tau}}{p}\right)^{k}\right\} * \mathcal{L}^{-1}\left\{\frac{e^{-p\tau}}{p}\right\}\right)(t)\\ &= \int_0^t \frac{(s-k\tau)^{k-1}}{(k-1)!}\sigma(s-k\tau)\sigma(t-s-\tau)ds\\ &= \int_{k\tau}^{t-\tau} \frac{(s-k\tau)^{k-1}}{(k-1)!}ds\, \sigma(t-(k+1)\tau) = \frac{(t-(k+1)\tau)^k}{k!}\sigma(t-(k+1)\tau)\end{aligned}$$

what was to be proved. □

Lemma 2.3. *Let* $n \in \mathbb{N}$, $0 < \tau_1, \tau_2, \dots, \tau_n \in \mathbb{R}$, $k_1, k_2, \dots, k_n \in \mathbb{N}_0$. *Then*

$$\begin{aligned}&\mathcal{L}^{-1}\left\{\prod_{m=1}^{n}\left(\frac{e^{-p\tau_m}}{p}\right)^{k_m}\right\}\\ &\quad= \begin{cases}\delta(t), & k_1 = k_2 = \dots = k_n = 0,\\ \frac{(t-\sum_{m=1}^n k_m\tau_m)^{\sum_{m=1}^n k_m - 1}}{(\sum_{m=1}^n k_m - 1)!}\,\sigma\left(t - \sum_{m=1}^{n} k_m\tau_m\right), & k_1 + k_2 + \dots + k_n \in \mathbb{N}.\end{cases}\end{aligned} \tag{2.1}$$

Proof. We shall prove the statement by mathematical induction with respect to n. For $n = 1$, (2.1) is obtained from Lemma 2.1.5 and Lemma 2.2.2. Now, suppose that the statement holds with $n = l$. For simplicity we denote L_n the left-hand side of (2.1). Then we expand as in the proof of Lemma 2.2.2,

$$L_{l+1} = \left(L_l * \mathcal{L}^{-1}\left\{\left(\frac{e^{-p\tau_{l+1}}}{p}\right)^{k_{l+1}}\right\}\right)(t). \tag{2.2}$$

Using the inductive hypothesis and Lemma 2.2.2, we subsequently consider four cases.

If $k_1 = k_2 = \cdots = k_{l+1} = 0$, we get $L_{l+1} = (\delta * \delta)(t) = \delta(t)$. If $k_1 = k_2 = \cdots = k_l = 0$, $k_{l+1} \in \mathbb{N}$, (2.2) gives

$$L_{l+1} = \frac{(t - k_{l+1}\tau_{l+1})^{k_{l+1}-1}}{(k_{l+1}-1)!}\,\sigma(t - k_{l+1}\tau_{l+1})$$

by the sifting property of δ function [3]. Similarly, if $k_1 + k_2 + \cdots + k_l \in \mathbb{N}$, $k_{l+1} = 0$, then $L_{l+1} = L_l$.

Finally, if $k_1 + k_2 + \cdots + k_l \in \mathbb{N}$ and $k_{l+1} \in \mathbb{N}$, it remains to rewrite the right-hand side of (2.2) as integral

$$\begin{aligned}
L_{l+1} &= \int_0^t \frac{\left(s - \sum_{m=1}^{l} k_m\tau_m\right)^{\sum_{m=1}^{l} k_m - 1}}{\left(\sum_{m=1}^{l} k_m - 1\right)!}\,\sigma\left(s - \sum_{m=1}^{l} k_m\tau_m\right) \\
&\quad \times \frac{(t - s - k_{l+1}\tau_{l+1})^{k_{l+1}-1}}{(k_{l+1}-1)!}\,\sigma(t - s - k_{l+1}\tau_{l+1})ds \\
&= \int_{\sum_{m=1}^{l} k_m\tau_m}^{t - k_{l+1}\tau_{l+1}} \frac{\left(s - \sum_{m=1}^{l} k_m\tau_m\right)^{\sum_{m=1}^{l} k_m - 1}}{\left(\sum_{m=1}^{l} k_m - 1\right)!}\,\frac{(t - s - k_{l+1}\tau_{l+1})^{k_{l+1}-1}}{(k_{l+1}-1)!} ds\,\sigma\left(t - \sum_{m=1}^{l+1} k_m\tau_m\right).
\end{aligned}$$

Now, take the substitution

$$s = \sum_{m=1}^{l} k_m\tau_m + \xi\left(t - \sum_{m=1}^{l+1} k_m\tau_m\right)$$

to obtain

$$L_{l+1} = \frac{\left(t - \sum_{m=1}^{l+1} k_m\tau_m\right)^{\sum_{m=1}^{l+1} k_m - 1}\,\sigma\left(t - \sum_{m=1}^{l+1} k_m\tau_m\right)}{\left(\sum_{m=1}^{l} k_m - 1\right)!\,(k_{l+1}-1)!}\,\mathrm{B}\left(\sum_{m=1}^{l} k_m, k_{l+1}\right)$$

where $\mathrm{B}(\cdot,\cdot)$ is the Euler beta function. Rewriting the beta function using gamma functions, $\mathrm{B}(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$ for any $u, v > 0$, and since $\Gamma(k) = (k-1)!$ for any $k \in \mathbb{N}$, one obtains

$$L_{l+1} = \frac{\left(t - \sum_{m=1}^{l+1} k_m\tau_m\right)^{\sum_{m=1}^{l+1} k_m - 1}\,\sigma\left(t - \sum_{m=1}^{l+1} k_m\tau_m\right)}{\left(\sum_{m=1}^{l+1} k_m - 1\right)!}.$$

The proof is finished. □

Remark 2.4. If $B_1, \ldots, B_n$ are $N \times N$ matrices and w is an N-dimensional vector, then the latter lemma yields

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\left(\prod_{m=1}^{n}\left(\frac{B_m \mathrm{e}^{-p\tau_m}}{p}\right)^{k_m}\right) w\right\} &= \mathcal{L}^{-1}\left\{\left(\prod_{m=1}^{n}\left(\frac{\mathrm{e}^{-p\tau_m}}{p}\right)^{k_m}\right)\left(\prod_{m=1}^{n} B_m^{k_m}\right) w\right\} \\
&= \mathcal{L}^{-1}\left\{\prod_{m=1}^{n}\left(\frac{\mathrm{e}^{-p\tau_m}}{p}\right)^{k_m}\right\}\left(\prod_{m=1}^{n} B_m^{k_m}\right) w \\
&= \begin{cases} \delta(t)w, & k_1 = k_2 = \cdots = k_n = 0, \\ \dfrac{(t - \sum_{m=1}^{n} k_m\tau_m)^{\sum_{m=1}^{n} k_m - 1}\left(\prod_{m=1}^{n} B_m^{k_m}\right) w}{(\sum_{m=1}^{n} k_m - 1)!}\,\sigma\left(t - \sum_{m=1}^{n} k_m\tau_m\right), & \\ & k_1 + k_2 + \cdots + k_n \in \mathbb{N}. \end{cases}
\end{aligned}$$

Next, we recall an estimation of the multi-delayed matrix exponential from [8].

Lemma 2.5. *Let $n \in \mathbb{N}$, $0 < \tau_1, \ldots, \tau_n \in \mathbb{R}$, $B_1, \ldots, B_n$ be pairwise permutable $N \times N$ matrices and $\mathrm{e}^{B_1,\ldots,B_n t}_{\tau_1,\ldots,\tau_n}$ be given by (1.5). If $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ are such that $\|B_i\| \leq \alpha_i \mathrm{e}^{\alpha_i \tau_i}$ for each $i = 1, \ldots, n$, then*

$$\left\| \mathrm{e}^{B_1,\ldots,B_n t}_{\tau_1,\ldots,\tau_n} \right\| \leq \mathrm{e}^{(\alpha_1+\cdots+\alpha_n)(t+\tau_n)}$$

for any $t \in \mathbb{R}$.

As a corollary we get a sufficient condition for $x(t)$ of (1.3) to be exponentially bounded.

Lemma 2.6. *Let the assumptions of Theorem 1.2 be fulfilled and the function f be exponentially bounded. Then the solution $x(t)$ of (1.2), (1.1) is exponentially bounded.*

Proof. Let $c_1, c_2 \in \mathbb{R}$ be such that $|f(t)| \leq c_1 \mathrm{e}^{c_2 t}$ for all $t \geq 0$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $\|B_i\| \leq \alpha_i \mathrm{e}^{\alpha_i \tau_i}$ for each $i = 1, \ldots, n$. We can suppose that $c_2 > 0$ (otherwise take $c_2 > 0$). By Lemma 2.5, $\|X_n(t)\| \leq \mathrm{e}^{\alpha t}$ for any $t \in \mathbb{R}$ where $\alpha = \sum_{i=1}^n \alpha_i$. Then denoting $\overline{\varphi} := \max_{t\in[-\tau,0]} |\varphi(t)|$, for $t \geq 0$ we obtain

$$\begin{aligned} |x(t)| &\leq \overline{\varphi}\mathrm{e}^{\alpha t} + \sum_{m=1}^{n} \|B_m\| \overline{\varphi} \int_0^t \mathrm{e}^{\alpha(t-s)} ds + c_1 \int_0^t \mathrm{e}^{\alpha(t-s)+c_2 s} ds \\ &\leq \overline{\varphi} \left(1 + \sum_{m=1}^{n} \frac{\|B_m\|}{\alpha} \right) \mathrm{e}^{\alpha t} + \frac{c_1 \mathrm{e}^{(\alpha+c_2)t}}{c_2} \leq C\mathrm{e}^{(\alpha+c_2)t} \end{aligned}$$

for a constant C. □

3 Main results

This section is devoted to main results of the present paper. First we suppose that the function f is exponentially bounded.

Theorem 3.1. *Let $n \in \mathbb{N}$, $0 < \tau_1, \ldots, \tau_n \in \mathbb{R}$, $\tau := \max\{\tau_1, \tau_2, \ldots, \tau_n\}$, $B_1, \ldots, B_n$ be pairwise permutable $N \times N$ matrices, i.e. $B_i B_j = B_j B_i$ for each $i, j \in \{1, \ldots, n\}$, $\varphi \in C([-\tau, 0], \mathbb{R}^N)$, and $f : [0, \infty) \to \mathbb{R}^N$ be a given exponentially bounded function, i.e., there exist constants $c_1, c_2 \in \mathbb{R}$ such that $|f(t)| \leq c_1 \mathrm{e}^{c_2 t}$ for all $t \geq 0$. Then the solution of the Cauchy problem (1.2), (1.1) has the form*

$$x(t) = \begin{cases} \varphi(t), & -\tau \leq t < 0, \\ \mathcal{A}(t)\varphi(0) + \sum_{j=1}^n B_j \int_0^{\tau_j} \mathcal{A}(t-s)\varphi(s-\tau_j) ds + \int_0^t \mathcal{A}(t-s) f(s) ds, & 0 \leq t \end{cases} \tag{3.1}$$

where

$$\mathcal{A}(t) = \sum_{\substack{\sum_{m=1}^n k_m \tau_m \leq t \\ k_1,\ldots,k_n \geq 0}} \frac{(t - \sum_{m=1}^n k_m \tau_m)^{\sum_{m=1}^n k_m}}{k_1! \ldots k_n!} \prod_{m=1}^{n} B_m^{k_m} \tag{3.2}$$

for any $t \in \mathbb{R}$.

Proof. By Lemma 2.6, the solution is exponentially bounded and we can apply the Laplace transform on the studied equation (1.2). By Lemma 2.1.4, we obtain

$$\begin{aligned} p\mathcal{L}\{x(t)\} - \varphi(0) &= \sum_{i=1}^{n} B_i \int_0^{\infty} \mathrm{e}^{-ps} x(s-\tau_i) ds + \mathcal{L}\{f(t)\} \\ &= \sum_{i=1}^{n} B_i \left(\int_0^{\tau_i} \mathrm{e}^{-ps} \varphi(s-\tau_i) ds + \int_{\tau_i}^{\infty} \mathrm{e}^{-ps} x(s-\tau_i) ds \right) + F(p) \\ &= \sum_{i=1}^{n} B_i \left(\int_0^{\infty} \mathrm{e}^{-ps} \psi(s-\tau_i) ds + \mathrm{e}^{-p\tau_i} \int_0^{\infty} \mathrm{e}^{-ps} x(s) ds \right) + F(p) \\ &= \sum_{i=1}^{n} \left(B_i \mathcal{L}\{\psi(t-\tau_i)\} + B_i \mathrm{e}^{-p\tau_i} \mathcal{L}\{x(t)\} \right) + F(p) \end{aligned}$$

for $\psi(t)$ given by (1.4). Therefrom, we get

$$\left(p\mathbb{I} - \sum_{i=1}^{n} B_i \mathrm{e}^{-p\tau_i} \right) \mathcal{L}\{x(t)\} = \varphi(0) + \sum_{i=1}^{n} B_i \mathcal{L}\{\psi(t-\tau_i)\} + F(p).$$

From theory of matrices we know (see e.g. [11, Proposition 7.5]) that, on suppose that p is sufficiently large, or more precisely, p is such that

$$\left\| \sum_{i=1}^{n} B_i \mathrm{e}^{-p\tau_i} \right\| < p$$

for a fixed induced norm $\|\cdot\|$, the matrix $\mathbb{I} - \sum_{i=1}^{n} \frac{B_i \mathrm{e}^{-p\tau_i}}{p}$ is invertible and it holds

$$\left(\mathbb{I} - \sum_{i=1}^{n} \frac{B_i \mathrm{e}^{-p\tau_i}}{p} \right)^{-1} = \sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} \frac{B_i \mathrm{e}^{-p\tau_i}}{p} \right)^{k}.$$

Hence

$$\mathcal{L}\{x(t)\} = \frac{1}{p} \left(\mathbb{I} - \sum_{i=1}^{n} \frac{B_i \mathrm{e}^{-p\tau_i}}{p} \right)^{-1} \left[\varphi(0) + \sum_{i=1}^{n} B_i \mathcal{L}\{\psi(t-\tau_i)\} + \mathcal{L}\{f(t)\} \right],$$

i.e.

$$x(t) = A_0 + \sum_{j=1}^{n} B_j A_j + A_f$$

where

$$\begin{aligned} A_0 &= \mathcal{L}^{-1} \left\{ \frac{1}{p} \left(\sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} \frac{B_i \mathrm{e}^{-p\tau_i}}{p} \right)^{k} \right) \varphi(0) \right\}, \\ A_j &= \mathcal{L}^{-1} \left\{ \frac{1}{p} \left(\sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} \frac{B_i \mathrm{e}^{-p\tau_i}}{p} \right)^{k} \right) \mathcal{L}\{\psi(t-\tau_j)\} \right\}, \qquad j = 1, \dots, n, \\ A_f &= \mathcal{L}^{-1} \left\{ \frac{1}{p} \left(\sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} \frac{B_i \mathrm{e}^{-p\tau_i}}{p} \right)^{k} \right) F(p) \right\}. \end{aligned}$$

Now, applying Lemma 2.1, we get

$$A_0 = \sum_{k=0}^{\infty} \mathcal{L}^{-1}\left\{\frac{1}{p}\left(\sum_{i=1}^{n}\frac{B_i e^{-p\tau_i}}{p}\right)^k \varphi(0)\right\} = \sum_{k=0}^{\infty}\left(\sigma * \mathcal{L}^{-1}\left\{\left(\sum_{i=1}^{n}\frac{B_i e^{-p\tau_i}}{p}\right)^k \varphi(0)\right\}\right)(t)$$
$$= (\sigma * \delta)(t)\varphi(0) + \sum_{k=1}^{\infty}\left(\sigma * \mathcal{L}^{-1}\left\{\left(\sum_{i=1}^{n}\frac{B_i e^{-p\tau_i}}{p}\right)^k \varphi(0)\right\}\right)(t).$$

Consequently, by multinomial theorem [1],

$$A_0 = \sigma(t)\varphi(0) + \sum_{k=1}^{\infty}\left(\sigma * \mathcal{L}^{-1}\left\{\left(\sum_{\substack{k_1+\cdots+k_n=k\\ k_1,\ldots,k_n\geq 0}}\binom{k}{k_1,\ldots,k_n}\prod_{m=1}^{n}\left(\frac{B_m e^{-p\tau_m}}{p}\right)^{k_m}\right)\varphi(0)\right\}\right)(t)$$
$$= \sigma(t)\varphi(0) + \sum_{k=1}^{\infty}\sum_{\substack{k_1+\cdots+k_n=k\\ k_1,\ldots,k_n\geq 0}}\binom{k}{k_1,\ldots,k_n}\left(\sigma * \mathcal{L}^{-1}\left\{\left(\prod_{m=1}^{n}\left(\frac{B_m e^{-p\tau_m}}{p}\right)^{k_m}\right)\varphi(0)\right\}\right)(t)$$

where

$$\binom{k}{k_1,\ldots,k_n} = \frac{k!}{k_1!\ldots k_n!}$$

is the multinomial coefficient. Finally, by Lemma 2.3 and Remark 2.4,

$$A_0 = \sigma(t)\varphi(0) + \sum_{k=1}^{\infty}\sum_{\substack{k_1+\cdots+k_n=k\\ k_1,\ldots,k_n\geq 0}}\binom{k}{k_1,\ldots,k_n}\int_0^t \frac{(s-\sum_{m=1}^{n}k_m\tau_m)^{\sum_{m=1}^{n}k_m-1}}{(\sum_{m=1}^{n}k_m-1)!}$$
$$\times\left(\prod_{m=1}^{n}B_m^{k_m}\right)\varphi(0)\sigma\left(s-\sum_{m=1}^{n}k_m\tau_m\right)\sigma(t-s)ds$$
$$= \sigma(t)\varphi(0) + \sum_{k=1}^{\infty}\sum_{\substack{k_1+\cdots+k_n=k\\ k_1,\ldots,k_n\geq 0}}\binom{k}{k_1,\ldots,k_n}\frac{(t-\sum_{m=1}^{n}k_m\tau_m)^k}{k!}$$
$$\times\sigma\left(t-\sum_{m=1}^{n}k_m\tau_m\right)\left(\prod_{m=1}^{n}B_m^{k_m}\right)\varphi(0) = \mathcal{A}(t)\varphi(0).$$

For each $j = 1,\ldots,n$ we apply Lemma 2.1,

$$A_j = \sum_{k=0}^{\infty}\left(\sigma * \mathcal{L}^{-1}\left\{\left(\sum_{i=1}^{n}\frac{B_i e^{-p\tau_i}}{p}\right)^k \mathcal{L}\{\psi(t-\tau_j)\}\right\}\right)(t),$$

multinomial theorem,

$$A_j = (\sigma * \psi(\cdot-\tau_j))(t) + \sum_{k=1}^{\infty}\sum_{\substack{k_1+\cdots+k_n=k\\ k_1,\ldots,k_n\geq 0}}\binom{k}{k_1,\ldots,k_n}$$
$$\times\left(\sigma * \mathcal{L}^{-1}\left\{\prod_{m=1}^{n}\left(\frac{e^{-p\tau_m}}{p}\right)^{k_m}\right\} * \left(\prod_{m=1}^{n}B_m^{k_m}\right)\psi(\cdot-\tau_j)\right)(t),$$

and Lemma 2.3,

$$A_j = \int_0^t \sigma(t-s)\psi(s-\tau_j)ds + \sum_{k=1}^{\infty} \sum_{\substack{k_1+\cdots+k_n=k \\ k_1,\ldots,k_n\geq 0}} \binom{k}{k_1,\ldots,k_n} \left(\prod_{m=1}^{n} B_m^{k_m}\right)$$
$$\times \left(\sigma * \frac{(\cdot - \sum_{m=1}^{n} k_m\tau_m)^{\sum_{m=1}^{n} k_m - 1}}{(\sum_{m=1}^{n} k_m - 1)!} \sigma\left(\cdot - \sum_{m=1}^{n} k_m\tau_m\right) * \psi(\cdot - \tau_j)\right)(t).$$

The double sum from the right-hand side of the above identity can be written as

$$\sum_{k=1}^{\infty} \sum_{\substack{k_1+\cdots+k_n=k \\ k_1,\ldots,k_n\geq 0}} \binom{k}{k_1,\ldots,k_n} \left(\prod_{m=1}^{n} B_m^{k_m}\right) \left(\frac{(\cdot - \sum_{m=1}^{n} k_m\tau_m)^k}{k!} \sigma\left(\cdot - \sum_{m=1}^{n} k_m\tau_m\right) * \psi(\cdot - \tau_j)\right)(t)$$
$$= \int_0^t \sum_{k=1}^{\infty} \sum_{\substack{k_1+\cdots+k_n=k \\ k_1,\ldots,k_n\geq 0}} \frac{\prod_{m=1}^{n} B_m^{k_m}}{k_1!\ldots k_n!} \left(t-s-\sum_{m=1}^{n} k_m\tau_m\right)^k \sigma\left(t-s-\sum_{m=1}^{n} k_m\tau_m\right) \psi(s-\tau_j)ds.$$

Hence,

$$A_j = \int_0^t \sum_{k=0}^{\infty} \sum_{\substack{k_1+\cdots+k_n=k \\ k_1,\ldots,k_n\geq 0}} \frac{\prod_{m=1}^{n} B_m^{k_m}}{k_1!\ldots k_n!} \left(t-s-\sum_{m=1}^{n} k_m\tau_m\right)^k \sigma\left(t-s-\sum_{m=1}^{n} k_m\tau_m\right) \psi(s-\tau_j)ds$$
$$= \int_0^t \sum_{\substack{\sum_{m=1}^{n} k_m\tau_m \leq t-s \\ k_1,\ldots,k_n\geq 0}} \frac{(t-s-\sum_{m=1}^{n} k_m\tau_m)^{\sum_{m=1}^{n} k_m}}{k_1!\ldots k_n!} \left(\prod_{m=1}^{n} B_m^{k_m}\right) \psi(s-\tau_j)ds$$

for each $j = 1,\ldots,n$. Note that the above integral can be shrink to $\int_0^{\tau_j}$ when $t > \tau_j$, along with $\psi(s-\tau_j) \to \varphi(s-\tau_j)$. On the other side, if $t < \tau_j$, it can be extended to $\int_0^{\tau_j}$, since $\int_t^{\tau_j} = 0$ because of the empty sum property. Therefore,

$$A_j = \int_0^{\tau_j} \mathcal{A}(t-s)\varphi(s-\tau_j)ds, \qquad j = 1,\ldots,n.$$

Finally, as for A_j we derive

$$A_f = \int_0^t \sum_{\substack{\sum_{m=1}^{n} k_m\tau_m \leq t-s \\ k_1,\ldots,k_n\geq 0}} \frac{(t-s-\sum_{m=1}^{n} k_m\tau_m)^{\sum_{m=1}^{n} k_m}}{k_1!\ldots k_n!} \left(\prod_{m=1}^{n} B_m^{k_m}\right) f(s)ds$$

which is exactly $\int_0^t \mathcal{A}(t-s)f(s)ds$. The statement is proved. □

As is shown below, the statement of the above theorem holds for a more general function f.

Corollary 3.2. *Theorem 3.1 remains valid if the function f is not exponentially bounded.*

Proof. On setting

$$\varphi(t) = \begin{cases} \Theta, & t \in [-\tau, 0), \\ \mathbb{I}, & t = 0, \end{cases}$$

$f \equiv \Theta$ and considering equation (1.2) as a matrix equation, one can see that $\mathcal{A}(t)$ of (3.2) is a matrix solution of this equation and initial condition, i.e.

$$\dot{\mathcal{A}}(t) = B_1\mathcal{A}(t-\tau_1) + \cdots + B_n\mathcal{A}(t-\tau_n), \qquad t \geq 0$$

considering the right-hand derivative at $t = 0$, and

$$\mathcal{A}(t) = \begin{cases} \Theta, & t \in [-\tau, 0), \\ \mathbb{I}, & t = 0. \end{cases}$$

Let $t \geq 0$ be arbitrary and fixed. Denote $M \subset \{1, \ldots, n\}$ the (possibly empty) set of all indices such that $t < \tau_j$ if and only if $j \in M$. Then from (3.1) we know that

$$x(t) = \mathcal{A}(t)\varphi(0) + \sum_{j \in M} B_j \int_0^t \mathcal{A}(t-s)\varphi(s-\tau_j)ds + \sum_{M \not\ni j=1}^{n} B_j \int_0^{\tau_j} \mathcal{A}(t-s)\varphi(s-\tau_j)ds + \int_0^t \mathcal{A}(t-s)f(s)ds.$$

Differentiating we obtain

$$\begin{aligned} \dot{x}(t) = {} & \sum_{m=1}^{n} B_m \mathcal{A}(t-\tau_m)\varphi(0) + \sum_{j \in M} B_j \left(\mathcal{A}(0)\varphi(t-\tau_j) + \int_0^t \sum_{m=1}^{n} B_m \mathcal{A}(t-\tau_m-s)\varphi(s-\tau_j)ds \right) \\ & + \sum_{M \not\ni j=1}^{n} B_j \int_0^{\tau_j} \sum_{m=1}^{n} B_m \mathcal{A}(t-\tau_m-s)\varphi(s-\tau_j)ds \\ & + \mathcal{A}(0)f(t) + \int_0^t \sum_{m=1}^{n} B_m \mathcal{A}(t-\tau_m-s)f(s)ds \\ = {} & \sum_{M \not\ni m=1}^{n} B_m \mathcal{A}(t-\tau_m)\varphi(0) + \sum_{j \in M} B_j \left(\varphi(t-\tau_j) + \int_0^{\tau_j} \sum_{M \not\ni m=1}^{n} B_m \mathcal{A}(t-\tau_m-s)\varphi(s-\tau_j)ds \right) \\ & + \sum_{M \not\ni j=1}^{n} B_j \int_0^{\tau_j} \sum_{M \not\ni m=1}^{n} B_m \mathcal{A}(t-\tau_m-s)\varphi(s-\tau_j)ds + f(t) \\ & + \int_0^t \sum_{M \not\ni m=1}^{n} B_m \mathcal{A}(t-\tau_m-s)f(s)ds \\ = {} & \sum_{j \in M} B_j \varphi(t-\tau_j) + \sum_{M \not\ni m=1}^{n} B_m \left(\mathcal{A}(t-\tau_m)\varphi(0) + \sum_{j=1}^{n} B_j \int_0^{\tau_j} \mathcal{A}(t-\tau_m-s)\varphi(s-\tau_j)ds \right. \\ & \left. + \int_0^{t-\tau_m} \mathcal{A}(t-\tau_m-s)f(s)ds \right) + f(t) \\ = {} & \sum_{j=1}^{n} B_j x(t-\tau_j) + f(t) \end{aligned}$$

since $x(t-\tau_j) = \varphi(t-\tau_j)$ if $j \in M$. This completes the proof. □

Taking a simple substitution we obtain the following result for delayed differential equations with a constant non-delayed term. The solution is understood in the sense analogous to Definition 1.3.

Theorem 3.3. *Let $n \in \mathbb{N}$, $0 < \tau_1, \ldots, \tau_n \in \mathbb{R}$, $\tau := \max\{\tau_1, \tau_2, \ldots, \tau_n\}$, $A, B_1, \ldots, B_n$ be pairwise permutable $N \times N$ matrices, $\varphi \in C([-\tau, 0], \mathbb{R}^N)$, and $f : [0, \infty) \to \mathbb{R}^N$ be a given function. Then the solution of the Cauchy problem* (1.6), (1.1) *has the form*

$$x(t) = \begin{cases} \varphi(t), & -\tau \leq t < 0, \\ \mathcal{B}(t)\varphi(0) + \sum_{j=1}^{n} B_j \int_0^{\tau_j} \mathcal{B}(t-s)\varphi(s-\tau_j)ds & \\ \quad + \int_0^t \mathcal{B}(t-s)f(s)ds, & 0 \leq t \end{cases} \tag{3.3}$$

where

$$\mathcal{B}(t) = \mathrm{e}^{At} \sum_{\substack{\sum_{m=1}^n k_m\tau_m \le t \\ k_1,\dots,k_n\ge 0}} \frac{(t-\sum_{m=1}^n k_m\tau_m)^{\sum_{m=1}^n k_m}}{k_1!\dots k_n!} \prod_{m=1}^n \widetilde{B}_m^{k_m} \tag{3.4}$$

for any $t\in\mathbb{R}$, *and* $\widetilde{B}_m = B_m\mathrm{e}^{-A\tau_m}$ *for each* $m=1,\dots,n$.

Proof. Let us set $y(t)=\mathrm{e}^{-At}x(t)$. Then y satisfies

$$\begin{aligned} \dot y(t) &= \widetilde{B}_1 y(t-\tau_1)+\dots+\widetilde{B}_n y(t-\tau_n)+\tilde f(t), \quad t\ge 0,\\ y(t) &= \mathrm{e}^{-At}\varphi(t) =: \widetilde{\varphi}(t), \quad -\tau\le t\le 0 \end{aligned}$$

for $\tilde f(t)=\mathrm{e}^{-At}f(t)$ for all $t\ge 0$. Application of Theorem 3.1 yields

$$y(t)=\begin{cases} \widetilde{\varphi}(t), & -\tau\le t<0,\\ \widetilde{\mathcal{A}}(t)\widetilde{\varphi}(0)+\sum_{j=1}^n \widetilde{B}_j\int_0^{\tau_j}\widetilde{\mathcal{A}}(t-s)\widetilde{\varphi}(s-\tau_j)ds & \\ \quad +\int_0^t \widetilde{\mathcal{A}}(t-s)\tilde f(s)ds, & 0\le t \end{cases}$$

with

$$\widetilde{\mathcal{A}}(t) = \sum_{\substack{\sum_{m=1}^n k_m\tau_m \le t \\ k_1,\dots,k_n\ge 0}} \frac{(t-\sum_{m=1}^n k_m\tau_m)^{\sum_{m=1}^n k_m}}{k_1!\dots k_n!} \prod_{m=1}^n \widetilde{B}_m^{k_m}.$$

Now, returning back to x and using $\widetilde{\varphi}(0)=\varphi(0)$,

$$\begin{aligned} \mathrm{e}^{At}\widetilde{B}_j\widetilde{\mathcal{A}}(t-s)\widetilde{\varphi}(s-\tau_j) &= B_j\mathrm{e}^{A(t-s)}\widetilde{\mathcal{A}}(t-s)\varphi(s-\tau_j),\\ \mathrm{e}^{At}\widetilde{\mathcal{A}}(t-s)\tilde f(s) &= \mathrm{e}^{A(t-s)}\widetilde{\mathcal{A}}(t-s)f(s) \end{aligned}$$

and $\mathcal{B}(t)=\mathrm{e}^{At}\widetilde{\mathcal{A}}(t)$, the statement follows immediately. □

Finally, we present an application of the results derived on a scalar equation. In practical computations, we rewrite the sum in (3.2) or (3.4) as

$$\sum_{\substack{\sum_{m=1}^n k_m\tau_m \le t \\ k_1,\dots,k_n\ge 0}} = \sum_{k_1=0}^{\left\lfloor \frac{t}{\tau_1}\right\rfloor} \sum_{k_2=0}^{\left\lfloor \frac{t-k_1\tau_1}{\tau_2}\right\rfloor} \cdots \sum_{k_n=0}^{\left\lfloor \frac{t-\sum_{m=1}^{n-1}k_m\tau_m}{\tau_n}\right\rfloor}.$$

Now the solution given by (3.1) or (3.3) can be computed by hand or using a software.

Example 3.4. Let us consider the following initial value problem

$$\begin{aligned} \dot x(t) &= -3.5x(t)+2x(t-1)-x(t-2)+3x(t-2.5)+1, \quad t\ge 0\\ x(t) &= -t, \quad t\in[-2.5,0]. \end{aligned} \tag{3.5}$$

The solution of this problem is found using Theorem 3.3 and is illustrated in Figure 3.1. We added a more detailed view at the interval $[2,3]$, as one may get an impression from the first part of Figure 3.1 that the solution is not differentiable at some point.

Acknowledgements

M. Pospíšil was supported by the Grant VEGA-SAV 2/0153/16.

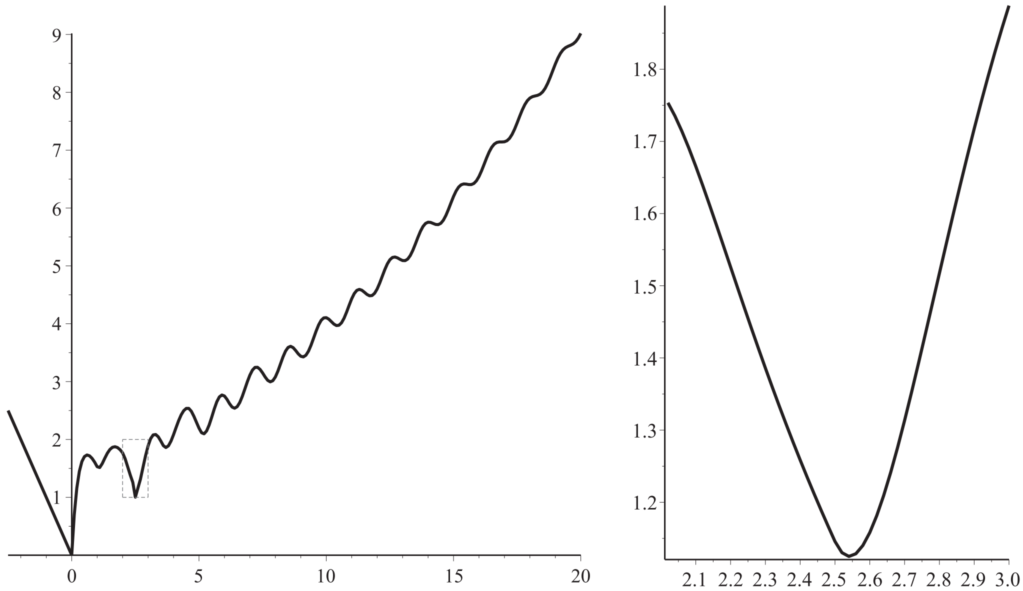

Figure 3.1: The solution of (3.5) with a delayed view at the interval $[2, 3]$.

References

[1] M. Abramowitz, I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, 10th printing, National Bureau of Standards, Washington, 1972. MR757537

[2] O. Arino, M. L. Hbid, E. Ait Dads, *Delay differential equations and applications*, Springer, Netherlands, 2006. MR2334175

[3] R. N. Bracewell, *The Fourier transform and its applications*, 3rd edition, McGraw Hill, Boston, 2000. MR0924577

[4] Ya. S. Bugrov, S. M. Nikolskij, *Vysshaya matematika. Differentsialnye uravneniya. Kratnye integraly. Ryady. Funktsii kompleksnogo peremennogo* (in Russian), Nauka, 1989.

[5] J. Hale, *Theory of functional differential equations*, Applied Mathematical Sciences, Vol. 3, Springer-Verlag, New York, 1977. MR0508721

[6] D. Ya. Khusainov, G. V. Shuklin, Linear autonomous time-delay system with permutation matrices solving, *Stud. Univ. Žilina Math. Ser.* **17**(2003), 101–108. MR2064983

[7] N. N. Lebedev, *Special functions and their applications*, Prentice Hall, Inc., New Jersey, 1965. MR0174795

[8] M. Medveď, M. Pospíšil, Sufficient conditions for the asymptotic stability of nonlinear multidelay differential equations with linear parts defined by pairwise permutable matrices, *Nonlinear Anal.* **75**(2012), 3348–3363. MR2891173; url

[9] M. Pospíšil, Representation and stability of solutions of systems of functional differential equations with multiple delays, *Electron. J. Qual. Theory Differ. Equ.* **2012**, No. 54, 1–30. MR2959044; url

[10] J. L. Schiff, *Laplace transform: theory and applications*, Springer-Verlag, New York, 1999. MR1716143

[11] D. Serre, *Matrices: theory and applications*, 2nd edition, Springer, New York, 2010. MR2744852