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Two weak solutions for some singular fourth order elliptic problems

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Abstract. In this paper, we establish the existence of at least two distinct weak solutions for some singular elliptic problems involving a p-biharmonic operator, subject to Navier boundary conditions in a smooth bounded domain in $\mathbb{R}^N$. A critical point result for differentiable functionals is exploited, in order to prove that the problem admits at least two distinct nontrivial weak solutions.

Keywords: singular problem, p-biharmonic operator, variational methods, critical point.

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1 Introduction and main result

Singular elliptic problems have been intensively studied in the last decades. Among others, we mention the works [1,7,11,12,14,20,21,25,26]. Stationary problems involving singular nonlinearities, as well as the associated evolution equations, describe naturally several physical phenomena and applied economical models. For instance, nonlinear singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids and boundary layer phenomena for viscous fluids. Moreover, nonlinear singular elliptic equations are also encountered in glacial advance, in transport of coal slurries down conveyor belts and in several other geophysical and industrial contents.

Recently, motivated by this large interest, the problem

$$\begin{cases} \Delta_p^2 u = \frac{|u|^{p-2}u}{|x|^{2p}} + g(\lambda, x, u) & \text{in } \Omega \\ u, \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $g\colon]0,+\infty[\times\Omega\times\mathbb{R}\to\mathbb{R}$ is a suitable function, has been extensively investigated.

For instance, when $p = 2$, Wang and Shen [25] considered the problem (1.1), assuming that the nonlinearity has the form $g(\lambda, x, u) = f(x, u)$. In this setting, the existence of non-trivial solutions by using variational methods is established. Successively, Berchio et al. [1] considered the case $g(\lambda, x, u) = (1 + u)^q$, study the behavior of extremal solutions to biharmonic

Gelfand-type equations under Steklov boundary conditions. Also in [7,20,21], the authors are interested in the existence and multiplicity solutions for this kind of singular elliptic problems. Precisely, the existence of multiple solutions is proved by Chung [7] through a variant of the three critical point theorem by Bonanno [2]. Pérez-Llanos and Primo [21] studied the optimal exponent q to have solvability of problem with $g(\lambda, x, u) = u^q + cf$. Sign-changing solutions is investigated by Pei and Zhang [20].

Also in presence of p-biharmonic operator, singular equations have been investigated. For instance, Xie and Wang, in [26] proved that the problem (1.1) has infinitely many solutions with positive energy levels. Later, Huang and Liu [11] obtained the existence of sign-changing solutions of p-biharmonic equations with Hardy potential by using the method of invariant sets of descending flow.

In this paper, we want to investigate the following problem

$$\begin{cases} \Delta_p^2 u + \frac{|u|^{p-2}u}{|x|^{2p}} = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

where $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ denotes the p-biharmonic operator, Ω is a bounded domain in $\mathbb{R}^N (N \geq 5)$ containing the origin and with smooth boundary $\partial\Omega$, $1 < p < N/2$, and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(f1) $|f(x,t)| \leq a_1 + a_2|t|^{q-1}, \quad \forall (x,t) \in \Omega \times \mathbb{R}$,

for some non-negative constants a_1, a_2 and $q \in]p, p^*[$, where

$$p^* := \frac{pN}{N-2p}.$$

In this work, our goal is to obtain the existence of two distinct weak solutions for problem ($\mathcal{P}$).

Recall that a function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be a Carathéodory function, if

(C1) the function $x \to f(x,t)$ is measurable for every $t \in \mathbb{R}$;

(C2) the function $t \to f(x,t)$ is continuous for a.e. $x \in \Omega$.

Now, we establish the main abstract result of this paper. We recall that c_q is the constant of the embedding $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$ for each $q \in [1, p^*[$, and c_1 stands for c_q with $q = 1$; see (2.2).

Theorem 1.1. *Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that condition (f1) holds. Moreover, assume that*

(f2) there exist $\theta > p$ and $M > 0$ such that

$$0 < \theta F(x,t) \leq t f(x,t),$$

for each $x \in \Omega$ and $|t| \geq M$. Then, for each $\lambda \in]0, \lambda^*[$, problem ($\mathcal{P}$) admits at least two distinct weak solutions, where

$$\lambda^* := \frac{q}{q a_1 c_1 p^{1/p} + a_2 c_q^q p^{q/p}}.$$

In conclusion we present a concrete example of application of Theorem 1.1 whose construction is motivated by [4, Example 4.1].

Example 1.2. We consider the function f defined by

$$f(x,t) := \begin{cases} c + dqt^{q-1}, & \text{if } x \in \Omega, t \geq 0, \\ c - dq(-t)^{q-1}, & \text{if } x \in \Omega, t < 0, \end{cases}$$

for each $(x,t) \in \Omega \times \mathbb{R}$, where $1 < p < q < p^*$ and c, d are two positive constants. Fixed $p < \theta < q$ and

$$r > \max\left\{ \left[\frac{(\theta-1)c}{d(q-\theta)}\right]^h, \left(\frac{c}{d}\right)^h \right\},$$

with $h = \frac{1}{q-1}$, we prove that f verifies the assumptions requested in Theorem 1.1. Condition (f1) of Theorem 1.1 is easily verified. We observe that

$$F(x,t) = ct + d|t|^q,$$

for each $(x,t) \in \Omega \times \mathbb{R}$. Taking into account that, condition (f2) is verified (see Example 4.1 of [4]) and clearly $f(x,0) \neq 0$ in Ω, problem ($\mathcal{P}$) has at least two nontrivial weak solutions for every $\lambda \in]0, \lambda^*[$, where λ^* is the constant introduced in the statement of Theorem 1.1.

Remark 1.3. Thanks to Talenti's inequality, it is possible to obtain an estimate of the embedding's constants c_1, c_q. By the Sobolev embedding theorem there exists a positive constant c such that

$$\|u\|_{L^{p^*}(\Omega)} \leq c\|u\|, \qquad \forall u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \tag{1.2}$$

see [24]. The best constant that appears in (1.2) is

$$c := \frac{1}{N^2\pi}\left(\frac{\Gamma^2\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N}{2p^*}\right)\Gamma\left(\left(\frac{N}{2}\right)-\left(\frac{N}{2p^*}\right)\right)}\right)^{2/N}\eta^{1-1/p}, \tag{1.3}$$

where

$$\eta := \frac{p-1}{p},$$

see, for instance [24].

Due to (1.3), as a simple consequence of Hölder's inequality, it follows that

$$c_q \leq \frac{\text{meas}(\Omega)^{\frac{p^*-q}{p^*q}}}{N^2\pi}\left(\frac{\Gamma^2\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N}{2p^*}\right)\Gamma\left(\left(\frac{N}{2}\right)-\left(\frac{N}{2p^*}\right)\right)}\right)^{2/N}\eta^{1-1/p},$$

where "meas(Ω)" denotes the Lebesgue measure of the set Ω.

A special case of our main result reads as follows.

Theorem 1.4. *Let $N = 5$ and $f(u) = (1+u^3)$. Then, there exists $\lambda^* > 0$, such that, for any $\lambda \in]0, \lambda^*[$ the following problem*

$$\begin{cases} \Delta_p^2 u + \frac{|u|^{p-2}u}{|x|^{2p}} = \lambda(1+u^3) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits two weak solutions.

Remark 1.5. Inspired by [4], we prove that, for small values of λ, problem ($\mathcal{P}$) admits at least two weak solutions requiring that the continuous and subcritical nonlinear term f satisfies the celebrated Ambrosetti–Rabinowitz condition without the usual additional assumption at zero, that is,

$$\lim_{t\to 0} \frac{f(x,t)}{t} = 0$$

uniformly for $x \in \Omega$.

For completeness, we recall that a careful and interesting analysis of singular elliptic problems was developed in the monograph [22] as well as the papers [5, 6, 9, 10, 15–18] and references therein and see also the recent monograph by Kristály, Rădulescu and Varga [13] as general reference for this topic.

2 Preliminaries and basic definitions

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 5$) containing the origin and with smooth boundary $\partial\Omega$. Further, denote by X the space $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega |\Delta u|^p dx\right)^{1/p}.$$

Let $1 < p < N/2$, we recall classical Hardy's inequality, which says that

$$\int_\Omega \frac{|u(x)|^p}{|x|^{2p}} dx \leq \frac{1}{H} \int_\Omega |\Delta u(x)|^p dx, \qquad \forall u \in X \tag{2.1}$$

where $H := \left(\frac{N(p-1)(N-2p)}{p^2}\right)^p$; see, for instance, the paper [19].

By the compact embedding $X \hookrightarrow L^q(\Omega)$ for each $q \in [1, p^*[$, there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)} \leq c_q \|u\|, \qquad \forall u \in X \tag{2.2}$$

where c_q is the best constant of the embedding.

Let us define $F(x,\xi) := \int_0^\xi f(x,t)dt$, for every (x,ξ) in $\Omega \times \mathbb{R}$. Moreover, we introduce the functional $I_\lambda \colon X \to \mathbb{R}$ associated with ($\mathcal{P}$),

$$I_\lambda := \Phi(u) - \lambda\Psi(u), \qquad \forall u \in X$$

where

$$\Phi(u) := \frac{1}{p}\left(\int_\Omega |\Delta u(x)|^p dx + \int_\Omega \frac{|u(x)|^p}{|x|^{2p}} dx\right), \qquad \Psi(u) := \int_\Omega F(x,u)dx.$$

From Hardy's inequality (2.1), it follows that

$$\frac{\|u\|^p}{p} \leq \Phi(u) \leq \left(\frac{H+1}{pH}\right) \|u\|^p, \tag{2.3}$$

for every $u \in X$.

Fixing the real parameter λ, a function $u \colon \Omega \to \mathbb{R}$ is said to be a weak solution of ($\mathcal{P}$) if $u \in X$ and

$$\int_\Omega |\Delta u|^{p-2} \Delta u \Delta v dx + \int_\Omega \frac{|u|^{p-2}}{|x|^{2p}} uv dx - \lambda \int_\Omega f(x,u) v dx = 0,$$

for every $v \in X$. Hence, the critical points of I_λ are exactly the weak solutions of ($\mathcal{P}$).

Definition 2.1 ([13]). A Gâteaux differentiable function I satisfies the Palais–Smale condition (in short (PS)-condition) if any sequence $\{u_n\}$ such that

(a) $\{I(u_n)\}$ is bounded,

(b) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$, where X^* denote the dual space of X,

has a convergent subsequence.

Definition 2.2 ([13]). Let X be a reflexive real Banach space. The operator $T\colon X \to X^*$ is said to satisfy the (S_+) condition if the assumptions $\limsup_{n\to+\infty}\langle T(u_n) - T(u_0), u_n - u_0\rangle \le 0$ and $u_n \rightharpoonup u_0$ in X imply $u_n \to u_0$ in X.

Proposition 2.3. *The operator $T\colon X \to X^*$ defined by*

$$\langle T(u), v\rangle := \int_\Omega |\Delta u|^{p-2}\Delta u \Delta v dx + \int_\Omega \frac{|u|^{p-2}}{|x|^{2p}} uv dx,$$

for every u, $v \in X$, is strictly monotone.

Proof. Clearly T is coercive. Taking into account (2.2) of [23] for $p > 1$ there exists a positive constant C_p such that if $p \ge 2$, then

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \ge C_p|x - y|^p,$$

if $1 < p < 2$, then

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \ge C_p \frac{|x - y|^p}{(|x| + |y|)^{2-p}},$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^N$. Thus, it is easy to see that, if $p \ge 2$, then, for any u, $v \in X$, with $u \ne v$, we have

$$\langle T(u) - T(v), u - v\rangle \ge C_p \int_\Omega |\Delta u - \Delta v|^p dx = C_p\|u - v\|^p > 0,$$

and if $1 < p < 2$, then

$$\langle T(u) - T(v), u - v\rangle \ge C_p \int_\Omega \frac{|\Delta u - \Delta v|^2}{(|\Delta u| + |\Delta v|)^{2-p}} dx > 0,$$

for every u, $v \in X$, which means that T is strictly monotone. □

Our main tool is the following critical point theorem.

Theorem 2.4 ([3, Theorem 3.2]). *Let X be a real Banach space and let $\Phi, \Psi\colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{\Phi(u)<r}\Psi(u) < +\infty$ and assume that, for each $\lambda \in \left]0, \frac{r}{\sup_{\Phi(u)<r}\Psi(u)}\right[$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies (PS)-condition and it is unbounded from below. Then, for each $\lambda \in \left]0, \frac{r}{\sup_{\Phi(u)<r}\Psi(u)}\right[$, the functional I_λ admits two distinct critical points.*

3 Proof of Theorem 1.1

Proof. Our aim is to apply Theorem 2.4 to problem ($\mathcal{P}$) in the case $r = 1$ to the space $X := W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ with the norm

$$\|u\| := \left(\int_\Omega |\Delta u|^p dx\right)^{1/p},$$

and to the functionals $\Phi, \Psi \colon X \to \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{p}\left(\int_\Omega |\Delta u(x)|^p dx + \int_\Omega \frac{|u(x)|^p}{|x|^{2p}} dx\right)$$

and

$$\Psi(u) := \int_\Omega F(x,u)dx,$$

for all $u \in X$. The functional Φ is in $C^1(X,\mathbb{R})$ and $\Phi' \colon X \to X^*$ is strictly monotone (see Proposition 2.3. Now we prove that Φ' is a mapping of type (S_+). Let $u_n \rightharpoonup u$ in X and $\limsup_{n\to+\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \le 0$. Since Φ' is strictly monotone, then

$$\limsup_{n\to+\infty}\langle K'(u_n) - K'(u), u_n - u\rangle \le 0,$$

where $K' \colon X \to X^*$ defined as

$$K(u) := \frac{1}{p}\int_\Omega |\Delta u|^p dx, \qquad \forall u \in X,$$

and

$$\langle K'(u), v\rangle = \int_\Omega |\Delta u|^{p-2}\Delta u \Delta v dx,$$

for every $v \in X$. Then $u_n \to u$ in X (see Theorem 3.1 of [8]). So, Φ' is a mapping of type (S_+). By Theorem 3.1 from [8], we get that $\Phi' \colon X \to X^*$ is a homeomorphism. Moreover, thanks to condition (f1) and to the compact embedding $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$, Ψ is $C^1(X,\mathbb{R})$ and has compact derivative and

$$\langle \Psi'(u), v\rangle = \int_\Omega f(x,u)v dx,$$

for every $v \in X$. Now we prove that $I_\lambda = \Phi - \lambda\Psi$ satisfies (PS)-condition for every $\lambda > 0$. Namely, we will prove that any sequence $\{u_n\} \subset X$ satisfying

$$d := \sup_n I_\lambda(u_n) < +\infty, \qquad \|I'_\lambda(u_n)\|_{X^*} \to 0, \tag{3.1}$$

contains a convergent subsequence. For n large enough, we have by (3.1)

$$d \ge I_\lambda(u_n) = \frac{1}{p}\left(\int_\Omega |\Delta u_n|^p dx + \int_\Omega \frac{|u_n|^p}{|x|^{2p}} dx\right) - \lambda\int_\Omega F(x,u_n)dx,$$

then

$$\begin{aligned} I_\lambda(u_n) &\ge \frac{1}{p}\left(\int_\Omega |\Delta u_n|^p dx + \int_\Omega \frac{|u_n|^p}{|x|^{2p}} dx\right) - \frac{\lambda}{\theta}\int_\Omega f(x,u_n)u_n dx \\ &> \left(\frac{1}{p} - \frac{1}{\theta}\right)\left(\int_\Omega |\Delta u_n|^p dx\right) + \frac{1}{\theta}\left(\int_\Omega |\Delta u_n|^p dx + \int_\Omega \frac{|u_n|^p}{|x|^{2p}} dx - \lambda\int_\Omega f(x,u_n)u_n dx\right) \\ &\ge \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p + \frac{1}{\theta}\langle I'(u_n), u_n\rangle. \end{aligned}$$

Due to (3.1), we can actually assume that $\left|\frac{1}{\theta}\langle I'_\lambda(u_n), u_n\rangle\right| \leq \|u_n\|$. Thus,

$$d + \|u_n\| \geq I_\lambda(u_n) - \frac{1}{\theta}\langle I'_\lambda(u_n), u_n\rangle \geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p.$$

It follows from this quadratic inequality that $\{\|u_n\|\}$ is bounded. By the Eberlian–Smulyan theorem, passing to a subsequence if necessary, we can assume that $u_n \rightharpoonup u$. Then $\Psi'(u_n) \to \Psi'(u)$ because of compactness. Since $I'_\lambda(u_n) = \Phi'(u_n) - \lambda\Psi'(u_n) \to 0$, then $\Phi'(u_n) \to \lambda\Psi'(u)$. Since Φ' is a homeomorphism, then $u_n \to u$ and so I_λ satisfies (PS)-condition.

From (f2), by standard computations, there is a positive constant C such that

$$F(x,t) \geq C|t|^\theta \tag{3.2}$$

for all $x \in \Omega$ and $|t| > M$. In fact, setting $a(x) := \min_{|\xi|=M} F(x,\xi)$ and

$$\varphi_t(s) := F(x,st), \qquad \forall s > 0, \tag{3.3}$$

by (f2), for every $x \in \Omega$ and $|t| > M$ one has

$$0 < \theta\varphi_t(s) = \theta F(x,st) \leq stf(x,st) = s\varphi'_t(s), \qquad \forall s > \frac{M}{|t|}.$$

Therefore,

$$\int_{M/|t|}^1 \frac{\varphi'_t(s)}{\varphi_t(s)}ds \geq \int_{M/|t|}^1 \frac{\theta}{s}ds.$$

Then

$$\varphi_t(1) \geq \varphi_t\left(\frac{M}{|t|}\right)\frac{|t|^\theta}{M^\theta}.$$

Taking into account of (3.3), we obtain

$$F(x,t) \geq F\left(x, \frac{M}{|t|t}\right)\frac{|t|^\theta}{M^\theta} \geq a(x)\frac{|t|^\theta}{M^\theta} \geq C|t|^\theta,$$

where $C > 0$ is a constant. Thus (3.2) is proved.

Fixed $u_0 \in X \setminus \{0\}$, for each $t > 1$ one has

$$I_\lambda(tu_0) \leq \frac{1}{p}t^p\|u_0\|^p - \lambda C t^\theta \int_\Omega |u_0|^\theta dx.$$

Since $\theta > p$, this condition guarantees that I_λ is unbounded from below. Fixed $\lambda \in]0, \lambda^*[$, from (2.3) it follows that

$$\|u\| < p^{1/p}, \tag{3.4}$$

for each $u \in X$ such that $u \in \Phi^{-1}(]-\infty, 1[)$. Moreover, the compact embedding $X \hookrightarrow L^1(\Omega)$, (f1), (3.4) and the compact embedding $X \hookrightarrow L^q(\Omega)$ imply that, for each $u \in \Phi^{-1}(]-\infty, 1[)$, we have

$$\begin{aligned}\Psi(u) &\leq a_1\|u\|_{L^1(\Omega)} + \frac{a_2}{q}\|u\|^q_{L^q(\Omega)}\\ &\leq a_1c_1\|u\| + \frac{a_2}{q}c_q^q\|u\|^q\\ &< a_1c_qp^{1/p} + \frac{a_2}{q}c_q^qp^{q/p},\end{aligned}$$

and so,

$$\sup_{\Phi(u)<1} \Psi(u) \leq a_1 c_q p^{1/p} + \frac{a_2}{q} c_q^q p^{q/p} = \frac{1}{\lambda^*} < \frac{1}{\lambda}. \tag{3.5}$$

From (3.5) one has

$$\lambda \in]0, \lambda^*[\subseteq \left] 0, \frac{1}{\sup_{\Phi(u)<1} \Psi(u)} \right[.$$

So all hypotheses of Theorem 2.4 are verified. Therefore, for each $\lambda \in]0, \lambda^*[$, the functional I_λ admits two distinct critical points that are weak solutions of problem ($\mathcal{P}$). □

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