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ON THE SINGULAR BEHAVIOR OF SOLUTIONS OF A TRANSMISSION PROBLEM IN A DIHEDRAL

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Abstract. In this paper, we study the singular behavior of solutions of a boundary value problem with mixed conditions in a neighborhood of an edge. The considered problem is defined in a nonhomogeneous body of $\mathbb{R}^3$, this is done in the general framework of weighted Sobolev spaces. Using the results of Benseridi-Dilmi, Grisvard and Aksentian, we show that the study of solutions' singularities in the spatial case becomes a study of two problems: a problem of plane deformation and the other is of normal plane deformation.

1 Introduction

Many research papers have been written recently, both on the singular behavior of solutions for elasticity system in a homogeneous polygon or a polyhedron, see for example [2, 6, 7, 11] and the references cited therein. In the homogeneous domain, in [14] it is introduced a unified and general approach to the asymptotic analysis of elliptic boundary value problems in singularly perturbed domains. The construction of this method capitalizes on the theory of elliptic boundary value problems with nonsmooth boundary. On the other hand, in [15] the authors developed an asymptotic theory of higher-order operator differential equations with nonsmooth nonlinearities.

The case of a nonhomogeneous polygon was already considered in [3]. The regularity of the solutions of transmission problem for the Laplace operator in $\mathbb{R}^3$ was studied in [4].

The aim of this paper, is to study the regularity of solutions for the following transmission problem:

$$(P_1)\begin{cases} \mu_i\Delta u_i + (\lambda_i+\mu_i)\nabla \operatorname{div} u_i = f_i & \text{in } \Omega_i, \\ u_1 = 0 & \text{on } \Gamma_1, \\ \sigma_2(u_2).\mathbf{N} = 0 & \text{on } \Gamma_2, \qquad i=1,2 \\ \left.\begin{array}{l} u_1 = u_2 = 0 \\ (\sigma_1(u_1)-\sigma_2(u_2)).\mathbf{N} = 0 \end{array}\right\} & \text{on } \Lambda\times\mathbb{R}, \end{cases}$$

where σ_i , $(i=1,2)$ designate the stress tensor with $\sigma_i = (\sigma_{ijk})$, j, $k = 1,2,3$ and $i = 1,2$. The σ_{ijk} elements are given by the Hooke's law

$$\sigma_{ijk}(u_i) = \mu_i\left(\frac{\partial u_{ik}}{\partial x_j} + \frac{\partial u_{ij}}{\partial x_k}\right) + \lambda_i \operatorname{div}(u_i)\delta_{jk},$$

and Ω_1, Ω_2 are two homogeneouse elastic and isotropic bodies occupying a domain of $\mathbb{R}^3$ with a polyhedral boundary. We suppose that the lateral surface Γ_2 forms an arbitrary angle ω_2 $(0<\omega_2\le 2\pi)$ to the surface Γ_1. In addition we suppose that Ω is an nonhomogeneous body constituted by two bodies $(\Omega_1\cup\Omega_2)$ rigidly joined along the cylindrical surface $\Lambda\times\mathbb{R}$, which passes through the edge A. The generator of this surface is inclined at an angle ω_1 $(0<\omega_1\le 2\pi)$ to the surface of the first body. For a function u, defined on Ω, we designate by u_1 (resp. u_2) its restriction on Ω_1(resp. Ω_2). Let μ_i and $\nu_i = \dfrac{\lambda_i}{2(\lambda_i+\mu_i)}$ $(i=1,2)$ be, respectively, the shear modulus and Poisson's ratio for the material of the body Ω_i, bounded by the surfaces Γ_i and $\Lambda\times\mathbb{R}$, $i=1,2$.

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The vector $\mathbf{N}$ (resp. τ) denotes the normal (resp. the tangent) on Λ toward the interior of Ω_1. B_i is the infinite subset of $\mathbb{R}^3$ defined by: $B_i = \mathbb{R}\times]0\,,\omega_i[\times\mathbb{R}$, $i = 1, 2$. Let θ_0, θ_∞ be two reals such that: $\theta_0 \leq \theta_\infty$, we put $\eta_0 = \theta_0 - 1$ and $\eta_\infty = \theta_\infty - 1$.

The paper is organised as follows: In section 1 we recall some definitions and properties of Sobolev spaces with double weights introduced by Pham The Lai [13]. In section 2 we transform the problem (P_1) using the partial complex Fourier transform with respect to the first variable, we obtain then a new problem. In section 3 we prove a result of existence and uniqueness of the $\eta-$ solutions according to boundary conditions and we find transcendental equations which govern the singular behavior of solution, then we compare these $\eta-$ solutions. This comparison will be very useful because it allows us to find a sufficient condition for the existence and the uniqueness of the solution of our initial problem. Finaly, we state our main result on the regularity for the problem (P_1).

2 Preliminary results and lemma

In this section we give some basic tools and properties of the weighted Sobolev spaces used in the next.

Definition 2.1. *For $s \in \mathbb{N}$, we define the spaces*

$$H^s_{\theta_0,\theta_\infty}(\Omega)=\left\{u \in L^2_{loc}(\Omega) : r^{\theta_0-s+|\alpha|}(1+r)^{\theta_\infty-\theta_0} D^\alpha u\ (x_1,x_2,x_3) \in L^2(\Omega),\ \forall\alpha \in \mathbb{N}^2,\ |\alpha| \leq s\right\},$$

equiped with the scalar product

$$\langle u, v\rangle = \sum_{|\alpha|\leq s} \iint_\Omega r^{2(\theta_0-s+|\alpha|)}(1+r)^{2(\theta_\infty-\theta_0)} D^\alpha u D^\alpha v\ dx_1dx_2dx_3.$$

$$H^s_{\theta_0,\theta_\infty}(B) = \left\{u \in L^2_{loc}(B)\ :\ e^{\theta_0\, t}\ (1+e^t)^{\theta_\infty-\theta_0} u\ (t,\theta,x_3) \in H^s(B)\right\},$$

equiped with the scalar product

$$\langle u, v\rangle = \sum_{|\alpha|\leq s} \iint_B D^\alpha \left(e^{\theta_0\, t}\ (1+e^t)^{\theta_\infty-\theta_0} u\ \right) D^\alpha \left(e^{\theta_0\, t}\ (1+e^t)^{\theta_\infty-\theta_0} v\right) dtd\theta dx_3.$$

Lemma 2.1 (cf. [5, 10]). *Let θ_1, θ_2 be two reals, we assume that $\theta_1 \leq \theta_2$. Let s be a positive integer, then $f \in H^s_{\theta_1,\theta_2}(\Omega)$, if and only if,*

$$f \in H^s_{\theta_1,\theta_1}(\Omega) \cap H^s_{\theta_2,\theta_2}(\Omega),$$

and we have

$$\|f\|_{H^s_{\theta_1,\theta_2}(\Omega)} \leq c\left[\|f\|_{H^s_{\theta_1,\theta_1}(\Omega)} + \|f\|_{H^s_{\theta_2,\theta_2}(\Omega)}\right],$$

c being a constant which depends only on θ_1, θ_2.

We define by the Fourier transform T with respect to the first variable in B.

The application $T : H^s(B) \longrightarrow V^s(B)$ is an isomorphism, where $V^s(B)$ is a Hilbert space define by

$$V^s(B) = \left\{u \in L^2(B) : (1+\xi^2)^{\frac{k}{2}} u \in L^2(\mathbb{R},\ H^{s-k}(]0,\omega[)),\ for\ k = 0, 1, ...s\right\}.$$

Proposition 2.1. *For $s \in \mathbb{N}$, $\theta_0 \leq \theta_\infty$, the application*

$$\begin{array}{ccc} \Omega & \longrightarrow & B \\ (x,y,z) & \longrightarrow & (t,\theta,x_3), \end{array}$$

defines *an isomorphism*

$$\begin{array}{ccc} H^s_{\theta_0,\theta_\infty}(\Omega) & \longrightarrow & H^s_{\theta_0-s+1,\theta_\infty-s+1}(B) \\ u & \longmapsto & \widetilde{u}, \end{array}$$

where

$$\widetilde{u}(t,\theta,x_3)=u(e^{-t}\cos\theta\ ,e^{-t}\sin\theta,x_3).$$

Proof. Use cylindrical coordinates together with the change of variable $r=e^{-t}$.

Definition 2.2. *The application*

$$\begin{array}{rcl} H^s_{\theta_0,\theta_\infty}(B) & \longrightarrow & H^s(B) \\ u & \longrightarrow & e^{\theta_0 t}\ (1+e^t)^{(\theta_\infty-\theta_0)}u, \end{array}$$

is an isomorphism.

3 Transformation of the problem (P_1)

We look for a possible solution $u=(u_1,u_2)$ in $H^2_{\theta_0,\theta_\infty}(\Omega_1)^3\times H^2_{\theta_0,\theta_\infty}(\Omega_2)^3$ for $f=(f_1,f_2)\in L^2_{\theta_0,\theta_\infty}(\Omega_1)^3\times L^2_{\theta_0,\theta_\infty}(\Omega_2)^3$ of the problem (P_1).

3.1 Use cylindrical coordinates

We put $x_1=r\cos\theta\,,\ x_2=r\sin\theta$ and $x_3=x_3$ with $r=e^{-t}$. Let us write the equations of the Lamé' system in this coordinates, the problem (P_1) becames

$$(P_2)\begin{cases} \dfrac{2(1-\nu_i)}{1-2\nu_i}(-u_{ir}+\dfrac{\partial^2u_{ir}}{\partial t^2})-\dfrac{3-4\nu_i}{1-2\nu_i}\dfrac{\partial u_{i\theta}}{\partial\theta}-\dfrac{1}{1-2\nu_i}\dfrac{\partial^2u_{i\theta}}{\partial t\partial\theta}+\dfrac{\partial^2u_{ir}}{\partial\theta^2}+\dfrac{1}{1-2\nu_i}e^{-t}\dfrac{\partial^2u_{ix_3}}{\partial t\partial x_3}+e^{-2t}\dfrac{\partial^2u_{ir}}{\partial x_3^2}=g_{i1} \\ \dfrac{2(1-\nu_i)}{1-2\nu_i}\dfrac{\partial^2u_{i\theta}}{\partial\theta^2}-\dfrac{1}{1-2\nu_i}\dfrac{\partial^2u_{ir}}{\partial t\partial\theta}-u_{i\theta}+\dfrac{3-4\nu_i}{1-2\nu_i}\dfrac{\partial u_{ir}}{\partial\theta}+\dfrac{\partial^2u_{ir}}{\partial t^2}+\dfrac{1}{1-2\nu_i}e^{-t}\dfrac{\partial^2u_{ix_3}}{\partial\theta\partial x_3}+e^{-2t}\dfrac{\partial^2u_{i\theta}}{\partial x_3^2}=g_{i2} \\ \dfrac{\partial^2u_{iz}}{\partial\theta^2}+\dfrac{\partial^2u_{iz}}{\partial t^2}-\dfrac{e^{-t}}{1-2\nu_i}(\dfrac{\partial^2u_{i\theta}}{\partial\theta\partial x_3}+\dfrac{\partial u_{ir}}{\partial x_3}-\dfrac{\partial^2u_{ir}}{\partial t\partial x_3})+\dfrac{2(1-\nu_i)}{1-2\nu_i}e^{-2t}\dfrac{\partial^2u_{ix_3}}{\partial x_3^2}=g_{i3} \\ u_1=0 \qquad \text{on } \mathbb{R}\times\{0\}\times\mathbb{R} \\ \sigma_2(u_2).\mathbf{N}=0 \qquad \text{on } \mathbb{R}\times\{\omega_2\}\times\mathbb{R} \\ \begin{pmatrix} u_1-u_2 \\ (\sigma_1(u_1)-\sigma_2(u_2))\,.\mathbf{N}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \qquad \text{on } \mathbb{R}\times\{\omega_1\}\times\mathbb{R}, \end{cases}$$

where

$$g_i(t,\theta,x_3)=e^{2t}f_i(e^{-t}\cos\theta,\ e^{-t}\sin\theta,\ x_3),$$

u_{ir}, $u_{i\theta}$ and u_{ix_3} are the components of the displacement vector, taken in the directions of the introduced coordinates.

Property 3.1. *For* $u_i(x_1,x_2,x_3)\in H^2_{\theta_0,\theta_\infty}(\Omega_i)^3$ *and* $f_i\in L^2_{\theta_0,\theta_\infty}(\Omega_i)^3$, $u_i(t,\theta,x_3)\in H^2_{\eta_0,\eta_\infty}(B_i)^3$ *and* $g_i\in L^2_{\eta_0,\eta_\infty}(\Omega_i)^3$, $i=1,\,2$.

Proof. For $s\in\mathbb{N}$ and $\theta_0\le\theta_\infty$, the application

$$\begin{array}{rcl} \Omega_i & \longrightarrow & B_i \\ (x_1,x_2,x_3) & \longrightarrow & (t,\theta,x_3), \end{array}$$

defines an isomorphism

$$\begin{array}{rcl} H^s_{\theta_0,\theta_\infty}(\Omega_i)^3 & \longrightarrow & H^s_{\theta_0-s+1,\theta_\infty-s+1}(B_i)^3 \\ u_i(x_1,x_2,x_3) & \longmapsto & u_i(t,\theta,x_3), \end{array}$$

which gives the result for $s=2$.
Property 3.2. *The problems* (P_1) *and* (P_2) *are equivalents.*
Proof. It follows from property 3.1.
Remark 3.1
1- To express the behavior of the solution of the boundary value problem far away from the vertex, noting that the neighborhood of A is sufficiently small so that terms containing the factor e^{-t} may be neglected.
2- According to the mixed condition it is shown that the surface Γ_2 is free of stresses while the surface Γ_1 is rigidly clamped. Since Γ_1, $\Lambda\times\mathbb{R}$ and Γ_2 are coordinate surfaces corresponding to $\theta=0$, $\theta=\omega_1$ and $\theta=\omega_2$ respectively.
3- The boundary conditions are

$$\begin{cases} \sigma_{1\theta\theta}=\tau_{1r\theta}=\tau_{1x_3\theta}=0 & \text{on } \Gamma_1 \\ u_{2r}=u_{2\theta}=u_{2x_3}=0 & \text{on } \Gamma_2 \\ \left.\begin{array}{l} \sigma_{1\theta\theta}=\sigma_{2\theta\theta},\ \tau_{1r\theta}=\tau_{2r\theta} \text{ and } \tau_{1x_3\theta}=\tau_{2x_3\theta} \\ u_{1r}=u_{2r},\ u_{1\theta}=u_{2\theta} \text{ and } u_{1x_3}=u_{2x_3} \end{array}\right\} & \text{on } \Lambda\times\mathbb{R}. \end{cases}$$

4- The indicated stresses, in terms of displacements in the above coordinate system, are given by:

$$\begin{cases} \sigma_{i\theta\theta}=\dfrac{2\mu_i\, e^t}{1-2\nu_i}\left((1-\nu_i)\dfrac{\partial u_{i\theta}}{\partial\theta}+(1-\nu_i)u_{ir}-\nu_i\dfrac{\partial u_{ir}}{\partial t}\right), \\ \tau_{ir\theta}=\mu_i\, e^t\left(\dfrac{\partial u_{ir}}{\partial\theta}-\dfrac{\partial u_{i\theta}}{\partial t}-u_{i\theta}\right), \\ \tau_{ix_3\theta}=\mu_i\, e^t\,\dfrac{\partial u_{ix_3}}{\partial\theta}, \end{cases}$$

where, $\tau_{ir\theta}$ and $\sigma_{i\theta\theta}$, are the tangential stress tensor and the normal stress tensor respectively.

3.2 Fourier transform of (P_2)

With the condition $f_i\in L^2_{\theta_0,\theta_\infty}(\Omega_i)^3$ the function $g_i(t,\theta,x_3)$ admits a Fourier transform $\widehat{g}_i(\xi,\theta,x_3)$ for any ξ in the strip C_{η_0,η_∞} defined by

$$C_{\eta_0,\eta_\infty}=\{\xi\in C\ \ /\ \eta_0\le\ Im\ \xi\ \ \le\eta_\infty\}\,.$$

This strip is not empty since it was assumed that $\theta_0\le\theta_\infty$. On the other hand $u_i(x_1,x_2,x_3)\in H^2_{\theta_0,\theta_\infty}(\Omega_i)^3$, u_i and its derivatives of order ≤ 2 admit a Fourier transform in the same strip.
Applying the Fourier transform on (P_2) and taking into account the smallness of the neighborhood, we obtain the following problem

$$(P_3)\begin{cases} (1-2\nu_i)\ \widehat{u}''_{ir}-2(1-\nu_i)(1+\xi^2)\ \widehat{u}_{ir}-(3-4\nu_i-i\xi)\ \widehat{u}'_{i\theta}=\widehat{g}_{i1} & \text{(I)} \\ 2(1-\nu_i)\ \widehat{u}''_{i\theta}-(1-2\nu_i)(1+\xi^2)\ \widehat{u}_{i\theta}+(3-4\nu_i+i\xi)\ \widehat{u}'_{ir}=\widehat{g}_{i2} & \text{(II)} \\ \widehat{u}''_{ix_3}-\xi^2\ \widehat{u}_{ix_3}=\widehat{g}_{i3} & \text{(III)} \\ \widehat{u}_1=0 & for\ \ \theta=0 \\ \widehat{\sigma}_2(u_2)=0 & for\ \ \theta=\omega_2 \\ \begin{pmatrix} \widehat{u}_1-\widehat{u}_2 \\ \widehat{\sigma}_1(u_1)-\ \widehat{\sigma}_2(u_2)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} & for\ \ \theta=\omega_1, \end{cases}$$

where $\widehat{u}_i$ and $\widehat{\sigma}_i$ are the Fourier transforms of u_i and σ_i respectively. More exactly we have:

$$\left\{\begin{array}{ll} \widehat{\sigma}_{1\theta\theta}=\widehat{\tau}_{1r\theta}=\widehat{\tau}_{1x_3\theta}=0 & \text{on} \quad \Gamma_1 \\ \widehat{u}_{2r}=\widehat{u}_{2\theta}=\widehat{u}_{2x_3}=0 & \text{on} \quad \Gamma_2 \\ \left.\begin{array}{l} \widehat{\sigma}_{1\theta\theta}=\widehat{\sigma}_{2\theta\theta},\ \widehat{\tau}_{1r\theta}=\widehat{\tau}_{2r\theta} \text{ and } \widehat{\tau}_{1x_3\theta}=\widehat{\tau}_{2x_3\theta} \\ \widehat{u}_{1r}=\widehat{u}_{2r},\ \widehat{u}_{1\theta}=\widehat{u}_{2\theta} \text{ and } \widehat{u}_{1x_3}=\widehat{u}_{2x_3} \end{array}\right\} & \text{on } \Lambda\times\mathbb{R} \end{array}\right. \tag{BC}$$

with

$$\left\{\begin{array}{l} \widehat{\sigma}_{i\theta}=0 \Leftrightarrow (1-\nu_i)\widehat{u}_{i\theta}^{'}+(1-\nu_i-i\xi\nu_i)\widehat{u}_{ir}=0, \\ \\ \widehat{\tau}_{ir\theta}=0 \Leftrightarrow \widehat{u}_{ir}^{'}-(1+i\xi)\widehat{u}_{i\theta}=0, \\ \\ \widehat{\tau}_{ix_3\theta}=0 \Leftrightarrow \widehat{u}_{ix_3}^{'}=0. \end{array}\right.$$

Remark 3.2
1- From equations of (P_3) it can be seen that the problem (P_1) can be divided into two problems: The first is a plane deformation to which correspond the two first equations (I) and (II), while the second is a normal plane deformation, expressed by the third equation (III).
2- Finally, we get the following problem: for a fixed ξ in the strip C_{η_0,η_∞} , we look for a possible solution $\widehat{u}=(\widehat{u}_1,\widehat{u}_2)$ in $H^2(]0,\omega_1[)^3\times H^2(]0,\omega_2[)^3$ for (P_3).

The study of the homogeneous problem corresponding to (P_3) gives the following results.
Proposition 3.1. *The transcendental equations governing the singular behavior of the problem* (P_3) *given by*:
Problem of plane deformation

$$\begin{gathered} \mu_2(1-\nu_2)^2(4\nu_1-3)\left(\sin^2\xi\omega_1-\frac{4(1-\nu_1)^2-\ \xi^2\sin^2\omega_1}{3-4\nu_1}\right)+ \\ (\mu_1-\mu_2)(3-4\nu_2)(1-\nu_2)(\sin^2\xi\omega_1-\xi^2\sin^2\omega_1)\sin^2\xi(\omega_2-\omega_1)+ \\ +\frac{1}{4}\mu_2^{-1}(\mu_1-\mu_2)^2(3-4\nu_2)^2(\sin^2\xi\omega_1-\xi^2\sin^2\omega_1)\sin^2\xi(\omega_2-\omega_1) \\ -2\mu_1(1-\nu_1)(1-\nu_2)(3-4\nu_2)\sin\xi\omega_1\sin\xi(\omega_2-\omega_1)\cos\xi(2\omega_1-\omega_2) \\ +(\mu_1-\mu_2)(1-\nu_1)(3-4\nu_2)^2\sin^2\xi\omega_1\sin^2\xi(\omega_2-\omega_1)+ \\ -\xi^2\tfrac{1}{4}\mu_2^{-1}(\mu_1-\mu_2)^2(\sin^2\xi\omega_1-\xi^2\sin^2\omega_1)\sin^2(\omega_2-\omega_1) \\ +4\mu_2(1-\nu_1)(1-\nu_2)(3-4\nu_2)(\sin\xi\omega_1\sin\xi(\omega_2-\omega_1))^2+ \\ -\xi^2(\mu_1-\mu_2)(1-\nu_1)\sin^2\xi\omega_1\ \sin^2(\omega_2-\omega_1)+ \\ -2\mu_1(1-\nu_1)(1-\nu_2)\ \xi^2\sin(\omega_2-\omega_1)\sin\omega_1\cos\omega_2 \\ -\mu_2(1-\nu_1)^2(3-4\nu_2)\sin^2\xi(\omega_2-\omega_1) \\ +\xi^2\mu_2(1-\nu_1)^2\sin^2(\omega_2-\omega_1)=0. \end{gathered} \tag{3.1}$$

Problem of normal plane deformation

$$\mu_1\sin\xi\omega_1\sin\xi(\omega_2-\omega_1)-\mu_2\cos\xi\omega_1\cos\xi(\omega_2-\omega_1)=0\,. \tag{3.2}$$

Proof. Using the boundary conditions on Γ_1, Γ_2 and $\Lambda\times\mathbb{R}$, we obtain a system of homogeneous equations. The condition of the vanishing of the system's determinant gives the transcendental equations with respect to ξ.
Proposition 3.2. *Let F and G be the zeros of* (3.1) *and* (3.2) *repectively, then the homogeneous problem* (P_3) *admits a unique solution, if and only if,* $\xi\notin(F\cup G)$.

Proof. It follows immediately from the proposition 3.1.
Proposition 3.3. *For all* $\xi \in \mathbb{C}/(F \cup G)$ *and* $\widehat{g}_i \in L^2(]0,\omega_i[)^3$, *there exists one and only one* $\widehat{u}_i \in H^2(]0,\omega_i[)^3$ *solution for the problem* (P_3). *In addition, the resolvant of* (P_3),

$$\begin{array}{rcl} R_\xi & : & L^2(]0,\omega_i[)^3 \longrightarrow H^2(]0,\omega_i[)^3 \\ \widehat{g}_i & \longmapsto & R_\xi(g_i) = \widehat{u}_i \end{array}$$

such that the map

$$\begin{array}{rcl} \mathbb{C}/(F \cup G) & \longrightarrow & L\left(L^2(]0,\omega_i[)^3 \longrightarrow H^2(]0,\omega_i[)^3\right) \\ \xi & \longmapsto & R_\xi \end{array}$$

is analytical.
Remark 3.3. The above proposition is similar to that of [5, 10].

4 The main result

In this section, we are going to prove a result of existence and uniqueness of the $\eta-$ solutions and then, we compare them $\eta-$ solutions. This comparison will be very useful because it allows us to find a sufficient condition for the existence and the uniqueness of the solution of our initial problem (P_1). It is important to introduce the following definition.
Defnition 4.1. *Let* $\eta \in [\eta_0, \eta_\infty]$, *we call* $\eta-$*solutions for the problem* (P_1), *all elements* $u = (u_1, u_2)$ *of* $H^2_{\eta+1,\eta+1}(\Omega_1)^3 \times H^2_{\eta+1,\eta+1}(\Omega_2)^3$, *verifying* (P_1).
The following property is a straightforward consequence of lemma 2.1.
Property 4.1. *u is a solution for the problem (P_1), iff, u is a η_0-solutions and $\eta_\infty-$solutions of (P_1).*
Proof. Let u be a solution of (P_1), then

$$u \in H^2_{\theta_0,\theta_\infty}(\Omega_1)^3 \times H^2_{\theta_0,\theta_\infty}(\Omega_2)^3 = H^2_{\eta_0+1,\eta_\infty+1}(\Omega_1)^3 \times H^2_{\eta_0+1,\eta_\infty+1}(\Omega_2)^3,$$

and from lemma 2.1, we have

$$\begin{array}{rcl} u & \in & H^2_{\eta_0+1,\eta_0+1}(\Omega_1)^3 \times H^2_{\eta_0+1,\eta_0+1}(\Omega_2)^3 \\ & & \textit{and} \\ u & \in & H^2_{\eta_\infty+1,\eta_\infty+1}(\Omega_1)^3 \times H^2_{\eta_\infty+1,\eta_\infty+1}(\Omega_2)^3. \end{array}$$

Then u *is a* η_0-*solution and* $\eta_\infty-$solution *of the* (P_1).■
Property 4.2. *If the transcendental equations* $(3.k)$, $k = 1, 2$ *have no zeros of imaginary part η, the problem (P_1) has a unique $\eta-$ solutions, in addition there exists a positive constant c such that*

$$\|u\|_{H^2_{\eta+1,\eta+1}(\Omega_1)^3 \times H^2_{\eta+1,\eta+1}(\Omega_2)^3} \leq c\,\|f\|_{L^2_{\theta_0,\theta_\infty}(\Omega_1)^3 \times L^2_{\theta_0,\theta_\infty}(\Omega_2)^3}\,.$$

The proof of this property is based on the following lemmas.
Lemma 4.1. *K is a compact containing no zeros of* $(3.k)$, $k = 1,\ 2$, *then there exist a constant c depending on K such that for all u and all $\xi \in K$:*

$$\|\widehat{u}_i\|_{H^2(]0,\omega_i[)^3} \leq c\,\|\digamma(\widehat{u}_{ir}, \widehat{u}_{i\theta}, \widehat{u}_{ix_3})\|_{L^2(]0,\omega_i[)^3}\,,$$

where

$$\digamma(\widehat{u}_{ir}, \widehat{u}_{i\theta}, \widehat{u}_{ix_3}) = \begin{pmatrix} (1-2\nu_i)\ \widehat{u}''_{ir} - 2(1-\nu_i)(1+\xi^2)\ \widehat{u}_{ir} - (3-4\nu_i - i\xi)\ \widehat{u}'_{i\theta} \\ 2(1-\nu_i)\ \widehat{u}''_{i\theta} - (1-2\nu_i)(1+\xi^2)\ \widehat{u}_{i\theta} + (3-4\nu_i + i\xi)\ \widehat{u}'_{ir} \\ \widehat{u}''_{ix_3} - \xi^2\ \widehat{u}_{ix_3} \end{pmatrix}.$$

Lemma 4.2. *Let $R>0$, there exists $\alpha>0$ and $c>0$ such that for any ξ verifying $|Re\xi|\geq\alpha$, $|Im\xi|\leq R$ and for all $\widehat{u}_i$ of $H^2(]0,\omega_i[)^3$, we have*

$$\|\widehat{u}_i\|_{H^2(]0,\omega_i[)^3}+|\xi|^4\|\widehat{u}_i\|_{L^2(]0,\omega_i[)^3}\leq c\|F(\widehat{u}_{ir},\widehat{u}_{i\theta},\widehat{u}_{ix_3})\|_{L^2(]0,\omega_i[)^3}.$$

Remark 4.1. For the proof of the two first lemmas we refer the reader to [10].
Lemme 4.3. *For a given η_1, $\eta_2\in\mathbb{R}$ such that, $\eta_1\leq\eta_2$. If $g\in L^2_{\eta_1,\eta_2}(B_1)^3\times L^2_{\eta_1,\eta_2}(B_2)^3$, one has*

$$\begin{cases}\forall\eta\in[\eta_1,\eta_2],\ e^{\eta\,t}g\in L^2(B_1)^3\times L^2(B_2)^3\\ \text{and}\\ \|e^{\eta\,t}g\|_{L^2(B_1)^3\times L^2(B_2)^3}\leq\|g\|_{L^2_{\eta_1,\eta_2}(B_1)^3\times L^2_{\eta_1,\eta_2}(B_2)^3}.\end{cases}$$

Proof. Let $g\in L^2_{\eta_1,\eta_2}(B_1)^3\times L^2_{\eta_1,\eta_2}(B_2)^3$, then

$$e^{\eta\,t}\left(1+e^t\right)^{\eta_2-\eta_1}g\in L^2(B_1)^3\times L^2(B_2)^3.$$

It suffices to show that

$$\left|e^{\eta\,t}g\right|\leq\left|e^{\eta_1\,t}\left(1+e^t\right)^{\eta_2-\eta_1}g\right|. \tag{4.1}$$

Indeed, for $t\in\mathbb{R}_+$, we have

$$(1+e^t)^{\eta_2-\eta_1}\geq e^{(\eta_2-\eta_1)\,t}\text{ and }e^{\eta_2\,t}\geq e^{\eta\,t},$$

as

$$\left|e^{\eta\,t}g\right|\leq\left|e^{\eta\,t}\left(1+e^t\right)^{\eta_2-\eta_1}g\right|,$$

and for $t\leq0$

$$(1+e^t)^{\eta_2-\eta_1}\geq1\text{ and }e^{\eta_1\,t}\geq e^{\eta\,t}.$$

Then

$$\left|e^{\eta\,t}g\right|\leq\left|e^{\eta\,t}\left(1+e^t\right)^{\eta_2-\eta_1}g\right|.$$

Hence the inequality (4.1).
Therefore,

$$e^{\eta\,t}g\in L^2(B_1)^3\times L^2(B_2)^3\ \text{ and }\ \left\|e^{\eta\,t}g\right\|_{L^2(B_1)^3\times L^2(B_2)^3}\leq\|g\|_{L^2_{\eta_1,\eta_2}(B_1)^3\times L^2_{\eta_1,\eta_2}(B_2)^3}.\blacksquare$$

Proof. (property 4.2)**.** This amounts to showing that the problem (P_2) admits a unique $\eta-$ solution, i.e. that there exists one and only one $u=(u_1,\,u_2)$ in $H^2_{\eta,\eta}(B_1)^3\times H^2_{\eta,\eta}(B_2)^3$ verifying (P_2).
<u>Existence</u>. The hypothesis that $(3.k)$ has no zeros on the half plane $\mathbb{R}+i\eta$ ensures that the problem (P_3) admits a solution

$$\widehat{u}\in H^2(]0,\omega_1[)^3\times H^2(]0,\omega_2[)^3,$$

where

$$\widehat{u}(\xi=\rho+i\eta,\theta,x_3)\in V^2(B_1)^3\times V^2(B_2)^3.$$

We set

$$u(t,\theta,x_3)=e^{-\eta\,t}T^{-1}(\widehat{u})(t,\theta,x_3),$$

where T^{-1} is the inverse Fourier transform with respect to ρ. One can easily verify that u is a solution of (P_2) and

$$u\in H^2_{\eta,\eta}(B_1)^3\times H^2_{\eta,\eta}(B_2)^3.$$

Uniqueness. Let u^1 and u^2 two solutions of the problem (P_1), then $\widehat{u}^1$ and $\widehat{u}^2$ are two solutions of (P_3). It follows from the proposition 3.3, that $\widehat{u}^1 = \widehat{u}^2$, now applying the inverse Fourier transform to both sides of this equality, we obtain $u^1 = u^2$, hence the uniqueness.
We show now that

$$\|u\|_{H^2_{\eta+1,\eta+1}(\Omega_1)^3\times H^2_{\eta+1,\eta+1}(\Omega_2)^3} \leq c\,\|f\|_{L^2_{\theta_0,\theta_\infty}(\Omega_1)^3\times L^2_{\theta_0,\theta_\infty}(\Omega_2)^3}\,.$$

For this, it suffies to show that

$$\|u\|_{H^2_{\eta,\eta}(B_1)^3\times H^2_{\eta,\eta}(B_2)^3} \leq c\,\|g\|_{L^2_{\eta_0,\eta_\infty}(B_1)^3\times L^2_{\eta_0,\eta_\infty}(B_2)^3}.$$

First recall that the application

$$\begin{array}{rcl} H^2_{\eta,\eta}(B_i) & \longrightarrow & V^2(B_i) \\ u & \longmapsto & \widehat{u}(\rho+i\eta,\theta,x_3) = T(e^{\eta t}u)(\rho+i\eta,\theta,x_3), \end{array}$$

is an isomorphism, this allows us to write

$$\|u\|_{H^2_{\eta,\eta}(B_1)^3\times H^2_{\eta,\eta}(B_2)^3} \leq c\,\|\widehat{u}\|_{V^2(B_1)^3\times V^2(B_2)^3}\,.$$

We have then

$$\begin{aligned} \|u\|_{H^2_{\eta,\eta}(B_1)^3\times H^2_{\eta,\eta}(B_2)^3} \leq & \sum_{j=1}^{2}\left(\int_{\mathbb{R}}\|\widehat{u}_j(\rho+i\eta,\theta,x_3)\|^2_{H^2(]0,\omega_j[)^3}d\rho\right)+ \\ & +|\xi|^4\sum_{j=1}^{2}\left(\int_{\mathbb{R}}\|\widehat{u}_j(\rho+i\eta,\theta,x_3)\|^2_{L^2(]0,\omega_j[)^3}d\rho\right). \end{aligned}$$

Let $R = |\eta|$ and α as defined in lemma 4.2, then for all ρ, $|\rho| \geq \alpha$

$$\begin{aligned} &\|\widehat{u}(\rho+i\eta,\theta,x_3)\|^2_{H^2(]0,\omega_1[)^3\times H^2(]0,\omega_2[)^3} + |\xi|^4\,\|\widehat{u}(\rho+i\eta,\theta,x_3)\|^2_{L^2(]0,\omega_1[)^3\times L^2(]0,\omega_2[)^3} \\ &\qquad \leq c\,\|\widehat{g}(\rho+i\eta,\theta,x_3)\|^2_{L^2(]0,\omega_1[)^3\times L^2(]0,\omega_2[)^3}\,. \end{aligned} \tag{4.2}$$

Set $K = \{\xi = \rho+i\eta : |\rho| \leq \alpha\}$, which is a compact set containing no zeros of $(3.k)$.
It comes from lemma 4.1 that

$$\|\widehat{u}(\rho+i\eta,\theta,x_3)\|^2_{H^2(]0,\omega_1[)^3\times H^2(]0,\omega_2[)^3} \leq c\,\|\widehat{g}(\rho+i\eta,\theta,x_3)\|^2_{L^2(]0,\omega_1[)^3\times L^2(]0,\omega_2[)^3}\,.$$

But

$$\|\widehat{u}(\rho+i\eta,\theta,x_3)\|^2_{L^2(]0,\omega_1[)^3\times L^2(]0,\omega_2[)^3} \leq \|\widehat{u}(\rho+i\eta,\theta,x_3)\|^2_{H^2(]0,\omega_1[)^3\times H^2(]0,\omega_2[)^3},$$

we deduce that (4.2) is valid for ρ such that $|\rho| \leq \alpha$, so it is also valid for any $\rho \in \mathbb{R}$.
By integrating both members of (4.2) with respect to ρ, we find

$$\|\widehat{u}\|_{V^2(B_1)^3\times V^2(B_2)^3} \leq c\,\|\widehat{g}\|_{L^2(B_1)^3\times L^2(B_2)^3}\,,$$

thus

$$\|u\|_{H^2_{\eta,\eta}(B_1)^3\times H^2_{\eta,\eta}(B_2)^3} \leq c\,\|g\|_{L^2_{\eta,\eta}(B_1)^3\times L^2_{\eta,\eta}(B_2)^3}.$$

Moreover, from lemma 4.3

$$\|g\|_{L^2_{\eta,\eta}(B_1)^3\times L^2_{\eta,\eta}(B_2)^3} \leq c\,\|g\|_{L^2_{\eta_0,\eta_\infty}(B_1)^3\times L^2_{\eta_0,\eta_\infty}(B_2)^3}\,.$$

Hence

$$\|u\|_{H^2_{\eta,\eta}(B_1)^3\times H^2_{\eta,\eta}(B_2)^3} \leq c\,\|g\|_{L^2_{\eta_0,\eta_\infty}(B_1)^3\times L^2_{\eta_0,\eta_\infty}(B_2)^3}\,.$$

Finally, from the proposition 2.1, we deduce that

$$\|u\|_{H^2_{\eta+1,\eta+1}(\Omega_1)^3\times H^2_{\eta+1,\eta+1}(\Omega_2)^3} \leq c\,\|f\|_{L^2_{\theta_0,\theta_\infty}(\Omega_1)^3\times L^2_{\theta_0,\theta_\infty}(\Omega_2)^3}\,.\blacksquare$$

The following proposition is devoted to the decomposition of the solution of the problem (P_1) to a singular and a regular parts.

Proposition 4.1. $\eta_1,\, \eta_2 \in [\eta_0,\eta_\infty]$, $\eta_1 \leq \eta_2$. *We assume that* $(3.k)$ *have no zeros of imaginary part* η_1 *or* η_2*, then*

$$u_{\eta_1} - u_{\eta_2} = i \sum_{\xi_0\in(F\cup G)\cap\{\eta_1\leq Im\xi\leq\eta_2\}} Res(e^{i\xi\, t} R_\xi(\widehat{g}))_{/_{\xi=\xi_0}}\,.$$

Proof. We note first that the sum has a meaning because the set $(F\cup G)\cap\{\eta_1\leq Im\xi\leq\eta_2\}$ is finite and the residuals are well defined.

Let γ be the domain defined in the half plane, by $\mathbb{R}+i\eta_1$ and $\mathbb{R}+i\eta_2$. We know that R_ξ is analytical on $\mathbb{C}/(F\cup G)$, hence

$$\int_\gamma e^{it\xi} R_\xi(\widehat{g})d\xi = 2\pi i \sum_{\xi_0\in(F\cup G)\cap\{\eta_1\leq Im\xi\leq\eta_2\}} Res(e^{i\,t\xi}R_\xi(\widehat{g}))\,|_{\xi=\xi_0}\ ,$$

and

$$\begin{aligned}\int_\gamma e^{it\xi} R_\xi(\widehat{g})d\xi \;&=\; \int_{[-\varepsilon+i\eta_1,\varepsilon+i\eta_1]} e^{it\xi} R_\xi(\widehat{g})d\xi + \int_{[\varepsilon+i\eta_1,\varepsilon+i\eta_2]} e^{it\xi} R_\xi(\widehat{g})d\xi \\ &\quad+ \int_{[\varepsilon+i\eta_2,-\varepsilon+i\eta_2]} e^{it\xi} R_\xi(\widehat{g})d\xi + \int_{[-\varepsilon+i\eta_2,-\varepsilon+i\eta_1]} e^{it\xi} R_\xi(\widehat{g})d\xi\end{aligned}$$

going to the limit when ε goes to infinity, we obtain

$$\lim_{\varepsilon\to\infty}\int_\gamma e^{it\xi} R_\xi(\widehat{g})d\xi = \int_{-\infty}^{+\infty} e^{it(\rho+i\eta_1)} R_{(\xi+i\eta_1)}(\widehat{g})d\rho - \int_{-\infty}^{+\infty} e^{it(\rho+i\eta_2)} R_{(\xi+i\eta_2)}(\widehat{g})d\rho.$$

The integrals $\int\limits_{[\varepsilon+i\eta_1,\varepsilon+i\eta_2]} e^{it\xi} R_\xi(\widehat{g})d\xi$ and $\int\limits_{[-\varepsilon+i\eta_2,-\varepsilon+i\eta_1]} e^{it\xi} R_\xi(\widehat{g})d\xi,$

tends to zero, thus

$$i\sum_{\xi_0\in(F\cup G)\cap\{\eta_1\leq Im\xi\leq\eta_2\}} Res(e^{i\,t\xi}R_\xi(\widehat{g}))\,|_{\xi=\xi_0} \;=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(i\xi-\eta_1)t} R_{(\rho+i\eta_1)}(\widehat{g})d\rho - \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(i\xi-\eta_2)t} R_{(\rho+i\eta_2)}(\widehat{g})d\rho$$

but

$$u_{\eta_1} = \;\frac{e^{-\eta_1 t}}{2\pi}\int_{-\infty}^{+\infty} e^{it\xi} R_{(\rho+i\eta_1)}(\widehat{g})d\rho \;\text{ and }\; u_{\eta_2} = \;\frac{e^{-\eta_2 t}}{2\pi}\int_{-\infty}^{+\infty} e^{it\xi} R_{(\rho+i\eta_2)}(\widehat{g})d\rho.$$

Which ends the proof.■

Now, our aim is to prove a theorem of existence, uniqueness and regularity of the solution of our initial problem (P_1).

Theorem 4.1. *Let θ_0, θ_∞ be two reals such that $\theta_0 \leq \theta_\infty$. We assume that* $(3.k)$, $k = 1, 2$ *have no zeros in the strip C_{η_0,η_∞}, then for all $f \in L^2_{\theta_0,\theta_\infty}(\Omega_1)^3 \times L^2_{\theta_0,\theta_\infty}(\Omega_2)^3$, there exists one and only one solution u in $H^2_{\theta_0,\theta_\infty}(\Omega_1)^3 \times H^2_{\theta_0,\theta_\infty}(\Omega_2)^3$ for the problem (P_1) and we have*

$$\|u\|_{H^2_{\theta_0,\theta_\infty}(\Omega_1)^3 \times H^2_{\theta_0,\theta_\infty}(\Omega_2)^3} \leq c \, \|f\|_{L^2_{\theta_0,\theta_\infty}(\Omega_1)^3 \times L^2_{\theta_0,\theta_\infty}(\Omega_2)^3} \, .$$

Proof. (1) *Existence*. The hypothesis that $(3.k)$ has no zeros on the strip C_{η_0,η_∞} ensures the existence of η_0-solution and the $\eta_\infty-$solution of (P_1), that we note u_{η_0}, u_{η_∞}.

In addition $(F \cup G) \cap \{\eta_0 \leq Im\xi \leq \eta_\infty\} = \emptyset$, the proposition 4.1 implies that

$$u_{\eta_0} - u_{\eta_\infty} = i \sum_{\xi_0 \in (F \cup G) \cap \{\eta_0 \leq Im\xi \leq \eta_\infty\}} Res(e^{i\,t\xi} R_\xi(\widehat{g})) \,|_{\xi=\xi_0} \, .$$

This shows that $u_{\eta_0} = u_{\eta_\infty}$. We put now $u = u_{\eta_0}$, it is clear that

$$u \in H^2_{\theta_0,\theta_0}(\Omega_1)^3 \times H^2_{\theta_0,\theta_0}(\Omega_2)^3 \ \text{ and } u \in H^2_{\theta_\infty,\theta_\infty}(\Omega_1)^3 \times H^2_{\theta_\infty,\theta_\infty}(\Omega_2)^3.$$

The lemma 2.1, shows that $u \in H^2_{\theta_0,\theta_\infty}(\Omega_1)^3 \times H^2_{\theta_0,\theta_\infty}(\Omega_2)^3$. Thus u is a solution of (P_1) by construction.

(2) *Uniqueness.* We assume that there exist two solutions u^1 and u^2 in $H^2_{\theta_0,\theta_\infty}(\Omega_1)^3 \times H^2_{\theta_0,\theta_\infty}(\Omega_2)^3$. Then u^1, u^2 are η_0-solutions and $\eta_\infty-$solutions (property 4.1). It follows from the uniqueness of $\eta-$solutions that $u^1 = u^2$.

(3) *Continuity with respect to the data*. We deduce from property 4.2, that

$$\begin{aligned}
\|u\|_{H^2_{\theta_0,\theta_0}(\Omega_1)^3 \times H^2_{\theta_0,\theta_0}(\Omega_2)^3} &\leq c \, \|f\|_{L^2_{\theta_0,\theta_\infty}(\Omega_1)^3 \times L^2_{\theta_0,\theta_\infty}(\Omega_2)^3} \, , \\
\|u\|_{H^2_{\theta_\infty,\theta_\infty}(\Omega_1)^3 \times H^2_{\theta_\infty,\theta_\infty}(\Omega_2)^3} &\leq c \, \|f\|_{L^2_{\theta_0,\theta_\infty}(\Omega_1)^3 \times L^2_{\theta_0,\theta_\infty}(\Omega_2)^3} \, ,
\end{aligned}$$

and from lemma 2.1, we get

$$\|u\|_{H^2_{\theta_0,\theta_\infty}(\Omega_1)^3 \times H^2_{\theta_0,\theta_\infty}(\Omega_2)^3} \leq c \, \|f\|_{L^2_{\theta_0,\theta_\infty}(\Omega_1)^3 \times L^2_{\theta_0,\theta_\infty}(\Omega_2)^3} \, .$$

Which proves the theorem.■

5 Singularity solutions of the homogeneous elasticity system

Let us now examine the case of a homogeneous plate, the side surface of which makes an angle ω with the plane of the face. This case may be obtained by setting: $\nu = \nu_1 = \nu_2$, $\mu = \mu_1 = \mu_2$ and $\omega = \omega_1 = \omega_2$ in the relations previously derived.

Proposition 5.1. *The transcendental equations governing the singular behavior of the problem (P_1) take the form*

$$\begin{cases} \sin^2 \xi\omega - \dfrac{4(1-\nu)^2 - \ \xi^2 \sin^2 \omega}{3 - 4\nu} = 0, & \textit{problem of plane deformation,} \\ \cos \xi\omega = 0, & \textit{problem of normal plane deformation.} \end{cases} \tag{5.1}$$

Proof. Setting in (3.1) and (3.2): $\nu = \nu_1 = \nu_2$, $\mu = \mu_1 = \mu_2$ and $\omega = \omega_1 = \omega_2$ we obtain the characteristic equations (5.1).

The singular solutions of the problem (P_1) are given in the following proposition:

Proposition 5.2. *Let ξ_l denote the zeros of the transcendental equation* (5.1), *then the singular solutions of the problem (P_1) are given by*

$$\Im_l(r,\theta,x_3)=\begin{cases} r^{\xi}\Psi_{\xi}(\theta,x_3), & \textit{if } \xi \textit{ is a simple root of } (5.1),\\ \Im'_l=\dfrac{\partial\left(r^{\xi}\Psi_{\xi}(\theta,x_3)\right)}{\partial\xi}, & \textit{if } \xi \textit{ is a double root of } (5.1).\end{cases}$$

a- $\underline{\omega\in]0,\pi[\cup]\pi,2\pi[}$

$$\begin{aligned}\Im(r,\theta,x_3) &= cr^{-i\xi}\begin{pmatrix}(4\nu-i\xi-3)\left(L_{\xi}(\omega)\cos(1+i\xi)\theta-M_{\xi}(\omega)\sin(1+i\xi)\theta\right)\\ (-4\nu-i\xi+3)\left(L_{\xi}(\omega)\sin(1+i\xi)\theta+M_{\xi}(\omega)\cos(1+i\xi)\theta\right)\\ \cos(i\xi\theta)\end{pmatrix}\\ &\quad -cr^{-i\xi}\begin{pmatrix}L_{\xi}(\omega)(1-i\xi)\cos(1-i\xi)\theta-M_{\xi}(\omega)(1+i\xi)\sin(1-i\xi)\theta\\ -L_{\xi}(\omega)(1-i\xi)\sin(1-i\xi)\theta-M_{\xi}(\omega)(1+i\xi)\cos(1-i\xi)\theta\\ 0\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}L_{\xi}(\omega) &= (2\nu-i\xi-2)\sin\omega\cos(i\xi\omega)\ -(1-2\nu)\cos(\omega)\sin(i\xi\omega).\\ M_{\xi}(\omega) &= -(2\nu-i\xi-1)\sin\omega\sin(i\xi\omega)-2(1-\nu)\cos(\omega)\cos(i\xi\omega)\,.\end{aligned}$$

b- $\underline{\omega=2\pi}$

$$\Im(r,\theta,x_3)=cr^{-i\xi}\begin{pmatrix}(4\nu-i\xi-3)\cos(1+i\xi)\theta-(1-i\xi)\cos(1-i\xi)\theta\\ -(4\nu+i\xi-3)\sin(1+i\xi)\theta+(1-i\xi)\sin(1-i\xi)\theta\\ r^{(\frac{1}{4}+i\xi)}\cos(\frac{\theta}{4})\end{pmatrix},$$

$$\Im'(r,\theta,x_3)=cr^{-i\xi}\begin{pmatrix}-(4\nu-i\xi-3)\sin(1+i\xi)\theta+(1+i\xi)\sin(1-i\xi)\theta\\ (4\nu+i\xi-3)\cos(1+i\xi)\theta+(1+i\xi)\cos(1-i\xi)\theta\\ 0\end{pmatrix}.$$

Proof. Let ξ_l denote the zeros of the equation (5.1) in the strip C_{η_0,η_∞}. A general solution of homogeneous system (P_3) is given by

$$\widehat{\widehat{u}}=\sum_{k=1}^{4}a_k e_k,$$

where

$$\begin{aligned}e_1 &= (ch\,(\xi-i)\,\theta,-i\ sh((\xi-i)\,\theta)\,,\\ e_2 &= (i\ sh\,(\xi-i)\,\theta,ch\,(\xi-i)\,\theta)\,,\\ e_3 &= \frac{1}{\xi}\left((A\ ch\,(\xi-i)\,\theta-B\ ch\,(\xi+i)),-i\ A(sh(\xi-i)\theta+sh(\xi+i)\theta)\right),\\ e_4 &= \frac{1}{\xi}\left(-iB\ (sh(\xi-i)\theta+sh(\xi+i)\theta),\ B\ ch(\xi-i)\theta-A\ ch(\xi+i)\theta\right),\end{aligned}$$

with

$$A=3-4\nu+i\xi,\quad B=3-4\nu-i\xi\ \text{ and }\ i^2=-1.$$

By setting $\theta=0$ and $\theta=\omega$ in the boundary conditions (BC), we obtain a system of homogeneous equations. The condition of the vanishing of the system's determinant gives the transcendental equations (5.1) with respect to ξ. So for any ξ a complex solution of (5.1), the solutions of this system give the singular solution $\Im(r,\theta,x_3)$ for $\omega\in]0,\pi[\cup]\pi,2\pi[$.

In the same way setting $\theta=2\pi$ in (BC), we obtain the component of the singular solution for $\omega=2\pi$.

This ends the proof.■

6 Conclusion and perspectives

The purpose of this paper is to study the singular behavior of solutions of a boundary value problem with mixed conditions in a neighborhood of an edge in the general framework of weighted Sobolev spaces. This work is an extension to similary ones in Sobolev spaces with null and single weight. In the non homogeneous case, it's not easy to solve the transcendental equations defined in the proposition 3.1, this does not permit us to find the singular solutions.

We will devote a further paper for the generalization of the results obtained here for the non-homogeneous case with presence of discontinuity of the boundary value on the intersection surface.

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