

ON Ψ -BOUNDED SOLUTIONS FOR NON-HOMOGENEOUS MATRIX LYAPUNOV SYSTEMS ON \mathbb{R}

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ABSTRACT. In this paper we provide necessary and sufficient conditions for the existence of at least one Ψ -bounded solution on \mathbb{R} for the system $X' = A(t)X + XB(t) + F(t)$, where $F(t)$ is a Lebesgue Ψ -integrable matrix valued function on \mathbb{R} . Further, we prove a result relating to the asymptotic behavior of the Ψ -bounded solutions of this system.

1. INTRODUCTION

The importance of matrix Lyapunov systems, which arise in a number of areas of control engineering problems, dynamical systems, and feedback systems are well known. This paper deals with the linear matrix differential system

$$(1.1) \quad X' = A(t)X + XB(t) + F(t)$$

where $A(t)$, $B(t)$ and $F(t)$ are continuous $n \times n$ matrix-valued functions on \mathbb{R} . The basic problem under consideration is the determination of necessary and sufficient conditions for the existence of a solution with some specified boundedness condition. A Classical result of this type, for system of differential equations is given by Coppel [4, Theorem 2, Chapter V].

The problem of Ψ -boundedness of the solutions for systems of ordinary differential equations has been studied in many papers, [1, 2, 3, 5, 9, 10]. Recently [11, 7], extended the concept of Ψ -boundedness of the solutions to Lyapunov matrix differential equations. In [6], the author obtained necessary and sufficient conditions for the non homogenous system $x' = A(t)x + f(t)$, to have at least one Ψ -bounded solution on \mathbb{R} for every Lebesgue Ψ -integrable function f on \mathbb{R} .

The aim of present paper is to give a necessary and sufficient condition so that the nonhomogeneous matrix Lyapunov system (1.1) has at least one Ψ -bounded solution on \mathbb{R} for every Lebesgue Ψ -integrable

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matrix function F on \mathbb{R} . The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the components of the solutions. Here, Ψ is a continuous matrix-valued function on \mathbb{R} . The results of this paper include results of Diamandescu [6], as a particular case when $B(t) = O_n$.

2. PRELIMINARIES

In this section we present some basic definitions, notations and results which are useful for later discussion.

Let \mathbb{R}^n be the Euclidean n -space. For $u = (u_1, u_2, u_3, \dots, u_n)^T \in \mathbb{R}^n$, let $\|u\| = \max\{|u_1|, |u_2|, |u_3|, \dots, |u_n|\}$ be the norm of u . Let $\mathbb{R}^{n \times n}$ be the linear space of all $n \times n$ real valued matrices. For a $n \times n$ real matrix $A = [a_{ij}]$, we define the norm $|A| = \sup_{\|u\| \leq 1} \|Au\|$. It is well-known that

$$|A| = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |a_{ij}| \right\}.$$

Let $\Psi_k : \mathbb{R} \rightarrow \mathbb{R} - \{0\}$ ($\mathbb{R} - \{0\}$ is the set of all nonzero real numbers), $k = 1, 2, \dots, n$, be continuous functions, and let

$$\Psi = \text{diag}[\Psi_1, \Psi_2, \dots, \Psi_n].$$

Then the matrix $\Psi(t)$ is an invertible square matrix of order n , for all $t \in \mathbb{R}$.

Definition 2.1. [8] Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ then the Kronecker product of A and B written $A \otimes B$ is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \dots & \dots & \dots & \dots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}$$

is an $mp \times nq$ matrix and is in $\mathbb{R}^{mp \times nq}$.

Definition 2.2. [8] Let $A = [a_{ij}] \in \mathbb{R}^{m \times n}$, then the vectorization operator

$Vec : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n^2}$, defined and denote by

$$\hat{A} = VecA = \begin{bmatrix} A_{.1} \\ A_{.2} \\ \vdots \\ A_{.n} \end{bmatrix}, \text{ where } A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad (1 \leq j \leq n).$$

Lemma 2.1. The vectorization operator $Vec : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n^2}$, is a linear and one-to-one operator. In addition, Vec and Vec^{-1} are continuous operators.

Proof. The fact that the vectorization operator is linear and one-to-one is immediate. Now, for $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, we have

$$\|Vec(A)\| = \max_{1 \leq i, j \leq n} \{|a_{ij}|\} \leq \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |a_{ij}| \right\} = |A|.$$

Thus, the vectorization operator is continuous and $\|Vec\| \leq 1$.

In addition, for $A = I_n$ (identity $n \times n$ matrix) we have $\|Vec(I_n)\| = 1 = |I_n|$ and then, $\|Vec\| = 1$.

Obviously, the inverse of the vectorization operator, $Vec^{-1} : \mathbb{R}^{n^2} \rightarrow \mathbb{R}^{n \times n}$, is defined by

$$Vec^{-1}(u) = \begin{bmatrix} u_1 & u_{n+1} & \cdot & \cdot & \cdot & u_{n^2-n+1} \\ u_2 & u_{n+2} & \cdot & \cdot & \cdot & u_{n^2-n+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ u_n & u_{2n} & \cdot & \cdot & \cdot & u_{n^2} \end{bmatrix}.$$

Where $u = (u_1, u_2, u_3, \dots, u_{n^2})^T \in \mathbb{R}^{n^2}$.

$$\text{We have } |Vec^{-1}(u)| = \max_{1 \leq i \leq n} \left\{ \sum_{j=0}^{n-1} |u_{n^2-j+i}| \right\} \leq n \cdot \max_{1 \leq i \leq n} \{|u_i|\} = n \cdot \|u\|.$$

Thus, Vec^{-1} is a continuous operator. Also, if we take $u = VecA$ in the above inequality, then the following inequality holds

$$|A| \leq n \|VecA\|,$$

for every $A \in \mathbb{R}$. □

Regarding properties and rules for Kronecker product of matrices we refer to [8].

Now by applying the Vec operator to the nonhomogeneous matrix Lyapunov system (1.1) and using Kronecker product properties, we have

$$(2.1) \quad \hat{X}'(t) = H(t)\hat{X}(t) + \hat{F}(t),$$

where $H(t) = (B^T \otimes I_n) + (I_n \otimes A)$ is a $n^2 \times n^2$ matrix and $\hat{F}(t) = \text{Vec}F(t)$ is a column matrix of order n^2 . System (2.1) is called the Kronecker product system associated with (1.1).

The corresponding homogeneous system of (2.1) is

$$(2.2) \quad \hat{X}'(t) = H(t)\hat{X}(t).$$

Definition 2.3. A function $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ is said to be Ψ - bounded on \mathbb{R} if $\Psi\gamma$ is bounded on \mathbb{R} (i.e., $\sup_{t \in \mathbb{R}} \|\Psi(t)\gamma(t)\| < +\infty$).

Extend this definition for matrix functions.

Definition 2.4. A matrix function $F : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ is said to be Ψ bounded on \mathbb{R} if the matrix function ΨF is bounded on \mathbb{R} (i.e., $\sup_{t \in \mathbb{R}} |\Psi(t)F(t)| < \infty$).

Definition 2.5. A function $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ is said to be Lebesgue Ψ - integrable on \mathbb{R} if γ is measurable and $\Psi\gamma$ is Lebesgue integrable on \mathbb{R} (i.e., $\int_{-\infty}^{\infty} \|\Psi(t)\gamma(t)\| dt < \infty$).

Extend this definition for matrix functions.

Definition 2.6. A matrix function $F : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ is said to be Lebesgue Ψ integrable on \mathbb{R} if F is measurable and ΨF is Lebesgue integrable on \mathbb{R}

$$\left(i.e., \int_{-\infty}^{\infty} |\Psi(t)F(t)| dt < \infty \right).$$

Now we shall assume that A and B are continuous $n \times n$ matrices on \mathbb{R} and F is a Lebesgue Ψ -integrable matrix function on \mathbb{R} .

By a solution of (1.1), we mean an absolutely continuous matrix function $W(t)$ satisfying the equation (1.1) for all most all $t \in \mathbb{R}$.

The following lemmas play a vital role in the proof of main result.

Lemma 2.2. The matrix function $F : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ is Lebesgue Ψ -integrable on \mathbb{R} if and only if the vector function $\text{Vec}F(t)$ is Lebesgue $I_n \otimes \Psi$ - integrable on \mathbb{R} .

Proof. From the proof of Lemma 2.1, it follows that

$$\frac{1}{n} |A| \leq \|\text{Vec}A\|_{\mathbb{R}^{n^2}} \leq |A|,$$

for every $A \in \mathbb{R}^{n \times n}$.

Put $A = \Psi(t)F(t)$ in the above inequality, we have

$$(2.3) \quad \frac{1}{n} |\Psi(t)F(t)| \leq \|(I_n \otimes \Psi(t)).\text{Vec}F(t)\|_{\mathbb{R}^{n^2}} \leq |\Psi(t)F(t)|,$$

$t \in \mathbb{R}$, for all matrix functions $F(t)$. Lemma follows from (2.3). \square

Lemma 2.3. The matrix function $F(t)$ is Ψ - bounded on \mathbb{R} if and only if the vector function $VecF(t)$ is $I_n \otimes \Psi$ - bounded on \mathbb{R} .

Proof. The proof easily follows from the inequality (2.3). \square

Lemma 2.4. Let $Y(t)$ and $Z(t)$ be the fundamental matrices for the systems

$$(2.4) \quad X'(t) = A(t)X(t),$$

and

$$(2.5) \quad X'(t) = B^T(t)X(t), \quad t \in \mathbb{R}$$

respectively. Then the matrix $Z(t) \otimes Y(t)$ is a fundamental matrix of (2.2).

Proof. Consider

$$\begin{aligned} (Z(t) \otimes Y(t))' &= (Z'(t) \otimes Y(t)) + (Z(t) \otimes Y'(t)) \\ &= (B^T(t)Z(t) \otimes Y(t)) + (Z(t) \otimes A(t)Y(t)) \\ &= (B^T(t) \otimes I_n)(Z(t) \otimes Y(t)) + (I_n \otimes A(t))(Z(t) \otimes Y(t)) \\ &= [B^T(t) \otimes I_n + I_n \otimes A(t)](Z(t) \otimes Y(t)) \\ &= H(t)(Z(t) \otimes Y(t)), \end{aligned}$$

for all $t \in \mathbb{R}$.

On the other hand, the matrix $Z(t) \otimes Y(t)$ is a nonsingular matrix for all $t \in \mathbb{R}$ (because $X(t)$ and $Y(t)$ are nonsingular matrices for all $t \in \mathbb{R}$). \square

Let the matrix space $\mathbb{R}^{n \times n}$ be represented as a direct sum of three subspaces X_-, X_0, X_+ such that a solution $W(t)$ of (1.1) is Ψ -bounded on \mathbb{R} if and only if $W(0) \in X_0$ and Ψ -bounded on \mathbb{R} if and only if $W(0) \in X_- \oplus X_0$. Also, let $\mathbb{P}_-, \mathbb{P}_0, \mathbb{P}_+$ denote the corresponding projection of $\mathbb{R}^{n \times n}$ onto X_-, X_0, X_+ respectively.

Then the vector space \mathbb{R}^{n^2} represents a direct sum of three subspaces S_-, S_0, S_+ such that a solution $\hat{W}(t) = VecW(t)$ of (2.1) is $I_n \otimes \Psi$ -bounded on \mathbb{R}^{n^2} if and only if $\hat{W}(0) \in S_0$ and $I_n \otimes \Psi$ -bounded on \mathbb{R} if and only if $\hat{W}(0) \in S_- \oplus S_0$ and also Q_-, Q_0, Q_+ denote the corresponding projection of \mathbb{R}^{n^2} onto S_-, S_0, S_+ respectively.

Theorem 2.1. Let $A(t)$, $B(t)$ and $F(t)$ be continuous matrix functions on \mathbb{R} . If $Y(t)$ and $Z(t)$ are the fundamental matrices for the systems (2.4) and (2.5), then

$$\begin{aligned}
 \hat{X}(t) = & \int_{-\infty}^t (Z(t) \otimes Y(t)) \mathbb{P}_-(Z^{-1}(s) \otimes Y^{-1}(s)) \hat{F}(s) ds \\
 & + \int_0^t (Z(t) \otimes Y(t)) \mathbb{P}_0(Z^{-1}(s) \otimes Y^{-1}(s)) \hat{F}(s) ds \\
 (2.6) \quad & - \int_t^{\infty} (Z(t) \otimes Y(t)) \mathbb{P}_+(Z^{-1}(s) \otimes Y^{-1}(s)) \hat{F}(s) ds
 \end{aligned}$$

is a solution of (2.1) on \mathbb{R} .

Proof. It is easily seen that \hat{X} is the solution of (2.1) on \mathbb{R} . □

The following theorems are useful in the proofs of our main results.

Theorem 2.2. [6] Let A be a continuous $n \times n$ real matrix on \mathbb{R} and let Y be the fundamental matrix of the homogeneous system $x' = A(t)x$ with $Y(0) = I_n$. Then the nonhomogeneous system

$$(2.7) \quad x' = A(t)x + f(t)$$

has at least one Ψ -bounded solution on \mathbb{R} for every Lebesgue Ψ -integrable function $f : \mathbb{R} \rightarrow \mathbb{R}^n$ on \mathbb{R} if and only if there exists a positive constant K such that

$$\begin{aligned}
 (2.8) \quad & |\Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)| \leq K \quad \text{for } t > 0, s \leq 0 \\
 & |\Psi(t)Y(t)(P_0 + P_-)Y^{-1}(s)\Psi^{-1}(s)| \leq K \quad \text{for } t > 0, s > 0, s < t \\
 & |\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)| \leq K \quad \text{for } t > 0, s > 0, s \geq t \\
 & |\Psi(t)Y(t)P_-Y^{-1}(s)\Psi^{-1}(s)| \leq K \quad \text{for } t \leq 0, s < t \\
 & |\Psi(t)Y(t)(P_0 + P_+)Y^{-1}(s)\Psi^{-1}(s)| \leq K \quad \text{for } t \leq 0, s \geq t, s < 0 \\
 & |\Psi(t)Y(t)P_+Y^{-1}(s)\Psi^{-1}(s)| \leq K \quad \text{for } t \leq 0, s \geq t, s \geq 0.
 \end{aligned}$$

Theorem 2.3. [6] Suppose that:

(1) the fundamental matrix $Y(t)$ of $x' = A(t)x$ satisfies:

(a) condition (2.8) is satisfied for some $K > 0$;

(b) the following conditions are satisfied:

- (i) $\lim_{t \rightarrow \pm\infty} |\Psi(t)Y(t)P_0| = 0;$
- (ii) $\lim_{t \rightarrow -\infty} |\Psi(t)Y(t)P_+| = 0;$
- (iii) $\lim_{t \rightarrow +\infty} |\Psi(t)Y(t)P_-| = 0;$

(2) the function $f : \mathbb{R} \rightarrow \mathbb{R}^n$ is Lebesgue Ψ -integrable on \mathbb{R} .

Then, every Ψ -bounded solution x of (2.7) is such that

$$\lim_{t \rightarrow \pm\infty} \|\Psi(t)x(t)\| = 0.$$

3. MAIN RESULT

The main theorms of this paper are proved in this section.

Theorem 3.1. *If A and B are continuous $n \times n$ real matrices on \mathbb{R} , then (1.1) has at least one Ψ -bounded solution on \mathbb{R} for every Lebesgue Ψ -integrable matrix function $F : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ on \mathbb{R} if and only if there exists a positive constant K such that*

(3.1)

$$|(Z(t) \otimes \Psi(t)Y(t))Q_-(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \leq K,$$

for $t > 0, s \leq 0$

$$|(Z(t) \otimes \Psi(t)Y(t))(Q_0 + Q_-)(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \leq K,$$

for $t > 0, s > 0, s < t$

$$|(Z(t) \otimes \Psi(t)Y(t))Q_+(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \leq K,$$

for $t > 0, s > 0, s \geq t$

$$|(Z(t) \otimes \Psi(t)Y(t))Q_-(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \leq K,$$

for $t \leq 0, s < t$

$$|(Z(t) \otimes \Psi(t)Y(t))(Q_0 + Q_+)(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \leq K,$$

for $t \leq 0, s \geq t, s < 0$

$$|(Z(t) \otimes \Psi(t)Y(t))Q_+(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| \leq K,$$

for $t \leq 0, s \geq t, s \geq 0$.

Proof. Suppose that the equation (1.1) has atleast one Ψ - bounded solution on \mathbb{R} for every Lebesgue Ψ - integrable matrix function

$F : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$.

Let $\hat{F} : \mathbb{R} \rightarrow \mathbb{R}^{n^2}$ be a Lebesgue $I_n \otimes \Psi$ - integrable function on \mathbb{R} . From Lemma 2.2, it follows that the matrix function $F(t) = Vec^{-1}\hat{F}(t)$ is Lebesgue Ψ - integrable matrix function on \mathbb{R} . From the hypothesis,

the system (1.1) has at least one Ψ - bounded solution $W(t)$ on \mathbb{R} . From Lemma 2.3, it follows that the vector valued function $\hat{W}(t) = VecW(t)$ is a $I_n \otimes \Psi$ - bounded solution of (2.1) on \mathbb{R} .

Thus, system (2.1) has at least one $I_n \otimes \Psi$ - bounded solution on \mathbb{R} for every Lebesgue $I_n \otimes \Psi$ - integrable function \hat{F} on \mathbb{R} .

From Theorem 2.2, there is a positive constant K such that the fundamental matrix $S(t) = Z(t) \otimes Y(t)$ of the system (2.2) satisfies the condition

$$|(I_n \otimes \Psi(t))S(t)Q_-S^{-1}(s)(I_n \otimes \Psi^{-1}(s))| \leq K, \\ \text{for } t > 0, s \leq 0$$

$$|(I_n \otimes \Psi(t))S(t)(Q_0 + Q_-)S^{-1}(s)(I_n \otimes \Psi^{-1}(s))| \leq K, \\ \text{for } t > 0, s > 0, s < t$$

$$|(I_n \otimes \Psi(t))S(t)Q_+S^{-1}(s)(I_n \otimes \Psi^{-1}(s))| \leq K, \\ \text{for } t > 0, s > 0, s \geq t$$

$$|(I_n \otimes \Psi(t))S(t)Q_-S^{-1}(s)(I_n \otimes \Psi^{-1}(s))| \leq K, \\ \text{for } t \leq 0, s < t$$

$$|(I_n \otimes \Psi(t))S(t)(Q_0 + Q_+)S^{-1}(s)(I_n \otimes \Psi^{-1}(s))| \leq K, \\ \text{for } t \leq 0, s \geq t, s < 0$$

$$|(I_n \otimes \Psi(t))S(t)Q_+S^{-1}(s)(I_n \otimes \Psi^{-1}(s))| \leq K, \\ \text{for } t \leq 0, s \geq t, s \geq 0.$$

Putting $S(t) = Z(t) \otimes Y(t)$ and using Kronecker product properties, (3.1) holds.

Conversly suppose that (3.1) holds for some $K \geq 0$.

Let $F : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ be a lebesgue Ψ - integrable matrix function on \mathbb{R} . From Lemma 2.2, it follows that the vector valued function $\hat{F}(t) = VecF(t)$ is a Lebesgue $I_n \otimes \Psi$ - integrable function on \mathbb{R} .

From Theorem 2.2, it follows the differential system (2.1) has at least one $I_n \otimes \Psi$ - bounded solution on \mathbb{R} . Let $w(t)$ be this solution.

From Lemma 2.3, it follows that the matrix function $W(t) = Vec^{-1}w(t)$ is a Ψ - bounded solution of the equation (1.1) on \mathbb{R} (because $F(t) = Vec^{-1}\hat{F}(t)$).

Thus the matrix Lyapunov system (1.1) has at least one Ψ - bounded solution on \mathbb{R} for every Lebesgue Ψ - integrable matrix function F on \mathbb{R} . \square

In a particular case, we have the following result.

Theorem 3.2. *If the homogeneous system ($F = O$ in (1.1)) has no nontrivial Ψ -bounded solution on \mathbb{R} , then the system (1.1) has a unique Ψ -bounded solution on \mathbb{R} for every Lebesgue Ψ -integrable matrix function $F : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ on \mathbb{R} if and only if there exists a positive constant K such that*

$$(3.2) \quad \begin{aligned} |(Z(t) \otimes \Psi(t)Y(t))Q_-(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| &\leq K \quad \text{for } s < t \\ |(Z(t) \otimes \Psi(t)Y(t))Q_+(Z^{-1}(s) \otimes Y^{-1}(s)\Psi^{-1}(s))| &\leq K \quad \text{for } t \leq s \end{aligned}$$

Proof. In this case, $Q_0 = O$. The proof is simple by putting $Q_0 = O$ in Theorem 3.1. \square

Next, we prove a theorem in which we will see that the asymptotic behavior of solutions to (1.1) is determined completely by the asymptotic behavior of the fundamental matrices $Y(t)$ and $Z(t)$ of (2.4) and (2.5) respectively.

Theorem 3.3. *Suppose that:*

(1) *the fundamental matrices $Y(t)$ and $Z(t)$ of (2.4) and (2.5) satisfies:*

- (a) *condition (3.1) is satisfied for some $K > 0$;*
- (b) *the following conditions are satisfied:*
 - (i) $\lim_{t \rightarrow \pm\infty} \|(Z(t) \otimes \Psi(t)Y(t))Q_0\| = 0$;
 - (ii) $\lim_{t \rightarrow -\infty} \|(Z(t) \otimes \Psi(t)Y(t))Q_+\| = 0$;
 - (iii) $\lim_{t \rightarrow +\infty} \|(Z(t) \otimes \Psi(t)Y(t))Q_-\| = 0$;

(2) *the matrix function $F : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ is Lebesgue Ψ -integrable on \mathbb{R} .*

Then, every Ψ -bounded solution X of (1.1) is such that

$$\lim_{t \rightarrow \pm\infty} |\Psi(t)X(t)| = 0.$$

Proof. Let $X(t)$ be a Ψ -bounded solution of (1.1). From Lemma 2.3, it follows that the function $\hat{X}(t) = \text{Vec}X(t)$ is a $I_n \otimes \Psi$ -bounded solution on \mathbb{R} of the differential system (2.1). Also from Lemma 2.2, the function $\hat{F}(t)$ is Lebesgue $I_n \otimes \Psi$ -integrable on \mathbb{R} . From the Theorem 2.2, it follows that

$$\lim_{t \rightarrow \pm\infty} \left\| (I_n \otimes \Psi(t)) \hat{X}(t) \right\| = 0.$$

Now, from the inequality (2.3) we have

$$|\Psi(t)X(t)| \leq n \left\| (I_n \otimes \Psi(t)) \hat{X}(t) \right\|, t \in \mathbb{R}$$

and, then

$$\lim_{t \rightarrow \pm\infty} |\Psi(t)X(t)| = 0.$$

□

The next result follows from Theorems 3.2 and 3.3.

Corollary 3.4. *Suppose that*

- (1) *the homogeneous system ($F = O$ in (1.1)) has no nontrivial Ψ -bounded solution on \mathbb{R} ;*
- (2) *the fundamental matrices $Y(t)$ and $Z(t)$ of (2.4) and (2.5) satisfies:*
 - (i) *the condition (3.2) for some $K > 0$.*
 - (ii) $\lim_{t \rightarrow -\infty} \|(Z(t) \otimes \Psi(t)Y(t))Q_+\| = 0$;
 - (iii) $\lim_{t \rightarrow +\infty} \|(Z(t) \otimes \Psi(t)Y(t))Q_-\| = 0$;
- (3) *the matrix function $F : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ is Lebesgue Ψ -integrable on \mathbb{R} .*

Then (1.1) has a unique solution X on \mathbb{R} such that

$$\lim_{t \rightarrow \pm\infty} \|\Psi(t)X(t)\| = 0.$$

Note that Theorem 3.3 is no longer true if we require that the function F be Ψ -bounded on \mathbb{R} (more, even $\lim_{t \rightarrow \pm\infty} |\Psi(t)F(t)| = 0$), instead of the condition (3) in the above Theorem. This is shown in the following example.

Example. Consider (1.1) with $A(t) = I_2$, $B(t) = -I_2$ and

$$F(t) = \begin{bmatrix} \sqrt{1+|t|} & \frac{1}{1+|t|} \\ 1 & 1 \end{bmatrix}.$$

Then, $Y(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}$, $Z(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix}$ are the fundamental matrices for (2.4) and (2.5) respectively. Consider

$$\Psi(t) = \begin{bmatrix} \frac{1}{1+|t|} & 0 \\ 0 & \frac{1}{(1+|t|)^2} \end{bmatrix}.$$

Therefore, $Q_- = I_2$, $Q_+ = O_2$ and $Q_0 = O_2$. The conditions (3.1) are satisfied with $K = 1$. In addition, the hypothesis (1b) of Theorem 3.3 is satisfied. Because

$$\Psi(t)F(t) = \begin{bmatrix} \frac{1}{\sqrt{1+|t|}} & \frac{1}{(1+|t|)^2} \\ \frac{1}{(1+|t|)^2} & \frac{1}{(1+|t|)^2} \end{bmatrix},$$

the matrix function F is not Lebesgue Ψ -integrable on \mathbb{R} , but it is Ψ -bounded on \mathbb{R} , with $\lim_{t \rightarrow \pm\infty} |\Psi(t)F(t)| = 0$. The solutions of the system (1.1) are

$$X(t) = \begin{bmatrix} p(t) + c_1 & q(t) + c_2 \\ t + c_3 & t + c_4 \end{bmatrix},$$

where

$$p(t) = \begin{cases} -\frac{2}{3}(1-t)^{3/2}, & t < 0 \\ \frac{2}{3}(1+t)^{3/2}, & t \geq 0 \end{cases}$$

and

$$q(t) = \begin{cases} -\ln(1-t), & t < 0 \\ \ln(1+t), & t \geq 0. \end{cases}$$

It is easily seen that $\lim_{t \rightarrow \pm\infty} \|\Psi(t)X(t)\| = +\infty$, for all $c_1, c_2, c_3, c_4 \in \mathbb{R}$. It follows that the solutions of the system (1.1) are Ψ -unbounded on \mathbb{R} .

Remark. If in the above example, $F(t) = \begin{bmatrix} 0 & \frac{1}{1+|t|} \\ 1 & 1 \end{bmatrix}$, then

$\int_{-\infty}^{+\infty} \|\Psi(t)F(t)\| dt = 2$. On the other hand, the solutions of (1.1) are

$$X(t) = \begin{bmatrix} c_1 & q(t) + c_2 \\ t + c_3 & t + c_4 \end{bmatrix},$$

where

$$q(t) = \begin{cases} -\ln(1-t), & t < 0 \\ \ln(1+t), & t \geq 0. \end{cases}$$

We observe that the asymptotic properties of the components of the solutions are not the same. The first row first column element is bounded and the remaining elements are unbounded on \mathbb{R} . However, all solutions of (1.1) are Ψ -bounded on \mathbb{R} and $\lim_{t \rightarrow \pm\infty} \|\Psi(t)X(t)\| = 0$. This shows that the asymptotic properties of the components of the solutions are the same, via the matrix function Ψ . This is obtained by using a matrix function Ψ rather than a scalar function.

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