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On some qualitative behaviors of solutions to a kind of third order nonlinear delay differential equations

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Abstract

Sufficiency criteria are established to ensure the asymptotic stability and boundedness of solutions to third-order nonlinear delay differential equations of the form

$$\dddot{x}(t) + e(x(t), \dot{x}(t), \ddot{x}(t))\ddot{x}(t) + g(x(t-r), \dot{x}(t-r)) + \psi(x(t-r))$$

$$= p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)).$$

By using Lyapunov's functional approach, we obtain two new results on the subject, which include and improve some related results in the relevant literature. Two examples are also given to illustrate the importance of results obtained.

Keywords : Delay differential equation of third order, Lyapunov functional, stability, boundedness.

AMS Subject Classification: 34K20.

1. Introduction

As is well-known, the area of differential equations is an old but durable subject that remains alive and useful to a wide variety of engineers, scientists, and mathematicians. Now, with over 300 years of history, the subject of differential equations represents a huge body of knowledge including many subfields and a vast array of applications in many disciplines. It is beyond exposition as a whole. It should be noted that principles of differential equations are largely related to the qualitative theory of ordinary differential equations. Qualitative theory refers to the study of behavior of solutions, for example the investigation of stability, instability, boundedness of solutions and etc., without determining explicit formulas for the solutions. Besides, stability and boundedness of solutions are also very important problems in the theory and applications of differential equations. In particular, the stability of motion in dynamical systems is an old but still active area of studies. At the close of the 19th century, three types of stability, Lyapunov stability, Poincare stability and Zhukovskij stability, were established for motion in continuous dynamical systems, i.e., for solutions of differential equations. Among them the Lyapunov stability and Poincare stability are most well-known. It is worth mentioning that if solutions of a differential equation describing a dynamical system or of any differential equation under consideration are known in closed form, one can determine the stability and boundedness properties of the system or the solutions of differential equation, appealing directly the definitions of stability and boundedness. As is well-known, in general, it is also not possible to find the solution of all linear and nonlinear differential equations, except numerically. Moreover, finding of solutions becomes more difficult for delay differential equations rather than the differential equations without delay. Therefore, it is very important to obtain information on the stability and boundedness behavior of solutions to differential equations when there is no analytical expression for solutions. So far, the most efficient tool to the study of stability and boundedness of solutions of a given nonlinear system is provided by Lyapunov's theory [12], that is, the Lyapunov's second (or direct) method. It is also worth mentioning that this theory became an important part of both mathematics and theoretical mechanics in twentieth century. By means of Lyapunov's second method, the stability in the large and boundedness of solutions can be obtained without any prior knowledge of solutions. That is, the method yields stability and boundedness information directly, without solving the differential equation. The chief characteristic of the method requires the construction of the scalar function and functional for the equation under study. Unfortunately, it is some times very difficult, even impossible, to find a proper Lyapunov function or functional for a given equation. However, within the past forty-five years and so, by using the Lyapunov's [12] second (or direct) method, many good results have been obtained and are still obtaining on the qualitative behaviors of solutions for various third order ordinary non-linear differential equations without delay. In particular, one can refer to the book of Reissig et al. [15] as a survey and the papers of Qian [14], Tunç ([20], [21]) and references quoted in these sources for some publications performed on the topic, which include the differential equations without delay. Besides, it is worth mentioning that, according to our observations, there are only a few papers on the same topics related to certain third order nonlinear differential equations with delay (See, Bereketoğlu and Karakoç [1], Chukwu [3], Sadek ([16], [17]), Sinha [18], Tejumola and Tchegnani [19], Tunç [22-26] and Zhu [28]). Perhaps, the possible difficulty raised to this case is due to the construction of Lyapunov functionals for delay differential equations. How to construct those Lyapunov functionals? So far, no author has discussed them. In fact, there is no general method to construct Lyapunov functionals. Clearly, it is also more difficult to construct Lyapunov's functional for higher order differential equations with delay than without delay. Meanwhile, especially, since 1960s many good books, most of them are in Russian literature, have been published on the delay differential equations (see for example the books of Burton [2], Èl'sgol'ts [4], Èl'sgol'ts and Norkin [5], Gopalsamy [6], Hale [7], Hale and Verduyn Lunel [8], Kolmanovskii and Myshkis [9], Kolmanovskii and Nosov [10], Krasovskii [11], Makay [13], Yoshizawa [27] and the references listed in these books).

In the present paper, we take into consideration the following nonlinear differential equation of third order with delay

$$\dddot{x}(t) + e(x(t),\dot{x}(t),\ddot{x}(t))\ddot{x}(t) + g(x(t-r),\dot{x}(t-r)) + \psi(x(t-r))$$
$$= p(t,x(t),x(t-r),x'(t),x'(t-r),x''(t)), \tag{1}$$

where $r > 0$ is a constant; $e(x,\dot{x},\ddot{x})$, $g(x,\dot{x})$, $\psi(x)$ and $p(t,x(t),x(t-r),x'(t),x'(t-r),x''(t))$ are continuous functions such that $g(x,0) = \psi(0) = 0$. This fact guarantees the existence of the solution of delay differential equation (1). The derivatives $\frac{d\psi}{dx} \equiv \psi'(x)$, $\frac{\partial}{\partial x}g(x,\dot{x}) \equiv g_x(x,\dot{x})$, $\frac{\partial}{\partial x}e(x,\dot{x},\ddot{x}) \equiv e_x(x,\dot{x},\ddot{x})$ and $\frac{\partial}{\partial \ddot{x}}e(x,\dot{x},\ddot{x}) \equiv e_{\ddot{x}}(x,\dot{x},\ddot{x})$ exist and are also continuous.

Now, equation (1) can be transformed into the equivalent system:

$$\dot{x}(t) = y(t),$$

$$\dot{y}(t) = z(t),$$

$$\dot{z}(t) = -\,e(x(t), y(t), z(t))z(t) - g(x(t), y(t)) - \psi(x(t)) + \int_{t-r}^{t} g_x(x(s), y(s))y(s)ds \qquad (2)$$

$$+\int_{t-r}^{t} g_y(x(s), y(s))z(s)ds + \int_{t-r}^{t} \psi'(x(s))y(s)ds$$

$$+p(t, x(t), x(t-r), y(t), y(t-r), z(t)),$$

where $x(t)$, $y(t)$ and $z(t)$ are respectively abbreviated as x, y and z throughout the paper. All solutions considered are assumed to be real valued. In addition, it is assumed that the functions $e(x(t), y(t), z(t))$, $g(x(t-r), y(t-r))$, $\psi(x(t-r))$ and $p(t, x(t), x(t-r), y(t), y(t-r), z(t))$ satisfy a Lipschitz condition in $x(t)$, $y(t)$, $z(t)$, $x(t-r)$ and $y(t-r)$. Then the solution is unique.

It should also be noted that here based of the result of Sinha [18] we establish our asymptotic stability result. Next, in view of the publication dates of the papers mentioned above, (see Sinha [18], Chukwu [3], Zhu [28], Sadek [16], Sadek [17], Bereketoğlu and Karakoç [1], it is very interesting that all authors did not make reference to the work that was carried out before their investigation except only the existence of reference [28] in Sadek ([16], [17]) and the reference [3] in [1], respectively. Finally, to the best of our knowledge from the literature, it is not found any boundedness result based on the result of Sinha [18].

2. Preliminaries

In order to reach our main result, first, we will give some basic definitions and some important stability criteria for the general non-autonomous and autonomous delay differential system (see also Burton [2], Èl'sgol'ts [4], Èl'sgol'ts and Norkin [5], Gopalsamy [6], Hale [7], Hale and Verduyn Lunel [8], Kolmanovskii and Myshkis [9], Kolmanovskii and Nosov [10], Krasovskii [11], Makay [13] and Yoshizawa [27]). Now, we consider the general non-autonomous delay differential system

$$\dot{x} = f(t, x_t), x_t(\theta) = x(t + \theta), -r \leq \theta \leq 0, t \geq 0, \tag{3}$$

where $f : [0,\ \infty) \times C_H \rightarrow \Re^n$ is a continuous mapping, $f(t, 0) = 0$, and we suppose that f takes closed bounded sets into bounded sets of $\Re^n$. Here $(C,\ \ \|.\ \|)$ is the Banach space of continuous function $\phi : [-r,\ 0] \rightarrow \Re^n$ with supremum norm, $r > 0$, C_H is the open H -ball in C ; $C_H := \{\phi \in (C\,[-r, 0]\,,\ \ \Re^n) : \|\phi\| < H\}$. Standard existence theory, see Burton [2], shows that if $\phi \in C_H$ and $t \geq 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ such that on $[t_0,\ t_0 + \alpha)$ satisfying equation (3) for $t > t_0$, $x_t(t, \phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B , then $\alpha = \infty$. Further, the symbol $|.\ |$ will denote the norm in $\Re^n$ with $|x| = \max_{1 \leq i \leq n} |x_i|$.

Definition 1. (See [2].) Let $f(t, 0) = 0$. The zero solution of equation (3) is:

(a) stable if for each $t_1 \geq t_0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $[\ \ \|\phi\| \leq \delta,\ \ t \geq t_1]$ imply that $|x(t, t_1, \phi)| < \varepsilon$.

(b) asymptotically stable if it is stable and if for each $t_1 \geq t_0$ there is an η such that $\|\phi\| \leq \eta$ implies that $x(t, t_0, \phi) \rightarrow 0$ as $t \rightarrow \infty$.

Definition 2. (See [2].) A continuous positive definite function $W : \Re^n \rightarrow [0,\ \infty)$ is called a wedge.

Definition3. (See [2].) A continuous function $W : [0,\ \infty) \rightarrow [0,\ \infty)$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i , where i an integer.)

Definition 4. (See [2].) Let D be an open set in $\Re^n$ with $0 \in D$. A function $V : [0,\ \infty) \times D \rightarrow [0,\ \infty)$ is called

(a) positive definite if $V(t, 0) = 0$ and if there is a wedge W_1 with $V(t, x) \geq W_1(|x|)$,

(b) decresent if there is a wedge W_2 with $V(t, x) \leq W_2(|x|)$.

Definition 5. (See [3].) Let $V(t, \phi)$ be a continuous functional defined for $t \geq 0$, $\phi \in C_H$. The derivative of V along solutions of (3) will be denoted by $\dot{V}_{(3)}$ and is defined by the following relation

$$\dot{V}_{(3)}(t,\phi) = \limsup_{h\to 0} \frac{V(t+h, x_{t+h}(t_0,\phi)) - V(t, x_t(t_0,\phi))}{h},$$

where $x(t_0,\phi)$ is the solution of (3) with $x_{t_0}(t_0,\phi) = \phi$.

Theorem 1. (See([17].) Suppose that there exists a Lyapunov functional $V(t,\phi)$ for (3) such that the following conditions are satisfied:

(i) $W_1(|\phi(0)|) \le V(t,\phi) \le W_2(\|\phi\|)$, (where $W_1(r)$ and $W_2(r)$ are wedges,) and

(ii) $\dot{V}(t,\phi) \le 0$.

Then, the zero solution of (3) is uniformly stable.

Now, we also consider the general autonomous delay differential system

$$\dot{x} = f(x_t), x_t(\theta) = x(t+\theta), -r \le \theta \le 0, t \ge 0, \tag{4}$$

where $f : C_H \to \Re^n$ is a continuous mapping, $f(0) = 0$, $C_H := \{\phi \in (C[-r,0]\,,\ \Re^n) : \|\phi\| < H\}$ and for $H_1 > H$, there exists $L(H_1) > 0$,with $|f(\phi)| \le L(H_1)$, when $\|\phi\| \le H_1$ (see also [26]). It is clear that the general autonomous delay differential system (4) is a special case of system (3), and the following definition and lemma are given.

Definition 6. (See [18].) Let V be a continuous scalar function on C_H^n , where C_H^n denotes the set of ϕ in C^n for which $\|\phi\| < H$, and C^n denotes the spaces of continuous functions mapping from the interval $[-r,0]$ into $\Re^n$ and for $\phi \in C_H$, $\|\phi\| = \sup_{-r\le\phi\le 0}\|\phi(\theta)\|$. The derivative of V along solutions of (4) will be denoted by $\dot{V}_{(4)}$ and is defined by the following relation

$$\dot{V}_{(4)}(\phi) = \limsup_{h\to 0^+} \frac{V(x_h(\phi)) - V(\phi))}{h}.$$

Lemma 1. (See[26].) Let $V(\phi) : C_H \to \Re$ be a continuous functional satisfying a local Lipschitz by condition , and assume that $V(0) = 0$ and that:

(i) $W_1(|\phi(0)|) \le V(\phi) \le W_2(\|\phi\|)$,where $W_1(r)$ and $W_2(r)$ are wedges;

(ii) $\dot{V}_{(4)}(\phi) \le 0$ for $\phi \in C_H$. Then the zero solution of (4) is uniformly stable. If we define $Z = \left\{\phi \in C_H : \dot{V}_{(4)}(\phi) = 0\right\}$, then the zero solution of (4) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

Example 1. Now,we consider the second order non-linear delay differential equation of the form

$$\ddot{x}(t) + \varphi(t, x(t), \dot{x}(t)) + \sin(x(t - r)) = 0, \tag{5}$$

where r is a positive constant; φ is a continuous function, $\varphi(t, x(t), 0) = 0$ and $sin(x(t - r))$ is a continuously differentiable function, which satisfy the following conditions

$$\varepsilon_1 + b \geq \frac{\sin x}{x} \geq b(x \neq 0) \text{ for all } x \text{ with } |x| < \pi \tag{6}$$

and

$$\frac{\varphi(t, x, y)}{y} \geq a \text{ for all } t \geq 0, x \text{ and } y(y \neq 0), \tag{7}$$

where a , b and ε_1 are some positive constants. Equation (5) can be rewritten as the following equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\varphi(t, x, y) - \sin x + \int_{t-r}^{t} \cos(x(s))y(s)ds. \end{aligned} \tag{8}$$

The following Lyapunov functional is defined

$$V(x_t, y_t) = \int_0^x \sin s ds + \frac{y^2}{2} + \lambda \int_{-r}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds \tag{9}$$

to verify the stability of trivial solution $x = 0$ of equation (5), where λ is a positive constant which will be determined later, and it is obvious that the term $\int_{-r}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds$ is non-negative. It is clear that the Lyapunov functional $V(x_t, y_t)$ in (9) is positive definite. Namely, $V(0, 0) = 0$, and we also have

$$\int_0^x \frac{\sin s}{s} s ds + \frac{1}{2} y^2 + \lambda \int_{-r}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds = V(x_t, y_t).$$

Making use of the assumption $0 < b \leq \frac{\sin x}{x}$ in (6), it follows that

$$\frac{b}{2} x^2 + \frac{1}{2} y^2 + \lambda \int_{-r}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds \leq V(x_t, y_t).$$

Similarly, in view of the above assumptions, it is readily verified from (9) that

$$V(x_t,y_t) \le \left(\frac{b+\varepsilon_1}{2}\right)x^2+\frac{1}{2}y^2+\lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds.$$

Thus,

$$\tfrac{b}{2}x^2(t)+\tfrac{1}{2}y^2(t)+\lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds \le V(x_t,y_t)$$

$$\le \left(\tfrac{b+\varepsilon_1}{2}\right)x^2(t)+\tfrac{1}{2}y^2(t)+\lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds.$$

Hence, there exist positive constants D_1 and D_2 such that

$$D_1\left(x^2(t)+y^2(t)\right)+\lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds \le V(x_t,y_t) \le D_2\left(x^2(t)+y^2(t)\right)+\lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds,$$

where $D_1=\min\left\{\frac{b}{2},\frac{1}{2}\right\}$ and $D_2=\max\left\{\frac{b+\varepsilon_1}{2},\frac{1}{2}\right\}$. Now, the existence of continuous functions $W_1(|\phi|)$ with $W_1(|\phi(0)|)\ge 0$ and $W_2(\|\phi\|)$ which satisfy the inequality $W_1(|\phi(0)|)\le V(\phi)\le W_2(\|\phi\|)$ is easily verified.

Finally, evaluating the time derivative of the functional $V(x_t,y_t)$, that is, $\dot V=\frac{d}{dt}V(x_t,y_t)$, it follows from (9) and (8) that

$$\dot V = -\varphi(t,x,y)y+\lambda y^2 r-\lambda\int_{t-r}^{t} y^2(s)ds = -\left(\frac{\varphi(t,x,y)}{y}\right)y^2+\lambda y^2 r-\lambda\int_{t-r}^{t} y^2(s)ds.$$

Now, making use of the assumption $\frac{\varphi(t,x,y)}{y}\ge a>0$ $(y\ne 0)$ and inequality $2|uv|\le u^2+v^2$, it follows that

$$\dot V \le -ay^2+\lambda y^2 r-\lambda\int_{t-r}^{t} y^2(s)ds = -(a-\lambda r)\,y^2-\lambda\int_{t-r}^{t} y^2(s)ds. \tag{10}$$

If we choose $\lambda=\frac{L}{2}$, (10) implies for some constant $\alpha>0$ that

$$\dot V \le -\alpha\, y^2 \le 0 \text{ provided } r<2aL^{-1}.$$

If we define $Z=\left\{\phi\in C_H : \dot V_{(8)}(\phi)=0\right\}$, then the zero solution of system (8) is asymptotically stable, provided that the largest invariant set in Z is $Q=\{0\}$. Thus, under the above discussion, one can say that the zero solution of equation (5) is asymptotically stable.

Meanwhile, it should be noted that under less restrictive conditions, it can be easily shown that all solutions of delay differential equation

$$\ddot{x}(t) + \varphi(t, x(t), \dot{x}(t)) + \sin(x(t-r)) = \frac{1}{1 + t^2 + x^2(t) + x^2(t-r) + x'^2(t) + x'^2(t-r)}$$

are bounded. Therefore, we omit details of related operations.

3. Main results

Let $p(t, x(t), x(t-r), y(t), y(t-r), z(t)) = 0$.

Our first main result is the following.

Theorem 2. In addition to the basic assumptions imposed on the functions e, g and ψ that appearing in (1), we assume that there are positive constants λ, α, a, b, c, L_1, L_2, L_3 and k_1 such that the following conditions hold:

(i) $e(x, y, z) \geq a + \frac{2\lambda}{\alpha} > 0$, $ye_x(x, y, 0) \leq 0$ and $ye_z(x, y, z) \leq 0$ for all x, y and z.

(ii) $\frac{g(x,y)}{y} \geq b + 2\lambda$, $(y \neq 0)$, $|g_y(x, y)| \leq L_2$ and $|g_x(x, y)| \leq L_3$ for all x and y .

(iii) $\frac{b}{c} > \alpha > \frac{1}{a}$, $c \geq \frac{\psi(x)}{x} > k_1$, $(x \neq 0)$, $|\psi'(x)| \leq L_1$, $\int_0^x \psi(\xi) d\xi \to +\infty$ as $|x| \to \infty$ and $b + 2\lambda - \alpha L_1 > 0$.

Then the zero solution of equation (1) is asymptotically stable, provided that

$$r < \min\left\{ \frac{2(b + 2\lambda - \alpha L_1)}{L_1 + L_2 + L_3 + (1 + \alpha)(L_1 + L_3)}, \ \frac{2(\alpha\, a + 2\lambda - 1)}{\alpha(L_1 + L_2 + L_3) + (1 + \alpha)L_2} \right\}$$

with $\gamma = \frac{(1+\alpha)(L_1+L_3)}{2}$ and $\mu = \frac{(1+\alpha)L_2}{2}$.

Proof. Our main tool for the proof of Theorem 2 is the Lyapunov functional $V_0 = V_0(x_t, y_t, z_t)$ defined by

$$V_0(x_t, y_t, z_t) = V_1(x, y) + \frac{1}{2} V_2(x, y, z) + \gamma \int_{-r}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds + \mu \int_{-r}^{0} \int_{t+s}^{t} z^2(\theta) d\theta ds, \qquad (11)$$

where

$$V_1(x, y) = \int_0^x \psi(\xi) d\xi + \alpha \psi(x) y + \alpha \int_0^y g(x, \eta) d\eta \qquad (12)$$

and

$$V_2(x,y,z)= a\, y^2 + 2yz + \alpha\, z^2 + 2\int_0^y [e(x,s,0)-a]\ sds. \tag{13}$$

Now, in view of conditions (ii) and (iii) of Theorem 2, it follows from (12) that

$$\begin{aligned}
V_1 &= \tfrac{1}{2}\left[2\alpha\int_0^y \tfrac{g(x,\eta)}{\eta}\eta d\eta + 2\alpha\psi(x)y\right] + \int_0^x \psi(\xi)d\xi \\
&\geq \tfrac{1}{2}\left[\alpha\, b\, y^2 + 2\alpha\psi(x)y\right] + \int_0^x \psi(\xi)d\xi \\
&= \tfrac{\alpha}{2b}\left[by+\psi(x)\right]^2 + \int_0^x \psi(\xi)d\xi - \tfrac{\alpha}{2b}\psi^2(x) \\
&= \tfrac{\alpha}{2b}\left[by+\psi(x)\right]^2 + \int_0^x \left[1-\tfrac{\alpha\psi'(\xi)}{b}\right]\psi(\xi)d\xi \\
&\geq \int_0^x \left[1-\tfrac{\alpha\, c}{b}\right]\psi(\xi)d\xi = \delta_1\int_0^x \tfrac{\psi(\xi)}{\xi}\xi d\xi > \tfrac{\delta_1 k_1}{2}x^2,
\end{aligned}$$

because $\delta_1 = \left(1-\frac{\alpha\, c}{b}\right) > 0$ and $\frac{\psi(x)}{x} > k_1$. Similarly, the function V_2 can be expressed as a quadratic form:

$$V_2 = (y,z)\begin{pmatrix} a & 1 \\ 1 & \alpha \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} + 2\int_0^y [e(x,s,0)-a]\ sds.$$

Now, $a\alpha > 1$. Hence the matrix $\begin{pmatrix} a & 1 \\ 1 & \alpha \end{pmatrix}$ is positive definite. Making use of the positive definiteness of above matrix and assumption $e(x,y,0) \geq a > 0$ of Theorem 2, we conclude that there exists a positive constant δ_2 such that

$$V_2 \geq \frac{\delta_2}{2}(y^2+z^2).$$

Therefore, subject to the above discussion, the existence of a continuous function $W_1(|\phi|)$ with $W_1(|\phi(0)|) \geq 0$ which satisfies the inequality $W_1(|\phi(0)|) \leq V_0(\phi)$ is easily verified, since $\gamma = \frac{(1+\alpha)L_1+(1+\alpha)L_3}{2} > 0$ and $\mu = \frac{(1+\alpha)L_2}{2} > 0$, and $\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$ and

$\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds$ are non-negative. Now, calculating the derivative of the functional $V_0(x_t, y_t, z_t)$ along a solution $(x(t), y(t), z(t))$ of system (2), we obtain

$$
\begin{aligned}
\dot{V}_0(x_t, y_t, z_t) = & -\left[\tfrac{g(x,y)}{y} - \alpha\psi'(x)\right] y^2 - \left[\alpha\ e(x,y,z) - 1\right] z^2 \\
& + y\int_0^y e_x(x,s,0)sds - \left[\tfrac{e(x,y,z)}{z} - \tfrac{e(x,y,0)}{z}\right] y\ z^2 \\
& +(y+\alpha\ z)\int_{t-r}^{t} g_y(x(s),y(s))z(s)ds + (y+\alpha\ z)\int_{t-r}^{t} g_x(x(s),y(s))y(s)ds \qquad (14) \\
& +(y+\alpha\ z)\int_{t-r}^{t} \psi'(x(s))y(s)ds + \gamma\ r\ y^2 - \gamma\int_{t-r}^{t} y^2(s)ds \\
& +\mu\ r\ z^2 - \mu\int_{t-r}^{t} z^2(s)ds.
\end{aligned}
$$

In the light of the hypothesis of Theorem 2 and the mean value theorem (for the derivative), it can be easily obtained the following inequalities for the first four terms in (14):

$$\left[\frac{e(x,y,z)}{z} - \frac{e(x,y,0)}{z}\right] y\ z^2 = y\ z^2 e_z(x,y,\theta z) \le 0,$$

where $0 \le \theta \le 1$. Next, the assumption $ye_x(x,y,0) \le 0$ of Theorem 2 shows that

$$y\int_0^y e_x(x,s,0)sds \le 0.$$

Finally,

$$\left[\frac{g(x,y)}{y} - \alpha\ \psi'(x)\right] y^2 \ge \left[b + 2\lambda - \alpha\ L_1\right] y^2$$

and

$$\left[\alpha\ e(x,y,z) - 1\right] z^2 \ge \left[\alpha a + 2\lambda - 1\right] z^2.$$

Combining the above estimates with that into (14), we get

$$
\begin{aligned}
\dot{V}_0(x_t, y_t, z_t) \;\; &\leq - \;\; [b + 2\lambda - \alpha\, L_1]\, y^2 - \;\; [\alpha\, a + 2\lambda - 1]\, z^2 \\
&+ (y + \alpha\, z) \int\limits_{t-r}^{t} g_y(x(s), y(s)) z(s) ds + (y + \alpha\, z) \int\limits_{t-r}^{t} g_x(x(s), y(s)) y(s) ds \\
&+ (y + \alpha\, z) \int\limits_{t-r}^{t} \psi'(x(s)) y(s) ds + \gamma\, r\, y^2 - \gamma \int\limits_{t-r}^{t} y^2(s) ds \\
&+ \mu\, r\, z^2 - \mu \int\limits_{t-r}^{t} z^2(s) ds.
\end{aligned} \tag{15}
$$

Now, in view of assumptions $|\psi'(x)| \leq L_1$, $|g_x(x, y)| \leq L_3$, $|g_y(x, y)| \leq L_2$ of Theorem 2 and inequality $2\,|uv| \leq u^2 + v^2$, we see from (15) that

$$
\begin{aligned}
\dot{V}_0(x_t, y_t, z_t) \;\; &\leq - \;\; \left[b + 2\lambda - \alpha\, L_1 - \tfrac{L_1 r}{2} - \tfrac{L_2 r}{2} - \tfrac{L_3 r}{2} - \gamma\, r\right]\; y^2 \\
&- \;\; \left[\alpha\, a + 2\lambda - \tfrac{\alpha L_1 r}{2} - \tfrac{\alpha L_2 r}{2} - \tfrac{\alpha L_3 r}{2} - 1 - \mu\, r\right]\; z^2 \\
&+ \left[\tfrac{(1+\alpha)(L_1+L_3)}{2} - \gamma\right] \int\limits_{t-r}^{t} y^2(s) ds \\
&+ \left[\tfrac{(1+\alpha)L_2}{2} - \mu\right] \int\limits_{t-r}^{t} z^2(s) ds.
\end{aligned} \tag{16}
$$

If we take $\gamma = \frac{(1+\alpha)(L_1+L_3)}{2}$ and $\mu = \frac{(1+\alpha)L_2}{2}$, then we have from (16) that

$$
\begin{aligned}
\dot{V}_0(x_t, y_t, z_t) \;\; &\leq - \;\; \left[b + 2\lambda - \alpha\, L_1 - \left(\tfrac{L_1+L_2+L_3+(1+\alpha)(L_1+L_3)}{2}\right) r\right]\; y^2 \\
&- \;\; \left[\alpha\, a + 2\lambda - 1 - \left(\tfrac{\alpha(L_1+L_2+L_3)+(1+\alpha)L_2}{2}\right) r\right] z^2.
\end{aligned}
$$

Hence, it follows that

$$
\frac{d}{dt} V_0(x_t, y_t, z_t) \leq -\rho(y^2 + z^2) \text{ for some constant } \rho > 0,
$$

provided that

$$r < \min\left\{\frac{2(b+2\lambda-\alpha L_1)}{L_1+L_2+L_3+(1+\alpha)(L_1+L_3)},\ \frac{2(\alpha\, a+2\lambda-1)}{\alpha(L_1+L_2+L_3)+(1+\alpha)L_2}\right\}$$

Now, using $\frac{d}{dt}V_0(x_t,y_t,z_t)=0$ and system (2), we can easily get $x=y=z=0$. Thus, $W_1(|\phi(0)|)\le V_0(\phi)$ and $\frac{d}{dt}V_0(x_t,y_t,z_t)\equiv 0$ if and only if $x=y=z=0$. Consequently, the zero solution of equation (1) is asymptotically stable (see also Sinha [18, Lemma1]).

This completes the proof of Theorem 2.

Example 2. Consider the equation

$$\dddot{x}(t)+[3+\exp(-x(t)\dot{x}(t)-\dot{x}(t)\ddot{x}(t))]\ddot{x}(t)+(4+\exp(-\dot{x}^2(t-r)))\dot{x}(t)+x(t)=0$$

or it is equivalent system

$$\dot{x}=y,\dot{y}=z,$$

$$\dot{z}=-[3+\exp(-xy-yz)]z-(4+\exp(-y^2(t-r)))y-x,$$

which is a special case of equation (1) provided that $e(x,y,z)=3+\exp(-xy-yz)$, $g(x(t-r),y(t-r))=(1+\exp(-y^2(t-r)))y$ and $\psi(x)=x$. Now, it is readily seen that

$$e(x,y,z)=3+\exp(-xy-yz)>0, ye_x(x,y,0)=-y^2\exp(-xy)\le 0,$$

$$ye_z(x,y,z)=-y^2\exp(-xy-yz)\le 0, \frac{g(x,y)}{y}=1+\exp(-y^2(t-r))>0,$$

$$g_x(x,y)=0, |g_y(x,y)|=1+\frac{1}{\exp(y^2(t-r))}+\frac{|2yy(t-r)|}{\exp(y^2(t-r))}\le 3$$

and

$$\frac{\psi(x)}{x}=1, \psi'(x)=1, \int_0^x \psi(\xi)d\xi=\int_0^x \xi d\xi=\frac{x^2}{2}\to\infty \text{ as } |x|\to\infty.$$

Therefore, all the hypotheses of Theorem 2 are satisfied.

Remark. Our result includes and improves the results of Sadek [16], Sinha [18], Tunç [22] and Zhu [28], which investigated the stability of solutions to third order nonlinear differential equations with delay. Because, equation (1) is more general than that considered in the above mentioned papers, and all papers were published without an explanatory example on the stability of solutions. Our assumptions and the Lyapunov functional constructed here are also completely different than that exist in Bereketoğlu and Karakoç [1] and Chukwu [3].

Now, let $p(t, x(t), x(t-r), y(t), y(t-r), z(t)) \neq 0$.

Our second and last main result is the following.

Theorem 3. Let us assume that assumptions (i)-(iii) of Theorem 2 hold. In addition, we assume that

$$|p(t, x(t), x(t-r), y(t), y(t-r), z(t))| \leq q(t) \text{ for all } t, x, x(t-r), y, y(t-r) \text{ and } z,$$

where $q \in L^1(0, \infty)$, $L^1(0, \infty)$ is space of integrable Lebesgue functions. Then, there exists a finite positive constant K such that the solution $x(t)$ of equation (1) defined by the initial functions

$$x(t) = \phi(t), x'(t) = \phi'(t), x''(t) = \phi''(t)$$

satisfies the inequalities

$$|x(t)| \leq K, |x'(t)| \leq K, |x''(t)| \leq K$$

for all $t \geq t_0$, where $\phi \in C^2([t_0 - r, t_0], \Re)$, provided that

$$r < \min\left\{\frac{2(b + 2\lambda - \alpha L_1)}{L_1 + L_2 + L_3 + (1+\alpha)(L_1 + L_3)}, \ \frac{2(\alpha\, a + 2\lambda - 1)}{\alpha(L_1 + L_2 + L_3) + (1+\alpha)L_2}\right\}$$

with $\gamma = \frac{(1+\alpha)(L_1+L_3)}{2}$ and $\mu = \frac{(1+\alpha)L_2}{2}$.

Proof. For the proof of this theorem, as in Theorem 2, we use the Lyapunov functional. $V_0 = V_0(x_t, y_t, z_t)$ given in (11). Obviously, it can be followed from the discussion of Theorem 2 that there exists a positive D_3 such that

$$D_3(x^2 + y^2 + z^2) \leq V_0(x_t, y_t, z_t),$$

where $D_3 = \min\left\{2^{-1}\delta_1 k_1, \ 2^{-1}\delta_2\right\}$.

Now, the time derivative of functional $V_0(x_t, y_t, z_t)$ along system (2) can be revised as:

$$\frac{d}{dt}V_0(x_t, y_t, z_t) \leq -\rho(y^2 + z^2) + (2y + 2\alpha z)p(t, x(t), x(t-r), y(t), y(t-r), z(t)).$$

By the assumptions of Theorem 3, we have

$$\frac{d}{dt}V_0(x_t, y_t, z_t) \le (2|y| + 2\alpha|z|)\ |p(t, x(t), x(t-r), y(t), y(t-r), z(t))|$$

$$\le 2(|y| + \alpha|z|)\, q(t) \ \le D_4(|y| + |z|)\, q(t),$$

where $D_4 = \max\{2, 2\alpha\}$. Hence

$$\frac{d}{dt}V_0(x_t, y_t, z_t) \le D_4\left(2 + y^2 + z^2\right)\ q(t).$$

Clearly,

$$(y^2 + z^2) \le D_3^{-1}\ V_0(x_t, y_t, z_t).$$

Therefore,

$$\begin{aligned}\frac{d}{dt}V_0(x_t, y_t, z_t) &\le D_4\left(2 + D_3^{-1}V_0(x_t, y_t, z_t)\right) q(t) \\ &= 2D_4 q(t) + D_4 D_3^{-1} V_0(x_t, y_t, z_t) q(t).\end{aligned} \tag{17}$$

Integrating (17) from 0 to t, using the assumption $q \in L^1(0, \infty)$ and Gronwall-Reid-Bellman inequality, we obtain

$$\begin{aligned}V_0(x_t, y_t, z_t) &\le V_0(x_0, y_0, z_0) + 2D_4 A + D_4 D_3^{-1}\int_0^t (V_0(x_s, y_s, z_s))\, q(s)ds \\ &\le (V_0(x_0, y_0, z_0) + 2D_4 A)\exp\left(D_4 D_3^{-1} A\right) = K_1 < \infty,\end{aligned} \tag{18}$$

where $K_1 > 0$ is a constant, $K_1 = (V_0(x_0, y_0, z_0) + 2D_4 A)\ \exp\left(D_4 D_3^{-1} A\right)$ and $A = \int_0^\infty q(s)ds$. The inequalities (17) and (18) together imply that

$$x^2(t) + y^2(t) + z^2(t) \le D_3^{-1} V_0(x_t, y_t, z_t) \le K,$$

where $K = D_3^{-1} K_1$. This completes the proof of Theorem 3.

Example 3. Consider the equation

$$\dddot{x}(t) + [3 + \exp(-x(t)\dot{x}(t) - \dot{x}(t)\ddot{x}(t))]\ddot{x}(t) + (4 + \exp(-\dot{x}^2(t-r)))\dot{x}(t) + x(t)$$

$$= \frac{1}{1+t^2+x^2(t)+x^2(t-r)+x'^2(t)+x'^2(t-r)+x''^2(t)}$$

whose associated system is

$$\dot{x} = y, \dot{y} = z,$$

$$\dot{z} = -[3 + \exp(-xy - yz)]z - (1 + \exp(-y^2(t - r)))y - x$$

$$+\frac{1}{1+t^2+x^2(t)+x^2(t-r)+y^2(t)+y^2(t-r)+z^2(t)}.$$

In addition to the observations in Example 1, we also have

$$\frac{1}{1 + t^2 + x^2(t) + x^2(t - r(t)) + y^2(t) + y^2(t - r(t)) + z^2(t)} \leq \frac{1}{1 + t^2},$$

and hence

$$\int_0^\infty q(s)ds = \int_0^\infty \frac{1}{1 + s^2}ds = \frac{\pi}{2} < \infty, \text{ that is}, q \in L^1(0, \infty).$$

Thus, all the assumptions of Theorem 2 hold. We omit details.

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