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Multiple solutions for a class of fractional equations

Ruichang Pei ✉ [1], **Jihui Zhang** [2] and **Caochuan Ma** [1]

[1]School of Mathematics and Statistics Tianshui Normal University, Tianshui, 741001, P. R. China
[2]Institute of Mathematics, School of Mathematics and Computer Sciences,
Nanjing Normal University, Nanjing, 210097, P. R. China

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Abstract. In this paper we study a class of fractional Laplace equations with asymptotically linear right-hand side. The existence results of three nontrivial solutions under the resonance and non-resonance conditions are established by using the minimax method and Morse theory.

Keywords: fractional Laplacian, multiple solutions, asymptotically linear, mountain pass theorem, Morse theory.

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1 Introduction

In this article, we are interested in the following non-local fractional equations:

$$\begin{cases} (-\Delta)^s u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $s \in (0,1)$ is a fixed parameter, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N > 2s$ and $(-\Delta)^s$ is the fractional Laplace operator.

In recent years, many papers are devoted to the study of non-local fractional Laplacian with superlinear and subcritical or critical growth (see [2,4,13,14,16,17] and references therein). Particularly, in [8], Fiscella et al. studied equations (1.1) with asymptotically linear right-hand side and obtained some existence results by using saddle point theorem; in [9], Iannizzotto et al. also studied fractional p-Laplacian equations with asymptotically p-linear and obtained two nontrivial solutions by the use of the mountain pass theorem.

There are many interesting problems in the standard framework of the Laplacian (or higher order Laplacian), widely studied in the literature. A natural question is whether or not the existence results of multiple solutions obtained in the classical context can be extended to the non-local framework of the fractional Laplacian operator. Chang et al. [7] showed the existence of three nontrivial solutions for asymptotically linear Dirichlet problem via the mountain pass

✉Corresponding author. Email: prc211@163.com

theorem and Morse theory. In [11] Qian et al. did similar work for fourth-order asymptotically linear elliptic problem.

Motivated by their work, we study the following non-local problem with homogeneous Dirichlet boundary conditions investigated by Servadei et al. [15] and the related works [12, 14]:

$$\begin{cases} -\mathcal{L}_k u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.2}$$

where $\mathcal{L}_k$ is the integro-differential operator defined as follows:

$$\mathcal{L}_k u(x) = \int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \qquad x \in \mathbb{R}^N, \tag{1.3}$$

with the kernel $K : \mathbb{R}^N \backslash 0 \to (0, +\infty)$ such that

$(B1)$ $mK \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^2, 1\}$,

$(B2)$ there exists $\theta > 0$ such that $K(x) \geq \theta |x|^{-(N+2s)}$ for any $x \in \mathbb{R}^N \backslash \{0\}$,

$(B3)$ $K(x) = K(-x)$ for any $x \in \mathbb{R}^N \backslash \{0\}$.

For narrative convenience, in this paper, we only consider the particular case of problem (1.2), i.e., we let K be given by the singular kernel $K(x) = |x|^{-(N+2s)}$ which leads to the fractional Laplace operator $-(-\Delta)^s$, which, up to normalization factors, may be defined as

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \qquad x \in \mathbb{R}^N. \tag{1.4}$$

Obviously, the corresponding fractional equation in the above model (1.2) changes problem (1.1). In fact, our methods and results in this paper also adapt for the general problem (1.2).

Let $f(x,0) = 0$ and $F(x,t) = \int_0^t f(x,s)ds$. Moreover, suppose that the non-linearity f satisfy the following conditions:

(f_1) $f \in C^1(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x,0) = 0$, $f(x,t)t \geq 0$ for all $x \in \Omega$, $t \in \mathbb{R}$,

(f_2) f' is subcritical in t, i.e. there is a constant $p \in (2, 2^*)$, $2^* = \frac{2N}{N-2s}$ such that

$$\lim_{t \to \infty} \frac{f_t(x,t)}{|t|^{p-1}} = 0 \quad \text{uniformly for } x \in \bar{\Omega},$$

(f_3) $\lim\limits_{|t| \to 0} \frac{f(x,t)}{t} = f_0$, $\lim\limits_{|t| \to \infty} \frac{f(x,t)}{t} = l$ uniformly for $x \in \Omega$, where f_0 and l are constants;

(f_4) $\lim\limits_{|t| \to \infty} [f(x,t)t - 2F(x,t)] = -\infty$.

Now, we give our main results.

Theorem 1.1. *Assume conditions (f_1)–(f_3) hold, $f_0 < \lambda_1$ and $l \in (\lambda_k, \lambda_{k+1})$ for some $k \geq 2$, then problem (1.1) has at least three nontrivial solutions.*

Theorem 1.2. *Assume conditions (f_1)–(f_4) hold, $f_0 < \lambda_1$ and $l = \lambda_k$ for some $k \geq 2$, then problem (1.1) has at least three nontrivial solutions.*

Here, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ are the eigenvalues of $(-\Delta)^s$ with homogeneous Dirichlet boundary data and $\phi_1(x) > 0$ be the eigenfunction corresponding to λ_1.

In view of the condition (f_3), problem (1.1) is called asymptotically linear at both zero and infinity, which means that usual Ambrosetti–Rabinowitz condition (see [1]) is not satisfied. This will bring some difficulty if the mountain pass theorem is used to seek nontrivial solutions of problem (1.1). For the standard Laplacian Dirichlet problem, Zhou [18] have overcome it by using some monotonicity condition. Novelties of our this paper are as following.

We consider multiple solutions of problem (1.1) in the cases of resonance and non-resonance by using the mountain pass theorem and Morse theory. First, we use the truncated technique and the mountain pass theorem to obtain a positive solution and a negative solution of problem (1.1) under our more general conditions (f_1), (f_2) and (f_3) with respect to the conditions (H_1) and (H_3) in [18]. In the course of proving the existence of a positive solution and a negative solution, the monotonicity condition (H_2) of [18] on the nonlinear term f is not necessary, this point is very important because we can directly prove existence of positive solution and negative solution by using Rabinowitz's mountain pass theorem. That is, the proof of our compact condition is more simple than that in [18]. Furthermore, we can obtain a nontrivial solution when the nonlinear term f is resonant or non-resonant at the infinity by using Morse theory.

The paper is organized as follows. In Section 2, we present some necessary preliminary knowledge about the working space. In Section 3, we prove some lemmas in order to prove our main results. In Section 4, we give the proofs for our main results.

2 Preliminaries

In this section, we give some preliminary results which will be used in the sequel. We briefly recall the related definitions and notes for functional space X_0 introduced in [15].

The functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and the map $(x,y) \longmapsto (g(x)-g(y))\sqrt{K(x-y)}$ is in $L^2(\mathbb{R}^N \times \mathbb{R}^N)\backslash(\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy)$ (here $\mathcal{C}\Omega = \mathbb{R}^N\backslash\Omega$). Also, we denote by X_0 the following linear subspace of X

$$X_0 := \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^N\backslash\Omega\}.$$

Note that X and X_0 are non-empty, since $C_0^2(\Omega) \subseteq X_0$ by [15]. Moreover, the space X is endowed with the norm defined as

$$\|g\|_X = |g|_{L^2(\Omega)} + \left(\int_Q |g(x)-g(y)|^2 K(x-y)dxdy\right)^{\frac{1}{2}}, \tag{2.1}$$

where $Q = (\mathbb{R}^N \times \mathbb{R}^N)\backslash\mathcal{O}$ and $\mathcal{O} = (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^N \times \mathbb{R}^N$. We equip X_0 with the following norm

$$\|g\|_{X_0} = \left(\int_Q |g(x)-g(y)|^2 K(x-y)dxdy\right)^{\frac{1}{2}}, \tag{2.2}$$

which is equivalent to the usual one defined in (2.1) (see [14]). It is easy to see that $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space with scalar product

$$\langle u,v\rangle_{X_0} = \int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)dxdy. \tag{2.3}$$

Denote by $H^s(\Omega)$ the usual fractional Sobolev space with respect to the Gagliardo norm

$$\|g\|_{H^s(\Omega)} = |g|_{L^2(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|g(x)-g(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}}. \tag{2.4}$$

Now, we give a basic fact which will be used later.

Lemma 2.1 ([14]). *The embedding $j : X_0 \hookrightarrow L^v(\Omega)$ is continuous for any $v \in [1, 2^*]$, while it is compact whenever $v \in [1, 2^*)$.*

Next, we state some propositions for the operator $(-\Delta)^s$. Let $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \cdots$ be the sequence of the eigenvalues of $(-\Delta)^s$ (see [8]) and ϕ_k be the k-th eigenfunction corresponding to the eigenvalues λ_k. Moreover, we will set

$$\mathbb{P}_{k+1} = \{u \in X_0 : \langle u, \phi_j\rangle_{X_0} = 0,\ \forall j = 1, 2, \dots, k\}$$

and

$$H_k = \operatorname{span}\{\phi_1, \dots, \phi_k\}.$$

Proposition 2.2 ([8]). *The following inequality holds true*

$$\|u\|^2_{X_0} \le \lambda_k |u|^2_{L^2(\Omega)}$$

for all $u \in H_k$ and $k \in N$.

Proposition 2.3 ([8]). *The following inequality holds true*

$$\|u\|^2_{X_0} \ge \lambda_{k+1} |u|^2_{L^2(\Omega)}$$

for all $u \in \mathbb{P}_{k+1}$ and any $k \in N$.

Next, we recall some definitions for compactness condition and a version of the mountain pass theorem.

Definition 2.4. Let $(X_0, \|\cdot\|_{X_0})$ be a real Banach space with its dual space $(X_0^*, \|\cdot\|_{X_0^*})$ and $\mathcal{J} \in C^1(X_0, \mathbb{R})$. For $c \in \mathbb{R}$, we say that $\mathcal{J}$ satisfies the $(PS)_c$ condition if for any sequence $\{x_n\} \subset X_0$ with

$$\mathcal{J}(x_n) \to c, \qquad D\mathcal{J}(x_n) \to 0 \quad \text{in } X_0^*,$$

there is a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in X_0. Also, we say that $\mathcal{J}$ satisfy the $(C)_c$ condition stated in [5] if for any sequence $\{x_n\} \subset X_0$ with

$$\mathcal{J}(x_n) \to c, \qquad \|D\mathcal{J}(x_n)\|_{X_0^*}(1 + \|x_n\|_{X_0}) \to 0,$$

there is subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly in X_0.

3 Some lemmas

First, we observe that problem (1.1) has a variational structure, indeed it is the Euler–Lagrange equation of the functional $\mathcal{J} : X_0 \to \mathbb{R}$ defined as follows:

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^2 K(x-y)dxdy - \int_\Omega F(x, u(x))dx.$$

It is well know that the functional $\mathcal{J}$ is Fréchet differentiable in X_0 and for any $\varphi \in X_0$

$$\langle \mathcal{J}'(u), \varphi \rangle = \int_{\mathbb{R}^N \times \mathbb{R}^N} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)dxdy - \int_\Omega f(x, u(x))\varphi(x)dx.$$

Thus, critical points of $\mathcal{J}$ are solutions of problem (1.1).

Consider the following problem

$$\begin{cases} (-\Delta)^s u = f_+(x,u), & \text{in } \Omega; \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where

$$f_+(x,t) = \begin{cases} f(x,t), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Define a functional $\mathcal{J}_+ : X_0 \to \mathbb{R}$ by

$$\mathcal{J}_+(u) = \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x) - u(y)|^2 K(x-y)dxdy - \int_\Omega F(x, u(x))dx,$$

where $F_+(x,t) = \int_0^t f_+(x,s)ds$, then $\mathcal{J}_+ \in C^{2-0}(X_0, \mathbb{R})$.

Lemma 3.1. *$\mathcal{J}_+$ satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset X_0$ be a sequence such that $|\mathcal{J}'_+(u_n)| \leq c$, $\langle \mathcal{J}'_+(u_n), \varphi \rangle \to 0$ as $n \to \infty$. Note that

$$\begin{aligned} \langle \mathcal{J}'_+(u_n), \varphi \rangle &= \int_{\mathbb{R}^N \times \mathbb{R}^N} (u_n(x) - u_n(y))(\varphi(x) - \varphi(y))K(x-y)dxdy - \int_\Omega f_+(x, u_n)\varphi dx \\ &= o(\|\varphi\|_{X_0}) \end{aligned} \tag{3.1}$$

for all $\varphi \in X_0$. Assume that $|u_n|_{L^2(\Omega)}$ is bounded, taking $\varphi = u_n$ in (3.1). By (f_3), there exists $c > 0$ such that $|f_+(x, u_n(x))| \leq c|u_n(x)|$, a.e. $x \in \Omega$. So u_n is bounded in X_0. If $|u_n|_{L^2(\Omega)} \to +\infty$, as $n \to \infty$, set $v_n = \frac{u_n}{|u_n|_{L^2(\Omega)}}$, then $|v_n|_{L^2(\Omega)} = 1$. Taking $\varphi = v_n$ in (3.1), it follows that $\|v_n\|_{X_0}$ is bounded. Without loss of generality, we assume that $v_n \rightharpoonup v$ in X_0, then $v_n \to v$ in $L^2(\Omega)$. Hence, $v_n \to v$ a.e. in Ω. Dividing both sides of (3.1) by $|u_n|_{L^2(\Omega)}$, we get

$$\begin{aligned} &\int_{\mathbb{R}^N \times \mathbb{R}^N} (v_n(x) - v_n(y))(\varphi(x) - \varphi(y))K(x-y)dxdy - \int_\Omega \frac{f_+(x, u_n)}{|u_n|_{L^2(\Omega)}} \varphi dx \\ &\quad = o\left(\frac{\|\varphi\|_{X_0}}{|u_n|_{L^2(\Omega)}} \right), \ \forall \varphi \in X_0. \end{aligned} \tag{3.2}$$

Then for a.e. $x \in \Omega$, we deduce that $\frac{f_+(x,u_n)}{|u_n|_{L^2(\Omega)}} \to lv_+$ as $n \to \infty$, where $v_+ = \max\{v, 0\}$. In fact, when $v(x) > 0$, by (f_3) we have

$$u_n(x) = v_n(x)|u_n|_{L^2(\Omega)} \to +\infty$$

and

$$\frac{f_+(x,u_n)}{|u_n|_{L^2(\Omega)}} = \frac{f_+(x,u_n)}{u_n} v_n \to lv.$$

When $v(x) = 0$, we have
$$\frac{f_+(x,u_n)}{|u_n|_{L^2(\Omega)}} \le c|v_n| \longrightarrow 0.$$
When $v(x) < 0$, we have
$$u_n(x) = v_n(x)|u_n|_{L^2(\Omega)} \longrightarrow -\infty$$
and
$$\frac{f_+(x,u_n)}{|u_n|_{L^2(\Omega)}} = 0.$$
Since $\frac{f_+(x,u_n)}{|u_n|_{L^2(\Omega)}} \le c|v_n|$, by (3.2) and the Lebesgue dominated convergence theorem, we arrive at
$$\int_{\mathbb{R}^N\times\mathbb{R}^N}(v(x)-v(y))(\varphi(x)-\varphi(y))K(x-y)dxdy - \int_\Omega lv_+\varphi dx = 0, \quad \text{for any } \varphi \in X_0. \tag{3.3}$$
From the strong maximum principle (see [9]) we deduce that $v > 0$. Choosing $\varphi = \phi_1$ in (3.3), we obtain
$$l\int_\Omega v\phi_1 dx = \lambda_1\int_\Omega v\phi_1 dx.$$
This is a contradiction. □

Lemma 3.2. *Let ϕ_1 be the eigenfunction corresponding to λ_1 with $\|\phi_1\| = 1$. If $f_0 < \lambda_1 < l$, then*

(a) there exist $\rho, \beta > 0$ such that $\mathcal{J}_+(u) \ge \beta$ for all $u \in X_0$ with $\|u\| = \rho$;

(b) $\mathcal{J}_+(t\phi_1) = -\infty$ as $t \to +\infty$.

Proof. By (f_1) and (f_3), if $l \in (\lambda_1, +\infty)$, for any $\varepsilon > 0$, there exist $A = A(\varepsilon) \ge 0$ and $B = B(\varepsilon)$ such that for all $(x,s) \in \Omega\times\mathbb{R}$,
$$F_+(x,s) \le \frac{1}{2}(f_0+\varepsilon)s^2 + As^{p+1}, \tag{3.4}$$
$$F_+(x,s) \ge \frac{1}{2}(l-\varepsilon)s^2 - B, \tag{3.5}$$
where $p \in (1, \frac{N+s}{N-s})$.

Choose $\varepsilon > 0$ such that $f_0 + \varepsilon < \lambda_1$. By (3.4) and Lemma 2.1, we get
$$\begin{aligned}\mathcal{J}_+(u) &= \frac{1}{2}\|u\|^2_{X_0} - \int_\Omega F(x,u)dx\\ &\ge \frac{1}{2}\|u\|^2_{X_0} - \frac{1}{2}\int_\Omega [(f_0+\varepsilon)u^2 + A|u|^{p+1}]dx\\ &= \frac{1}{2}\left(1-\frac{f_0+\varepsilon}{\lambda_1}\right)\|u\|^2_{X_0} - c\|u\|^{p+1}_{X_0}.\end{aligned}$$
So, part (a) holds if we choose $\|u\|_{X_0} = \rho > 0$ small enough.

On the other hand, if $l \in (\lambda_1, +\infty)$, take $\varepsilon > 0$ such that $l - \varepsilon > \lambda_1$. By (3.5), we have
$$\mathcal{J}_+(u) \le \frac{1}{2}\|u\|^2_{X_0} - \frac{l-\varepsilon}{2}|u|^2_{L^2(\Omega)} + B|\Omega|.$$
Since $l - \varepsilon > \lambda_1$ and $\|\phi_1\|_{X_0} = 1$, it is easy to see that
$$\mathcal{J}_+(t\phi_1) \le \frac{1}{2}\left(1-\frac{l-\varepsilon}{\lambda_1}\right)t^2 + B|\Omega| \to -\infty \quad \text{as } t \to +\infty$$
and part (b) is proved. □

Lemma 3.3. *Let $X_0 = H_k \oplus \mathbb{P}_{k+1}$. If f satisfies $(f_1), (f_3)$ and (f_4) then*

(i) the functional $\mathcal{J}$ is coercive on $\mathbb{P}_{k+1}$, that is

$$\mathcal{J}(u) \to +\infty \quad \text{as } \|u\|_{X_0} \to +\infty,\ u \in \mathbb{P}_{k+1}$$

and bounded from below on $\mathbb{P}_{k+1}$,

(ii) the functional $\mathcal{J}$ is anti-coercive on H_k.

Proof. For $u \in \mathbb{P}_{k+1}$, by (f_3), for any $\varepsilon > 0$, there exists $B_1 = B_1(\varepsilon)$ such that for all $(x,s) \in \Omega \times \mathbb{R}$,

$$F(x,s) \leq \frac{1}{2}(l+\varepsilon)s^2 + B_1. \tag{3.6}$$

So, from Proposition 2.3 we have

$$\begin{aligned}\mathcal{J}(u) &= \frac{1}{2}\|u\|^2_{X_0} - \int_\Omega F(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2_{X_0} - \frac{1}{2}(l+\varepsilon)|u|^2_{L^2(\Omega)} - B_1|\Omega| \\ &\geq \frac{1}{2}\left(1 - \frac{l+\varepsilon}{\lambda_{k+1}}\right)\|u\|^2_{X_0} - B_1|\Omega|.\end{aligned}$$

Choose $\varepsilon > 0$ such that $l + \varepsilon < \lambda_{k+1}$. This proves (i).
(ii) We firstly consider the case $l = \lambda_k$.
Write $G(x,t) = F(x,t) - \frac{1}{2}\lambda_k t^2$, $g(x,t) = f(x,t) - \lambda_k t$. Then (f_3) and (f_4) imply that

$$\lim_{|t|\to\infty} [g(x,t)t - 2G(x,t)] = -\infty \tag{3.7}$$

and

$$\lim_{|t|\to\infty} \frac{2G(x,t)}{t^2} = 0. \tag{3.8}$$

It follows from (3.7) that for every $M > 0$, there exists a constant $T > 0$ such that

$$g(x,t)t - 2G(x,t) \leq -M, \qquad \forall t \in \mathbb{R},\ |t| \geq T,\ \text{a.e. } x \in \Omega. \tag{3.9}$$

For $\tau > 0$, we have

$$\frac{d}{d\tau}\frac{G(x,\tau)}{\tau^2} = \frac{g(x,\tau)\tau - 2G(x,\tau)}{\tau^3}. \tag{3.10}$$

Integrating (3.10) over $[t,s] \subset [T,+\infty)$, we deduce that

$$\frac{G(x,s)}{s^2} - \frac{G(x,t)}{t^2} \leq \frac{M}{2}\left(\frac{1}{s^2} - \frac{1}{t^2}\right). \tag{3.11}$$

Letting $s \to +\infty$ and using (3.8), we see that $G(x,t) \geq \frac{M}{2}$, for $t \in \mathbb{R}$, $t \geq T$, a.e. $x \in \Omega$. A similar argument shows that $G(x,t) \geq \frac{M}{2}$, for $t \in \mathbb{R}$, $t \leq -T$, a.e. $x \in \Omega$. Hence

$$\lim_{|t|\to\infty} G(x,t) \to +\infty, \quad \text{a.e. } x \in \Omega. \tag{3.12}$$

By (3.12) and Proposition 2.2, we get

$$\begin{aligned}\mathcal{J}(v) &= \frac{1}{2}\|v\|_{X_0}^2 - \int_\Omega F(x,v)dx \\ &= \frac{1}{2}\|v\|_{X_0}^2 - \frac{1}{2}\lambda_k\int_\Omega v^2dx - \int_\Omega G(x,v)dx \\ &\leq -\delta\|v^-\|_{X_0}^2 - \int_\Omega G(x,v)dx \to -\infty\end{aligned}$$

for $v \in V$ with $\|v\|_{X_0} \to +\infty$, where $v^- \in H_{k-1}$.

In the case of $\lambda_k < l < \lambda_{k+1}$, we needn't the assumption (f_4) and it is easy to see that the conclusion also holds . □

Lemma 3.4. *If $\lambda_k < l < \lambda_{k+1}$, then $\mathcal{J}$ satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset X_0$ be a sequence such that $|\mathcal{J}(u_n)| \leq c$, $\langle \mathcal{J}'(u_n), \varphi\rangle \to 0$. Since

$$\begin{aligned}\langle \mathcal{J}'(u_n), \varphi\rangle &= \int_{\mathbb{R}^N\times\mathbb{R}^N}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))K(x-y)dxdy - \int_\Omega f(x,u_n)\varphi dx \\ &= o(\|\varphi\|_{X_0}).\end{aligned} \tag{3.13}$$

for all $\varphi \in X_0$. If $|u_n|_{L^2(\Omega)}$ is bounded, we can take $\varphi = u_n$. By (f_3), there exists a constant $c > 0$ such that $|f(x,u_n(x))| \leq c|u_n(x)|$, a.e. $x \in \Omega$. So u_n is bounded in X_0. If $|u_n|_{L^2(\Omega)} \to +\infty$, as $n \to \infty$, set $v_n = \frac{u_n}{|u_n|_{L^2(\Omega)}}$, then $|v_n|_{L^2(\Omega)} = 1$. Taking $\varphi = v_n$ in (3.13), it follows that $\|v_n\|_{X_0}$ is bounded. Without loss of generality, we assume $v_n \rightharpoonup v$ in X_0, then $v_n \to v$ in $L^2(\Omega)$. Hence, $v_n \to v$ a.e. in Ω. Dividing both sides of (3.13) by $|u_n|_{L^2(\Omega)}$, we get

$$\begin{aligned}&\int_{\mathbb{R}^N\times\mathbb{R}^N}(v_n(x)-v_n(y))(\varphi(x)-\varphi(y))K(x-y)dxdy - \int_\Omega \frac{f(x,u_n)}{|u_n|_{L^2(\Omega)}}\varphi dx \\ &= o\left(\frac{\|\varphi\|_{X_0}}{|u_n|_{L^2(\Omega)}}\right), \quad \forall \varphi \in X_0.\end{aligned} \tag{3.14}$$

Then for a.e. $x \in \Omega$, we have $\frac{f(x,u_n)}{|u_n|_{L^2(\Omega)}} \to lv$ as $n \to \infty$. In fact, if $v(x) \neq 0$, by (f_3), we have

$$|u_n(x)| = |v_n(x)||u_n|_{L^2(\Omega)} \to +\infty$$

and

$$\frac{f(x,u_n)}{|u_n|_{L^2(\Omega)}} = \frac{f(x,u_n)}{u_n}v_n \to lv.$$

If $v(x) = 0$, we have

$$\frac{|f(x,u_n)|}{|u_n|_{L^2(\Omega)}} \leq c|v_n| \longrightarrow 0.$$

Since $\frac{|f(x,u_n)|}{|u_n|_{L^2(\Omega)}} \leq c|v_n|$, by (3.14) and the Lebesgue dominated convergence theorem, we arrive at

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}(v(x)-v(y))(\varphi(x)-\varphi(y))K(x-y)dxdy - \int_\Omega lv\varphi dx = 0, \quad \text{for any } \varphi \in X_0.$$

Obviously $v \neq 0$, hence, l is an eigenvalue of $(-\Delta)^s$. This contradicts our assumption. □

Lemma 3.5. *Suppose $l = \lambda_k$ and f satisfies (f_4). Then the functional $\mathcal{J}$ satisfies the (C) condition.*

Proof. Suppose $u_n \in X_0$ satisfies

$$\mathcal{J}(u_n) \to c \in \mathbb{R}, \quad (1 + \|u_n\|)\|\mathcal{J}'(u_n)\| \to 0 \quad \text{as } n \to \infty. \tag{3.15}$$

In view of (f_3), it suffices to prove that u_n is bounded in X_0. Similar to the proof of Lemma 3.4, we have

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} (v(x) - v(y))(\varphi(x) - \varphi(y))K(x - y)dxdy - \int_\Omega lv\varphi dx = 0, \quad \text{for any } \varphi \in X_0. \tag{3.16}$$

Therefore $v \neq 0$ is an eigenfunction of λ_k, then $|u_n(x)| \to \infty$ for a.e. $x \in \Omega_0$ ($\Omega_0 \subset \Omega$) with positive measure. It follows from (f_4) that

$$\lim_{n \to +\infty} [f(x, u_n(x))u_n(x) - 2F(x, u_n(x))] = -\infty$$

holds uniformly in $x \in \Omega_0$, which implies that

$$\int_\Omega (f(x, u_n)u_n - 2F(x, u_n))dx \to -\infty \quad \text{as } n \to \infty. \tag{3.17}$$

On the other hand, (3.15) implies that

$$2\mathcal{J}(u_n) - \langle \mathcal{J}'(u_n), u_n \rangle \to 2c \quad \text{as } n \to \infty.$$

Thus

$$\int_\Omega (f(x, u_n)u_n - 2F(x, u_n))dx \to 2c \quad \text{as } n \to \infty,$$

which contradicts (3.17). Hence u_n is bounded. □

It is well known that critical groups and Morse theory are the main tools in solving elliptic partial differential equation. Let us recall some results which will be used later. We refer the readers to the book [6] for more information on Morse theory.

Let X be a Hilbert space and $\mathcal{J} \in C^1(X, \mathbb{R})$ be a functional satisfying the (PS) condition or (C) condition, and $H_q(X, Y)$ be the q-th singular relative homology group with integer coefficients. Let u_0 be an isolated critical point of $\mathcal{J}$ with $\mathcal{J}(u_0) = c$, $c \in \mathbb{R}$, and U be a neighborhood of u_0. The group

$$C_q(\mathcal{J}, u_0) := H_q(\mathcal{J}^c \cap U, \mathcal{J}^c \cap U \backslash \{u_0\}), \qquad q \in Z$$

is said to be the q-th critical group of $\mathcal{J}$ at u_0, where $\mathcal{J}^c = \{u \in X : \mathcal{J}(u) \leq c\}$.

Let $K := \{u \in X : \mathcal{J}'(u) = 0\}$ be the set of critical points of $\mathcal{J}$ and $a < \inf \mathcal{J}(K)$, the critical groups of $\mathcal{J}$ at infinity are formally defined by (see [3])

$$C_q(\mathcal{J}, \infty) := H_q(X, \mathcal{J}^a), \qquad q \in Z.$$

The following result comes from [3,6] and will be used to prove the results in this paper.

Proposition 3.6 ([3]). *Assume that $X = V \oplus W$, $\mathcal{J}$ is bounded from below on W and $\mathcal{J}(u) \to -\infty$ as $\|u\| \to \infty$ with $u \in V$. Then*

$$C_k(\mathcal{J}, \infty) \ncong 0, \quad \text{if } k = \dim V < \infty. \tag{3.18}$$

4 Proof of the main results

Proof of Theorem 1.1. By Lemmas 3.1, 3.2 and the mountain pass theorem, the functional $\mathcal{J}_+$ has a critical point u_1 satisfying $\mathcal{J}_+(u_1) \geq \beta$. Since $\mathcal{J}_+(0) = 0$, $u_1 \neq 0$ and by the maximum principle (see [9]), we get $u_1 > 0$. Hence u_1 is a positive solution of the problem (1.1) and satisfies

$$C_1(\mathcal{J}_+, u_1) \neq 0, \qquad u_1 > 0. \tag{4.1}$$

By (f_2), the functional $\mathcal{J}$ is C^2. Using the results in [6,10], we obtain

$$C_q(\mathcal{J}, u_1) = C_q(\mathcal{J}_{C_d^0(\overline{\Omega})}, u_1) = C_q(\mathcal{J}_+|_{C_d^0(\overline{\Omega})}, u_1) = C_q(\mathcal{J}_+, u_1) = \delta_{q1}Z. \tag{4.2}$$

Here

$$C_d^0(\overline{\Omega}) = \{u \in C^0(\overline{\Omega}) : ud^{-\gamma} \in C^0(\overline{\Omega})\},$$

where $d(x) = \text{dist}(x, \partial\Omega)$ for all $x \in \overline{\Omega}$ and $0 < \gamma < 1$. More detailed topology knowledge will be seen in [9] and we omit it.

Similarly, we can obtain another negative critical point u_2 of $\mathcal{J}$ satisfying

$$C_q(\mathcal{J}, u_2) = \delta_{q,1}Z. \tag{4.3}$$

Since $f_0 < \lambda_1$, the zero function is a local minimizer of $\mathcal{J}$, then

$$C_q(\mathcal{J}, 0) = \delta_{q,0}Z. \tag{4.4}$$

On the other hand, by Lemmas 3.3, 3.4 and Proposition 3.6, we have

$$C_k(\mathcal{J}, \infty) \ncong 0. \tag{4.5}$$

Hence $\mathcal{J}$ has a critical point u_3 satisfying

$$C_k(\mathcal{J}, u_3) \ncong 0. \tag{4.6}$$

Since $k \geq 2$, it follows from (4.2)–(4.6) that u_1, u_2 and u_3 are three different nontrivial solutions of the problem (1.1). □

Proof of Theorem 1.2. By Lemmas 3.3, 3.5 and Proposition 3.6, we can prove the conclusion (4.5). The other proof is similar to that of Theorem 1.1. □

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