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A generalized Picard–Lindelöf theorem

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Abstract. We generalize the Picard–Lindelöf theorem on the unique solvability of initial value problems $\dot{x} = f(t, x)$, $x(t_0) = x_0$, by replacing the sufficient classical Lipschitz condition of f with respect to x with a more general Lipschitz condition along hyperspaces of the (t, x)-space. A comparison with known results is provided and the generality of the new criterion is shown by an example.

Keywords: Picard–Lindelöf theorem, initial value problem, generalized Lipschitz condition, unique solvability.

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1 Introduction

We consider the initial value problem

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0, \tag{1.1}$$

where $f \colon D \to \mathbb{R}^n$ is defined on an open set $D \subseteq \mathbb{R} \times \mathbb{R}^n$ and $(t_0, x_0) \in D$. We assume throughout the paper that f is continuous. Problem (1.1) is called *locally uniquely solvable* if there exists an open interval I containing t_0 such that (1.1) has exactly one solution on I.

The unique solvability problem of (1.1) is not fully solved up to now as simple examples show (see [2] and the references therein, see also [1]). The classical Lipschitz condition measures the vector field differences with respect to the x variable and is assumed in the classical Picard–Lindelöf theorem to prove unique solvability for (1.1). By introducing a Lipschitz condition along a hyperspace of the extended state space $\mathbb{R} \times \mathbb{R}^n$, we establish a new uniqueness theorem which generalizes the classical Picard–Lindelöf theorem and Theorem 3.2 in the paper by Cid [2]. It is also an n-dimensional generalization of the scalar criterion in [6] and of the uniqueness theorem in [3] if the functions φ and ψ are constants. The advantage of our result is shown by an example.

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Definition 1.1 (Lipschitz continuity along a hyperspace)**.** Let $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be open, $f\colon D \to \mathbb{R}^n$ be continuous and let $\mathcal{V} \subset \mathbb{R} \times \mathbb{R}^n$ be a hyperspace, i.e. $\mathcal{V}$ is an n-dimensional linear subspace of $\mathbb{R}^{1+n}$. We say that f is *Lipschitz continuous along* $\mathcal{V}$ on an open set $U \subseteq D$ if there exists a constant $L \geq 0$ such that for all $(t,x),(s,y) \in U$

$$\|f(t,x) - f(s,y)\| \leq L\|(t,x) - (s,y)\| \quad \text{if } (t,x) - (s,y) \in \mathcal{V}.$$

2 Main result

In the following let $F(t,x) = (1, f(t,x))^T$ be the vector of the direction field of (1.1) determined by f at the point $(t,x) \in D$.

Theorem 2.1 (Generalized Picard–Lindelöf theorem)**.** *Consider the initial value problem* (1.1)*, let* $\mathcal{V} \subset \mathbb{R} \times \mathbb{R}^n$ *be a hyperspace and assume that the following two conditions hold:*

(A1) Transversality condition: $F(t_0, x_0) \notin \mathcal{V}$,

(A2) Generalized Lipschitz condition: f *is Lipschitz continuous along* $\mathcal{V}$ *on an open neighborhood* $U \subseteq D$ *of* (t_0, x_0).

Then (1.1) *is locally uniquely solvable.*

The proof of Theorem 2.1 uses only Peano's theorem and the implicit function theorem. Since the classical Picard–Lindelöf theorem is a special case of Theorem 2.1, the following proof also offers an alternative proof of Picard–Lindelöf's theorem.

Proof. Let $\|\cdot\|$ denote the Euclidean norm and its induced matrix norm, respectively. Since $\mathcal{V}$ is a hyperspace in $\mathbb{R}^{1+n}$, there exist linearly independent vectors $v^{(1)}, \dots, v^{(n)} \in \mathbb{R}^{1+n}$ with $\mathcal{V} = \operatorname{span}\{v^{(1)}, \dots, v^{(n)}\} \subseteq \mathbb{R}^{1+n}$. Write

$$v^{(i)} = (v_t^{(i)}, v_1^{(i)}, \dots, v_n^{(i)})^T \quad \text{for } i = 1, \dots, n,$$

and define $v_t := (v_t^{(1)}, \dots, v_t^{(n)}) \in \mathbb{R}^n$, $v_x^{(i)} := (v_1^{(i)}, \dots, v_n^{(i)})^T \in \mathbb{R}^n$, $V_x := (v_x^{(1)}|\cdots|v_x^{(n)}) \in \mathbb{R}^{n\times n}$. Then for

$$V := \left(v^{(1)} \mid \cdots \mid v^{(n)}\right) = \begin{pmatrix} v_t^{(1)} & \cdots & v_t^{(n)} \\ \hline v_1^{(1)} & \cdots & v_1^{(n)} \\ \vdots & & \vdots \\ v_n^{(1)} & \cdots & v_n^{(n)} \end{pmatrix} = \begin{pmatrix} v_t^{(1)} & \cdots & v_t^{(n)} \\ \hline v_x^{(1)} & |\cdots| & v_x^{(n)} \end{pmatrix} = \begin{pmatrix} v_t \\ V_x \end{pmatrix}$$

we have $V \in \mathbb{R}^{(1+n)\times n}$ and $\operatorname{rank} V = n$. Peano's theorem guarantees that (1.1) has at least one solution $x\colon [t_0 - \alpha, t_0 + \alpha] \to \mathbb{R}^n$ for some $\alpha > 0$. By shrinking $\alpha > 0$ if necessary, we can assume that $\operatorname{graph} x \subset U$ and, by assumption (A1) and continuity of f, $F(t, x(t)) \notin \mathcal{V}$ for all $t \in I := (t_0 - \alpha, t_0 + \alpha)$. To prove that (1.1) is locally uniquely solvable with solution x on I, assume to the contrary that there exists a solution $y\colon I \to \mathbb{R}^n$ of (1.1) and $x \not\equiv y$ on $[t_0, t_0 + \alpha)$ (the case $x \not\equiv y$ on $(t_0 - \alpha, t_0]$ is treated similarly). For $t_1 := \sup\{t \in [t_0, t_0 + \alpha) : x(s) = y(s) \text{ for } s \in [t_0, t]\}$ we have $t_1 \in [t_0, t_0 + \alpha)$, $x(t_1) = y(t_1) =: x_1$ by continuity and $F(t_1, x_1) \notin \mathcal{V}$.

We show that the equation

$$y(t+v_t k(t)) = x(t) + V_x k(t) \tag{2.1}$$

is uniquely solvable with respect to $k = k(t) = (k_1(t), \dots, k_n(t))^T$ on a subinterval of I which contains t_1. The problem suggests to apply the implicit function theorem. Choose $\varepsilon > 0$ such that

$$H(t,k) := y(t+v_t k) - x(t) - V_x k$$

is well-defined on $[t_1-\varepsilon, t_1+\varepsilon] \times [-\varepsilon, \varepsilon]^n$. Then $H(t_1, 0) = 0$,

$$\frac{\partial H}{\partial k}(t,k) = \Big(f_i(t+v_t k, y(t+v_t k)) v_t^{(j)} - v_i^{(j)} \Big)_{i,j=1,\dots,n}$$

and therefore $\partial H(t_1,0)/\partial k = WV$ with

$$W := \left(f(t_1, x_1) \,\middle|\, \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} \right) \in \mathbb{R}^{n\times(1+n)}.$$

By the rank-nullity theorem (see e.g. [4, p. 199]) $\dim \operatorname{im}(V) + \dim \ker(V) = n$ and, using the fact that $\dim \operatorname{im}(V) = \operatorname{rank} V = n$, we get $\ker V = \{0\}$. Assume that WV is not invertible. Then there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $WVv = 0$. Hence $w := Vv \neq 0$ and $w \in \mathcal{V}$, as well as $w \in \ker W = \operatorname{span}\{F(t_1, x_1)\}$. Therefore $F(t_1, x_1) \in \mathcal{V}$ leads to a contradiction, proving that WV is invertible.

The implicit function theorem (cf. e.g. [5, Theorem 9.28]) yields a unique C^1 function $k\colon J \to [-\varepsilon, \varepsilon]^n$ on an open interval $J \subseteq I$ containing t_1 such that $k(t_1) = 0$ and $H(t, k(t)) = 0$ for all $t \in J$. Using the fact that $\partial H(t_1, 0)/\partial k$ is invertible, we get by shrinking J if necessary, that $(\partial H(t,k(t))/\partial k)^{-1}$ exists and is bounded for t in J, i.e. there exists $\eta \geq 0$ such that

$$\left\| \frac{\partial H}{\partial k}(t, k(t))^{-1} \right\| \leq \eta \quad \text{for } t \in J.$$

Since $\partial H(t,k)/\partial t = f(t+v_t k, y(t+v_t k)) - f(t, x(t))$, (A2) implies, together with (2.1) and $Vk(t) \in \mathcal{V}$, that

$$\left\| \frac{\partial H}{\partial t}(t, k(t)) \right\| \leq L \|Vk(t)\|.$$

Now we consider $u(t) := \|k(t)\|^2 = \langle k(t), k(t) \rangle$. We get

$$\dot{u}(t) = \frac{d}{dt} \langle k(t), k(t) \rangle = 2 \langle k(t), \dot{k}(t) \rangle\,.$$

Using the fact that

$$\dot{k}(t) = -\frac{\partial H}{\partial k}(t, k(t))^{-1} \frac{\partial H}{\partial t}(t, k(t)),$$

we conclude that

$$\dot{u}(t) \leq \left\| 2k(t)^T \frac{\partial H}{\partial k}(t, k(t))^{-1} \frac{\partial H}{\partial t}(t, k(t)) \right\| \leq 2\|k(t)\| \eta L \|V\| \|k(t)\|$$

and hence

$$\dot{u}(t) \leq 2\eta L \|V\| u(t)$$

which is equivalent to

$$\frac{d}{dt}\left[e^{-2\eta L\|V\|(t-t_1)}u(t)\right] \leq 0.$$

Since $u(t_1) = \|k(t_1)\|^2 = 0$, we get $u(t) = \|k(t)\|^2 \equiv 0$, and hence from (2.1) we conclude $x(t) \equiv y(t)$ on J, which contradicts the definition of t_1. □

Remark 2.2. (a) The classical Picard–Lindelöf theorem which requires a Lipschitz condition with respect to x is a special case of Theorem 2.1 with

$$V = \begin{pmatrix} v_t \\ V_x \end{pmatrix}, \qquad v_t = 0 \in \mathbb{R}^n \quad \text{and} \quad V_x = I_n, \tag{2.2}$$

where I_n denotes the $n \times n$ identity matrix. Cid [2] introduces the notion of *Lipschitz continuity when fixing component* $i_0 \in \{0, 1, \ldots, n\}$ where the component $i_0 = 0$ corresponds to the variable t, i.e. Lipschitz continuity when fixing $i_0 = 0$ is equivalent to Lipschitz continuity with respect to x. Lipschitz continuity when fixing another component is defined similarly. Under the assumption that f is Lipschitz continuous when fixing a component i_0, Cid can show uniqueness provided that either $i_0 = 0$ or $f_{i_0} \neq 0$. Thus Theorem 3.2 by Cid can be interpreted as a special case of our Theorem 2.1 with matrices V of the form (2.2) where in the case of $i_0 \neq 0$ the corresponding column of V is replaced by a vector $v^{(i_0)}$ with $v_t^{(i_0)} = 1$ and all other components equal 0. Note that [3, Theorem 1] is a special case of Theorem 2.1 for $n = 1$ if the functions φ and ψ are constants.

(b) Let $\mathcal{V} = \operatorname{span}\{v^{(1)}, \ldots, v^{(n)}\} \subset \mathbb{R}^{1+n}$ and $U \subseteq D$ be a convex open neighborhood of $(t_0, x_0) \in D \subseteq \mathbb{R} \times \mathbb{R}^n$. If the directional derivatives

$$\frac{\partial f}{\partial v}(t,x) = \lim_{h\to 0} \frac{f((t,x)+hv) - f(t,x)}{h\|v\|}, \qquad v \in \mathcal{V},$$

exist and are continuous and bounded on U, then f is Lipschitz continuous along $\mathcal{V}$ on U.

Proof. With $(t,x) = (s,y) + v, v \in \mathcal{V}$, and $g(\tau) := f((s,y) + \tau v)$ we get

$$\begin{aligned}
f(t,x) - f(s,y) = g(1) - g(0) &= \int_0^1 g'(\tau)d\tau \\
&= \int_0^1 \lim_{h\to 0} \frac{g(\tau+h) - g(\tau)}{h} d\tau \\
&= \int_0^1 \lim_{h\to 0} \frac{f((s,y)+(\tau+h)v) - f((s,y)+\tau v)}{h} d\tau \\
&= \int_0^1 \left(\lim_{h\to 0} \frac{f((s,y)+(\tau+h)v) - f((s,y)+\tau v)}{h\|v\|}\right) \|v\| d\tau \\
&= \int_0^1 \frac{\partial f}{\partial v}((s,y)+\tau v)\|v\| d\tau
\end{aligned}$$

and therefore

$$\|f(t,x) - f(s,y)\| \leq L\|v\|, \qquad L := \sup_{\tau\in[0,1]} \frac{\partial f}{\partial v}((s,y)+\tau v).$$ □

Example 2.3. Consider the 2-dimensional initial value problem

$$\dot{x} = f(t,x), \qquad x(0) = 0,$$

where $f(t,x) = (f_1(t,x_1,x_2), f_2(t,x_1,x_2))^T$ with

$$f_1(t,x_1,x_2) = \begin{cases} x_1 + g(x_2), & x_1 < t, \\ x_1 + g(x_2) + \sqrt[3]{x_1 - t}, & x_1 \geq t, \end{cases}$$
$$f_2(t,x_1,x_2) = 1 + h(x_1),$$

$g(x_2)$ and $h(x_1)$ are Lipschitz continuous functions and $g(0) \neq 1$. The classical Lipschitz condition is not fulfilled, and we cannot show uniqueness with the hyperspace $\mathcal{V}$ being the (t,x_1)-plane or (t,x_2)-plane. Therefore the result by Cid cannot be applied.

With the basis vectors $v^{(1)} = (1,1,0)^T$, $v^{(2)} = (0,0,1)^T$ and $\mathcal{V} = \operatorname{span}\{v^{(1)}, v^{(2)}\}$ we can show uniqueness of the given problem.

(A1) is satisfied, as $(1, g(0), 1+h(0))^T \notin \mathcal{V}$ if $g(0) \neq 1$. The only numbers α, β, γ, satisfying $\alpha(1, f(0,0))^T + \beta v^{(1)} + \gamma v^{(2)} = 0$ are $\alpha = \beta = \gamma = 0$ if $g(0) \neq 1$.

Now (A2) is shown. With $v_t = (1,0)$ and $V_x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ we have to show that

$$\begin{aligned} \|f(t + v_t k, x + V_x k) - f(t,x)\| &= \|f(t + k_1, x_1 + k_1, x_2 + k_2) - f(t, x_1, x_2)\| \\ &\leq L\|(v_t k, V_x k)^T\| \end{aligned}$$

with $k = (k_1, k_2)^T$. For $x_1 < t$ we get

$$\left\| \begin{pmatrix} x_1 + k_1 + g(x_2 + k_2) - x_1 - g(x_2) \\ 1 + h(x_1 + k_1) - 1 - h(x_1) \end{pmatrix} \right\|$$

which can be estimated by $L\|(k_1, k_1, k_2)^T\|$ with $L \geq 0$. For $x_1 \geq t$ we get

$$\left\| \begin{pmatrix} x_1 + k_1 + g(x_2 + k_2) + \sqrt[3]{x_1 + k_1 - t - k_1} - x_1 - g(x_2) - \sqrt[3]{x_1 - t} \\ 1 + h(x_1 + k_1) - 1 - h(x_1) \end{pmatrix} \right\|$$

which can also be estimated by $L\|(k_1, k_1, k_2)^T\|$ with $L \geq 0$.

3 Alternative proof

We provide an alternative proof for Theorem 2.1 by transforming (1.1) into a system to which the classical Picard–Lindelöf theorem can be applied.

Alternative proof of Theorem 2.1. Choose a unit vector $a_0 \in \mathbb{R}^{1+n}$ such that $\mathcal{V} = a_0^{\perp}$ and also $\langle a_0, F(t_0, x_0) \rangle > 0$, which is possible due to assumption (A1). Since $\mathbb{R}^{1+n} = \langle a_0 \rangle \oplus \mathcal{V}$ is the direct sum of $\langle a_0 \rangle = \{ s a_0 \in \mathbb{R}^{1+n} : s \in \mathbb{R} \}$ and $\mathcal{V}$, there exist unique $s_0 \in \mathbb{R}$ and $v_0 \in \mathcal{V}$ with $(t_0, x_0) = s_0 a_0 + v_0$. We divide the proof into three steps.

Step 1: We show that the nonautonomous initial value problem on $\mathcal{V}$

$$\frac{dv}{ds} = g(s,v) := \frac{F(s a_0 + v) - \sigma(s,v) a_0}{\sigma(s,v)}, \qquad v(s_0) = v_0, \tag{3.1}$$

with $\sigma(s,v) := \langle a_0, F(s a_0 + v) \rangle$ is well-posed and locally uniquely solvable.

The function

$$\sigma \colon \mathbb{R} \times \mathcal{V} \to \mathbb{R}, \qquad (s,v) \mapsto \sigma(s,v) = \langle a_0, F(s a_0 + v) \rangle$$

is continuous and satisfies $\sigma(s_0, v_0) = \langle a_0, F(s_0 a_0 + v_0)\rangle = \langle a_0, F(t_0, x_0)\rangle > 0$. As a consequence there exists an $\eta > 0$ and a bounded open neighborhood $U \subseteq \mathbb{R} \times \mathcal{V}$ of (s_0, v_0) such that $\sigma(s, v) \geq \eta$ for all $(s, v) \in U$.

Using assumption (A2) and by shrinking U if necessary, we can w.l.o.g. assume that f is Lipschitz continuous along $\mathcal{V}$ on the open neighborhood $\{sa_0 + v \in \mathbb{R}^{1+n} : (s, v) \in U\}$ of (t_0, x_0). Using this fact, we get for $(s, v), (s, \bar{v}) \in U$

$$\begin{aligned}|\sigma(s,v) - \sigma(s,\bar{v})| &= |\langle a_0, F(sa_0 + v)\rangle - \langle a_0, F(sa_0 + \bar{v})\rangle| \\ &= |\langle a_0, F(sa_0 + v) - F(sa_0 + \bar{v})\rangle| \leq \|a_0\| \cdot \|F(sa_0 + v) - F(sa_0 + \bar{v})\| \\ &= \|F(sa_0 + v) - F(sa_0 + \bar{v})\| = \|f(sa_0 + v) - f(sa_0 + \bar{v})\| \\ &\leq L\|v - \bar{v}\|,\end{aligned}$$

proving that σ is Lipschitz continuous on U. With σ also the quotient $1/\sigma$ is Lipschitz continuous with respect to v. Thus we get

$$\begin{aligned}\|g(s,v) - g(s,\bar{v})\| &= \left\| \frac{F(sa_0 + v)}{\sigma(s,v)} - \frac{F(sa_0 + \bar{v})}{\sigma(s,\bar{v})} \right\| \\ &\leq \left| \frac{1}{\sigma(s,v)} \right| \cdot \|F(sa_0 + v) - F(sa_0 + \bar{v})\| \\ &\quad + \left| \frac{1}{\sigma(s,v)} - \frac{1}{\sigma(s,\bar{v})} \right| \cdot \|F(sa_0 + \bar{v})\| .\end{aligned}$$

By shrinking U again if necessary, we can assume w.l.o.g. that $\bar{U} \subseteq D$. Then boundedness of F and of $1/\sigma$ on $\bar{U}$ imply Lipschitz continuity of g with respect to v on the neighborhood U of (s_0, v_0). Since $\mathcal{V}$ is isomorphic to $\mathbb{R}^n$, the classical Picard–Lindelöf theorem can be applied to (3.1) to prove local unique solvability.

Step 2: We show that the autonomous initial value problem on $\mathbb{R} \times \mathcal{V}$

$$\begin{aligned}\dot{s} &= \sigma(s, v), & s(t_0) &= s_0, \\ \dot{v} &= F(sa_0 + v) - \sigma(s, v)a_0, & v(t_0) &= v_0,\end{aligned} \tag{3.2}$$

is locally uniquely solvable.

By Peano's theorem (3.2) admits a solution. Assume that $(\hat{s}_1, \hat{v}_1), (\hat{s}_2, \hat{v}_2) : J \to \mathbb{R} \times \mathcal{V}$, are two solutions of (3.2) on an open interval J containing t_0. Then the solution identities

$$\begin{aligned}\dot{\hat{s}}_i(t) &= \sigma(\hat{s}_i(t), \hat{v}_i(t)), \\ \dot{\hat{v}}_i(t) &= F(\hat{s}_i(t)a_0 + \hat{v}_i(t)) - \sigma(\hat{s}_i(t), \hat{v}_i(t))a_0\end{aligned} \tag{3.3}$$

for $t \in J$ and the initial conditions

$$\hat{s}_i(t_0) = s_0, \qquad \hat{v}_i(t_0) = v_0 \tag{3.4}$$

are fulfilled for $i = 1, 2$. By shrinking J if necessary, we can w.l.o.g. assume that $(\hat{s}_i(t), \hat{v}_i(t)) \in U$ and therefore $\dot{\hat{s}}_i(t) = \sigma(\hat{s}_i(t), \hat{v}_i(t)) \geq \eta$ for $t \in J$. As a consequence the functions $\hat{s}_i : J \to \mathbb{R}$ are strictly monotonically increasing, and hence the inverse functions $\hat{s}_i^{-1} : \hat{s}_i(J) \to J$ exist and satisfy

$$\hat{s}_i^{-1}(s_0) = t_0 \tag{3.5}$$

for $i = 1, 2$. With the bijection $t = \hat{s}_i^{-1}(s)$ both solution curves through (s_0, v_0) can be reparametrized in the form

$$\begin{aligned}\big\{\big(\hat{s}_i(t), \hat{v}_i(t)\big) : t \in J\big\} &= \big\{\big(\hat{s}_i(\hat{s}_i^{-1}(s)), \hat{v}_i(\hat{s}_i^{-1}(s))\big) : s \in \hat{s}_i(J)\big\} \\ &= \big\{\big(s, \hat{v}_i(\hat{s}_i^{-1}(s))\big) : s \in \hat{s}_i(J)\big\}\end{aligned}$$

for $i = 1, 2$. Then

$$v_i \colon \hat{s}_i(J) \to \mathcal{V}, \qquad v_i(s) := \hat{v}_i(\hat{s}_i^{-1}(s)),$$

solve (3.1) for $i = 1, 2$, since

$$\frac{dv_i}{ds}(s) = \frac{\dot{\hat{v}}_i(\hat{s}_i^{-1}(s))}{\dot{\hat{s}}_i(\hat{s}_i^{-1}(s))} \overset{(3.3)}{=} \frac{F(sa_0 + v_i) - \sigma(s, v_i)a_0}{\sigma(s, v_i)}$$

and

$$v_i(s_0) = \hat{v}_i(\hat{s}_i^{-1}(s_0)) \overset{(3.5)}{=} \hat{v}_i(t_0) \overset{(3.4)}{=} v_0.$$

By shrinking J if necessary, we can apply Step 1 to conclude that $v_1 = v_2$ on J and hence $\hat{v}_1(\hat{s}_1^{-1}(s)) = \hat{v}_2(\hat{s}_2^{-1}(s))$ for all $s \in \hat{s}_1(J) \cap \hat{s}_2(J)$, proving that $\hat{s}_1 = \hat{s}_2$ and $\hat{v}_1 = \hat{v}_1$ on J.

Step 3: We show that (1.1) is locally uniquely solvable.

By Peano's theorem (1.1) admits a solution. Assume that $x_1, x_2 \colon I \to \mathbb{R}^n$ are two solutions of (1.1). For $t \in I$ we have $X_i(t) := (1, x_i(t)) \in \mathbb{R}^{1+n} = \langle a_0 \rangle \oplus \mathcal{V}$ and therefore there exist unique functions $s_i \colon I \to \mathbb{R}$ and $v_i \colon I \to \mathcal{V}$ such that

$$X_i(t) = s_i(t)a_0 + v_i(t).$$

Moreover, $(s_i(t_0), v_i(t_0)) = (s_0, v_0)$, and using the fact that $\|a_0\| = 1$ and $a_0^\perp = \mathcal{V}$, $s_i(t) = \langle a_0, X_i(t)\rangle$ and $v_i(t) = X_i(t) - s_i(t)a_0$ for $t \in I$ and $i = 1, 2$. Now $(s_i, v_i) \colon I \to \mathbb{R} \times \mathcal{V}$ solve (3.2), since

$$\begin{aligned}\dot{s}_i(t) &= \langle a_0, \dot{X}_i(t)\rangle = \langle a_0, F(t, x_i(t))\rangle = \langle a_0, F(s_i(t)a_0 + v_i(t))\rangle \\ &= \sigma(s_i(t), v_i(t)), \\ \dot{v}_i(t) &= \dot{X}_i(t) - \langle a_0, \dot{X}_i(t)\rangle a_0 = F(t, x_i(t)) - \langle a_0, F(t, x_i(t))\rangle a_0 \\ &= F(s_i(t)a_0 + v_i(t)) - \langle a_0, F(s_i(t)a_0 + v_i(t))\rangle a_0 \\ &= F(s_i(t)a_0 + v_i(t)) - \sigma(s_i(t), v_i(t))a_0\end{aligned}$$

for $t \in I$ and $i = 1, 2$. By shrinking I if necessary, we can apply Step 2 to conclude that $s_1 = s_2$ and $v_1 = v_2$ on I, proving that $x_1 = x_2$. □

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