Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **20**, 1–15; doi: 10.14232/ejqtde.2016.1.20 http://www.math.u-szeged.hu/ejqtde/

Decay mild solutions of the nonlocal Cauchy problem for second order evolution equations with memory

Vu Trong Luong ✉

Department of Mathematics, Tay Bac University, Quyettam, Sonla, Vietnam

Received 16 August 2015, appeared 20 April 2016
Communicated by Gennaro Infante

Abstract. Our aim in this paper is to find decay mild solutions of the nonlocal Cauchy problem for a class of second order evolution equations with memory. By constructing a suitable measure of noncompactness on the space of solutions, we prove the existence of a compact set containing decay mild solutions to the mentioned problem.

Keywords: decay mild solution, evolution equations with memory, nonlocal condition, measure of noncompactness.

2010 Mathematics Subject Classification: 34K20, 34K30, 45K05, 47H08.

1 Introduction

Let X be a Hilbert space, A an unbounded, selfadjoint, positive definite operator in X and let $\beta \in L_1(\mathbb{R}_+)$ be locally absolutely continuous in $(0,\infty)$, nonnegative, nonincreasing and such that $\int_0^\infty \beta(t)dt < 1$.

In this paper we consider the following nonlocal Cauchy problem:

$$u''(t) + Au(t) - \int_0^t \beta(t-s)Au(s)ds = f(t,u(t)), \qquad t > 0, \tag{1.1}$$

$$u(0) + g(u) = x_0, \qquad u'(0) + h(u) = y_0, \tag{1.2}$$

where $f : \mathbb{R}_+ \times X \to X$, $g, h : C(\mathbb{R}_+, X) \to X$, and $x_0, y_0 \in X$ are given data.

The above abstract model arises in several applied fields. For example, in viscolasticity, the operator $A = -\Delta, X = L_2(\Omega)$, Eq. (1.1) is a nonlinear wave equation with memory. When the problem is linear $(f, g, h \equiv 0)$, Eq. (1.1) can be rewritten as an integral equation. In this case, the theory developed by Prüss in [8] provides a general framework for the existence and uniqueness of solutions. In [9] Prüss considers the following problems:

$$\begin{cases} u''(t) + Au(t) - \int_0^t \beta(t-s)Au(s)ds = f(t), \quad t > 0, \\ u(0) = x_0, \quad u'(0) = y_0, \end{cases} \tag{1.3}$$

✉ Email: vutrongluong@gmail.com; luongvt@utb.edu.vn

and

$$\begin{cases} u''(t) + Au(t) - \int_0^t \beta(t-s)Au(s)ds = f(u(t), u'(t)), \quad t > 0, \\ u(0) = x_0, \quad u'(0) = y_0, \end{cases} \tag{1.4}$$

where $f : D(\sqrt{A}) \times X \to X$ is Lipschitz in a neighborhood of 0, with $f(0) = 0$ and a sufficiently small constant. With these problems, Prüss obtained stable properties of the solutions of (1.3) and (1.4), decay of polynomial or exponential type in particular.

Recently the Cauchy problem for Eq. (1.1) also has been studied in [2, 4] with the replacement of $f(t, u(t))$ by $\nabla F(u(t))$ or $\nabla F(u(t)) + g(t)$ and ∇F is Lipschitz in a neighborhood of 0. Motivated by the above work of [2, 4], this paper also deals with problem (1.1)–(1.2). Here, the nonlinear f is more general than the one in [2, 4] and the initial conditions are nonlocal. The concept of nonlocal initial conditions is introduced to extend the classical theory of initial value problems. This notion is more appropriate than the classical one in describing natural phenomena because it allows us to consider additional information (see, e.g., [5, 7, 11] and their references).

In this work, we will prove the existence of decay mild solutions for problem (1.1)–(1.2), basing on the fixed point theorem for condensing map for measure of noncompactness (MNC) in [6].

The rest of the paper is organized as follows. Section 2 introduces some useful preliminaries. In addition, we construct a regular MNC on $BC(\mathbb{R}_+; X)$ and give a fixed point principle. In Section 3, we prove the existence of mild solutions on $[0, T]$, $T > 0$, for problem (1.1)–(1.2). Section 4 is devoted to show the decay mild solutions. In the last section, we give an example to illustrate the abtract results obtained in the paper.

2 Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper. First, we consider the problem

$$\begin{aligned} &u''(t) + Au(t) - \int_0^t \beta(t-s)Au(s)ds = F(t), \qquad t > 0, \\ &u(0) + g(u) = x_0, \qquad u'(0) + h(u) = y_0, \end{aligned} \tag{2.1}$$

where $F : \mathbb{R}_+ \to X$ is continuous, and g, h, x_0, y_0 are given.

Definition 2.1 ([8]). A family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ of bounded linear operators in X is called a resolvent for (2.1) if the following conditions are satisfied.

(S1) $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;

(S2) $S(t)$ commutes with A, which means that $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;

(S3) the resolvent equation holds

$$S(t)x = x + \int_0^t a(t-s)AS(s)xds, \quad \text{for all } \ x \in D(A),\ t \geq 0.$$

Integrating (2.1) twice we obtain the equivalent problem

$$u(t) + (a * Au)(t) = [x_0 - g(u)] + t[y_0 - h(u)] + (t * F)(t), \qquad t \geq 0,$$

where

$$a(t) = t - t * \beta(t) = (1 - b_0)t + (1 * b)(t), b(t) = \int_t^\infty \beta(s)ds, \qquad t \geq 0,\ b_0 = b(0),$$

and the star indicates the convolution. By a similar argument as in [8, p. 160], we get the mild solution given by the formula

$$u(t) = S(t)[x_0 - g(u)] + R(t)[y_0 - h(u)] + (R * F)(t), \qquad t \geq 0, \tag{2.2}$$

where $S(t)$ is the resolvent of (2.1) and $R(t) = \int_0^t S(s)ds$ is its integral. Moreover, since $\beta(t)$ is real and A is selfadjoint, we obtain $S(t)$ and $R(t)$ are selfadjoint as well (cf. [8, Corollary 2.1]).

The following results are direct consequences of [9, Proposition 2.1 and Theorem 3.1].

Proposition 2.2. *Let A be a selfadjoint, positive definite operator in the Hilbert X, and let $\beta \in L_1(\mathbb{R}_+)$ a locally absolutely continuous, nonnegative, nonincreasing map such that $b_0 = \int_0^\infty \beta(t)dt < 1$. Then the resolvent $S(t)$ and its integral $R(t)$, satisfy*

(i) $\|S(t)\| \leq 1,\ \|A^{1/2}R(t)\| \leq \frac{1}{\sqrt{1+b_0}}, \quad t \geq 0,$

(ii) *$S(t), A^{1/2}R(t)$ are strongly integrable, and converge strongly to 0 as $t \to \infty$.*

A typical example of kernel considered in [9] is as follows:

$$\beta(t) = k_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-\gamma t}, \qquad t > 0,$$

where $\gamma > 0,\ \alpha \in (0, 1)$ and $0 < k_0 < \gamma^\alpha$.

Next, we recall the knowledge of the measure of noncompactness in Banach spaces. Among them the Hausdorff measure of noncompactness is important. Next, we mention the condensing map and the fixed point principle for condensing maps. We denote the collection of all nonempty bounded subsets in X by B_X, and the norm of space $C([0, T]; X)$ by $\|\cdot\|_C$, with $\|u\|_C = \sup_{t\in[0,T]} \|u(t)\|_X$.

Definition 2.3. A function $\Phi : B_X \longrightarrow [0, +\infty)$ is called a measure of noncompactness (MNC) in X if

$$\Phi(\overline{\text{co}}\,\Omega) = \Phi(\Omega), \qquad \forall \Omega \in B_X,$$

where $\overline{\text{co}}\,\Omega$ is the closure of the convex hull of Ω. An MNC Φ in X is called

(i) monotone if for $\forall\, \Omega_1, \Omega_2 \in B_X,\ \Omega_1 \subset \Omega_2$ implies $\Phi(\Omega_1) \leq \Phi(\Omega_2)$;

(ii) nonsingular if $\Phi(\{x\} \cup \Omega) = \Phi(\Omega)$ for $\forall\, x \in X,\ \forall\, \Omega \in B_X$;

(iii) invariant with respect to union with compact set if $\Phi(K \cup \Omega) = \Phi(\Omega)$ for every relatively compact $K \subset X$ and $\Omega \in B_X$;

(iv) algebraically semi-additive if $\Phi(\Omega_1 + \Omega_2) \leq \Phi(\Omega_1) + \Phi(\Omega_2)$ for any $\Omega_1, \Omega_2 \in B_X$;

(v) regular if $\Phi(\Omega) = 0$ is equivalent to the relative compactness of Ω.

An important example of MNC is the Hausdorff MNC $\chi(\cdot)$ which is defined as follows

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\} \tag{2.3}$$

for $\forall\, \Omega \in B_X$.

For $T > 0$, let χ_T be the Hausdorff MNC in $C([0, T]; X)$. We recall the following facts (see [6]): for each bounded $D \subset C([0, T]; X)$, we have

- $\chi(D(t)) \leq \chi_T(D)$ for all $t \in [0,T]$, where $D(t) = \{x(t) : x \in D\}$.
- If D is an equicontinuous set on $[0,T]$, then

$$\chi_T(D) = \sup_{t\in[0,T]} \chi(D(t)).$$

Consider the space $BC(\mathbb{R}_+; X)$ of bounded continuous functions on $\mathbb{R}_+$ taking values on X. Denote by π_T the restriction operator on $[0,T]$, $\pi_T(u)$ is the restriction of u on $[0,T]$. Then

$$\chi_\infty(D) = \sup_{T>0} \chi_T\big(\pi_T(D)\big), \qquad D \subset BC(\mathbb{R}_+; X), \tag{2.4}$$

is an MNC. We give some measures of noncompactness as follows

$$d_T(D) = \sup_{u\in D}\sup_{t\geq T} \|u(t)\|_X, \tag{2.5}$$

$$d_\infty(D) = \lim_{T\to\infty} d_T(D), \tag{2.6}$$

$$\chi^*(D) = \chi_\infty(D) + d_\infty(D). \tag{2.7}$$

The regularity of MNC χ^* is proved in [3, Lemma 2.6].

In the sequel, we need some basic MNC estimates. Recall that one can define the sequential MNC χ_0 as follows:

$$\chi_0(\Omega) = \sup\{\chi(D) : D \in \Delta(\Omega)\},$$

where $\Delta(\Omega)$ is the collection of all at-most-countable subsets of Ω (see [1]). We know that

$$\frac{1}{2}\chi(\Omega) \leq \chi_0(\Omega) \leq \chi(\Omega),$$

for all bounded set $\Omega \subset X$. Then the following property is evident.

Proposition 2.4. *Let χ be the Hausdorff MNC on Banach space $X, \Omega \in B_X$. Then there exists a sequence $\{x_n\}_{n=1}^\infty \subset \Omega$ such that*

$$\chi(\Omega) \leq 2\chi\big(\{x_n\}_{n=1}^\infty\big) + \varepsilon, \qquad \forall\varepsilon > 0. \tag{2.8}$$

We have the following estimate whose proof can be found in [6].

Proposition 2.5 **([6]).** *Let χ be the Hausdorff MNC on Banach space X, sequence $\{u_n\}_{n=1}^\infty \subset L_1(0,T;X)$ such that $\|u_n(t)\|_X \leq v(t)$, for every $n \in \mathbb{N}^*$ and a.e. $t \in [0,T]$, for some $v \in L_1(0,T)$. Then we have*

$$\chi\left(\left\{\int_0^t u_n(s)dx\right\}\right) \leq 2\int_0^t \chi\big(\{u_n(t)\}\big)ds, \tag{2.9}$$

for $t \in [0,T]$.

To end this section, we recall a fixed point principle for condensing maps that will be used in the next sections.

Definition 2.6. Let X be a Banach space, χ is an MNC on X, and $\emptyset \neq D \subset X$. A continuous map $\Phi : D \longrightarrow X$ is said to be condensing with respect to χ (χ-condensing) if for all $\Omega \in B_D$, the relation

$$\chi(\Omega) \leq \chi(\Phi(\Omega))$$

implies the relative compactness of Ω.

Theorem 2.7 ([6]). *Let X be a Banach space, χ is an MNC on X, D is a bounded convex closed subset of X and let $\Phi : D \longrightarrow D$ be a χ-condensing map. Then the fixed point set of Φ*

$$\text{Fix}(\Phi) = \{x \in D : x = \Phi(x)\}$$

is a nonempty compact set.

3 Existence result

It should be noted that $X_{\frac{1}{2}} = D(\sqrt{A})$ is a Hilbert space equipped with the scalar product $(x, y)_{\frac{1}{2}} = \big(\sqrt{A}x, \sqrt{A}y\big)$, $x, y \in X_{\frac{1}{2}}$, where $(\cdot, \cdot)$ is a inner product in X. Denote $\|\cdot\|_{1/2} := \|\cdot\|_{X_{\frac{1}{2}}}$, $\|x\|_C := \sup_{t\in[0,T]} \|x(t)\|_{1/2}$, $x \in C([0, T]; X_{\frac{1}{2}})$. Let χ and χ_T be Hausdorff MNC on $X_{\frac{1}{2}}$ and $C([0, T], X_{\frac{1}{2}})$, respectively.

In formulation of problem (1.1)–(1.2), we assume that

(G) The function $g : C\big([0, T]; X_{\frac{1}{2}}\big) \longrightarrow X_{\frac{1}{2}}$ obeys the following conditions:

(i) g is continuous and

$$\|g(u)\|_{1/2} \leq \theta_g(\|u\|_C), \tag{3.1}$$

for all $u \in C([0, T]; X_{\frac{1}{2}})$, where $\theta_g : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing.

(ii) There exist non-negative constants η_g such that

$$\chi\big(g(\Omega)\big) \leq \eta_g \chi_T(\Omega), \tag{3.2}$$

for all bounded set $\Omega \subset C([0, T]; X_{\frac{1}{2}})$.

(H) The function $h : C([0, T]; X_{\frac{1}{2}}) \longrightarrow X$ satisfies the following conditions:

(i) h is continuous and

$$\|h(u)\|_X \leq \theta_h(\|u\|_C), \tag{3.3}$$

for all $u \in C([0, T]; X_{\frac{1}{2}})$ where $\theta_h : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing function.

(ii) There exists a function $\eta_h \in L_1(0, T)$ such that for all bounded set $\Omega \subset C([0, T]; X_{\frac{1}{2}})$,

$$\chi\big(R(t)h(\Omega)\big) \leq \eta_h(t)\chi_T(\Omega), \tag{3.4}$$

for a.e. $t \in [0, T]$.

(F) The nonlinear function $f : \mathbb{R}_+ \times X_{\frac{1}{2}} \longrightarrow X$ satisfies:

(i) $f\big(\cdot, u(\cdot)\big)$ is measurable for each $u(\cdot) \in X_{\frac{1}{2}}$, $f(t, \cdot)$ is continuous for a.e. $t \in [0, T]$, and

$$\|f\big(t, v\big)\|_X \leq m(t)\theta_f(\|v\|_{1/2}), \tag{3.5}$$

for all $v \in X_{\frac{1}{2}}$ where $m \in L_1(0, T)$, $\theta_f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing function.

(ii) There exists $\eta_f : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ such that $\eta_f(t,\cdot) \in L_1(0,T)$ and for all bounded set $\Omega \subset X_{\frac{1}{2}}$,

$$\chi\big(R(t-s)f(s,\Omega)\big) \le \eta_f(t,s)\chi(\Omega), \tag{3.6}$$

for a.e. $t,s \in [0,T], s \le t$.

Remark 3.1. Let us give some comments on the assumptions **(G)**(ii), **(H)**(ii) and **(F)**(ii).

1. If g,h are Lipschitz, then (3.2) and (3.4) are satisfied. These conditions are also satisfied with $\eta_g = \eta_h = 0$ if g,h are completely continuous.

2. If $f(t,\cdot)$ satisfies the Lipschitzian condition respect to the second variable, i.e.,

$$\|f(t,v_1) - f(t,v_2)\| \le k_f(t)\|v_1 - v_2\|_{1/2}, \qquad v_1, v_2 \in X_{1/2},$$

for some $k_f \in L_1(0,T)$, then (3.6) is satisfied. In fact, we have

$$\begin{aligned}\|R(t-s)(f(s,v_1) - f(s,v_2))\|_{1/2} &= \|A^{1/2}R(t-s)(f(s,v_1) - f(s,v_2))\| \\ &\le \frac{k_f(s)}{\sqrt{1+b_0}}\|v_1 - v_2\|_{1/2}.\end{aligned}$$

It implies that for all bounded set $\Omega \subset X_{\frac{1}{2}}$,

$$\chi\big(R(t-s)f(s,\Omega)\big) \le \frac{k_f(s)}{\sqrt{1+b_0}}\chi(\Omega),$$

for a.e. $t,s \in [0,T],\ s \le t$.
Furthermore, if $f(t,\cdot)$ is completely continuous (for each fixed t), then (3.6) is obviously fulfilled with $\eta_f = 0$.

Let $x_0 \in X_{\frac{1}{2}}, y_0 \in X$. Motivated by (2.2), we say that a function $u \in C(\mathbb{R}_+, D(\sqrt{A}))$ is a *mild solution* of problem (1.1)–(1.2) on $[0,T], T > 0$, if it satisfies the integral equation

$$u(t) = S(t)[x_0 - g(u)] + R(t)[y_0 - h(u)] + \int_0^t R(t-\tau)f(\tau,u(\tau))d\tau, \qquad \forall t \in [0,T]. \tag{3.7}$$

Here $S(t)$ is the resolvent for the linear equation

$$u''(t) + Au(t) - \int_0^t \beta(t-s)Au(s)ds = 0, \qquad t > 0,$$

and $R(t) = \int_0^t S(\tau)d\tau$ its integral.
We denote

$$\mathcal{M} = \Big\{u \in C\Big([0,T]; X_{\frac{1}{2}}\Big) : \|u\|_C \le R\Big\},$$

where $R > 0$ given. We conclude that $\mathcal{M}$ is a bounded convex closed subset of $C([0,T]; X_{\frac{1}{2}})$. For each $u \in \mathcal{M}$, we define the solution operator $\Phi : \mathcal{M} \to C([0,T]; X_{\frac{1}{2}})$ as follows:

$$\Phi(u)(t) = S(t)[x_0 - g(u)] + R(t)[y_0 - h(u)] + \int_0^t R(t-\tau)f(\tau,u(\tau))d\tau. \tag{3.8}$$

Then u is a mild solution of problem (1.1)–(1.2) if it is a fixed point of solution operator Φ. Thanks to the assumptions imposed of g,h,f, then Φ is continuous on $\mathcal{M}$.

Lemma 3.2. *Let the assumptions of Proposition 2.2 and the hypothesis* **(F)***(i) be satisfied, then*

$$Qu(t) = \int_0^t R(t-\tau)f(\tau, u(\tau))d\tau, \qquad u \in \mathcal{M},$$

is equicontinuous on $[0, T]$.

Proof. For $u \in \mathcal{M}$, $0 \le t \le s \le T$, we have

$$\begin{aligned}
\|Qu(t) - Qu(s)\|_{\frac{1}{2}} &= \left\| \int_0^t A^{1/2}R(t-\tau)f(\tau, u(\tau))d\tau - \int_0^s A^{1/2}R(s-\tau)f(\tau, u(\tau))d\tau \right\| \\
&\le \left\| \int_0^t A^{1/2}(R(t-\tau) - R(s-\tau))f(\tau, u(\tau))d\tau \right\| \\
&\quad + \left\| \int_t^s A^{1/2}R(s-\tau)f(\tau, u(\tau))d\tau \right\| \\
&\le \theta(R) \int_0^t \left\| A^{1/2}(R(t-\tau) - R(s-\tau)) \right\| m(\tau)d\tau \\
&\quad + \theta(R)/\sqrt{1+b_0} \int_t^s m(\tau)d\tau \\
&\to 0 \quad \text{as } |t-s| \to 0,
\end{aligned}$$

by the strong continuity of $A^{1/2}R(t),\ t \ge 0$ and $m \in L_1(0, T)$. Therefore, $\{Qu : u \in \mathcal{M}\}$ is equicontinuous on $[0, T]$. □

Lemma 3.3. *Let the assumptions of Proposition 2.2 and the hypothesis* **(G)***(i),* **(H)***(i),* **(F)***(i) be satisfied. Then there exists* $R > 0$ *such that* $\Phi(\mathcal{M}) \subset \mathcal{M}$*, provided that*

$$\lim_{n\to\infty} \frac{1}{n}\left[\theta_g(n) + \frac{1}{\sqrt{1+b_0}}(\theta_h(n) + \|m\|\theta_f(n))\right] < 1. \tag{3.9}$$

Proof. Assuming to the contrary that for each $n \in \mathbb{N}$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{M}$ with $\|u_n\|_C \le n$ and $\|\Phi(u_n)\|_C > n$. From the formulation of Φ, we have

$$\begin{aligned}
\|\Phi(u_n)(t)\|_{1/2} &\le \|A^{1/2}S(t)[x_0 - g(u_n)]\| + \|A^{1/2}R(t)[y_0 - h(u_n)]\| \\
&\quad + \left\| A^{1/2} \int_0^t R(t-\tau)f(\tau, u_n(\tau))d\tau \right\| \\
&\le \|x_0\|_{1/2} + \theta_g(n) + \frac{\|y_0\|}{\sqrt{1+b_0}} + \frac{\theta_h(n)}{\sqrt{1+b_0}} + \frac{\|m\|\theta_f(n)}{\sqrt{1+b_0}}, \qquad \forall t \in [0, T].
\end{aligned}$$

The above inequality leads to

$$n < \|\Phi(u_n)\|_C \le \|x_0\|_{1/2} + \theta_g(n) + \frac{\|y_0\|}{\sqrt{1+b_0}} + \frac{\theta_h(n)}{\sqrt{1+b_0}} + \frac{\|m\|\theta_f(n)}{\sqrt{1+b_0}}.$$

Therefore,

$$1 < \frac{1}{n}\left[|x_0\|_{1/2} + \theta_g(n) + \frac{\|y_0\|}{\sqrt{1+b_0}} + \frac{\theta_h(n)}{\sqrt{1+b_0}} + \frac{\|m\|\theta_f(n)}{\sqrt{1+b_0}}\right].$$

Passing to the limit as $n \to \infty$, we get a contradiction to (3.9). □

In order to apply the fixed point theory for condensing maps, we will establish the so-called MNC-estimate for the solution operator Φ.

Lemma 3.4. *Let the assumptions of Proposition 2.2 and the hypothesis* **(G)***(ii),* **(H)***(ii),* **(F)***(ii) be satisfied, then*

$$\chi_T\big(\Phi(D)\big) \le \big[\eta_g + \sup_{t\in[0,T]}(\eta_h(t) + 4\|\eta_f(t,\cdot)\|)\big]\chi_T(D), \tag{3.10}$$

for all bounded sets $D \subset \mathcal{M}$, here $\|\eta_f\| = \|\eta_f\|_{L_1(0,T)}$.

Proof. Setting

$$\begin{aligned}
\Phi_1(u)(t) &= S(t)[x_0 - g(u)],\\
\Phi_2(u)(t) &= R(t)[y_0 - h(u)],\\
\Phi_3(u)(t) &= \int_0^t R(t-\tau)f(\tau,u(\tau))d\tau.
\end{aligned}$$

We have

$$\chi_T\big(\Phi(D)\big) \le \chi_T\big(\Phi_1(D)\big) + \chi_T\big(\Phi_2(D)\big) + \chi_T\big(\Phi_3(D)\big). \tag{3.11}$$

1. For every $z_1, z_2 \in \Phi_1(D)$, there exist $u_1, u_2 \in D$ such that for $t \in [0,T]$,

$$z_i(t) = \Phi_1(u_i)(t), \qquad (i = 1,2).$$

We have

$$\|z_1(t) - z_2(t)\|_{1/2} \le \|S(t)\|\|A^{1/2}(g(u_1) - g(u_2))\| \le \|g(u_1) - g(u_2)\|_{1/2}.$$

It implies that

$$\|z_1 - z_2\|_C \le \|g(u_2) - g(u_1)\|_{1/2}.$$

Hence,

$$\chi_T\big(\Phi_1(D)\big) \le \chi\big(g(D)\big) \le \eta_g\chi_T(D). \tag{3.12}$$

2. By similar arguments as above, we get

$$\chi_T\big(\Phi_2(D)\big) \le \sup_{t\in[0,T]} \eta_h(t)\chi_T(D). \tag{3.13}$$

3. Applying Proposition 2.4, there exists $\{u_n\}_{n=1}^\infty \subset D$ such that for every $\varepsilon > 0$, we obtain

$$\chi_T\big(\Phi_3(D)\big) \le 2\chi_T\big(\{\Phi_3(u_n)\}_{n=1}^\infty\big) + \varepsilon. \tag{3.14}$$

We invoke Proposition 2.5 to deduce that

$$\begin{aligned}
\chi_T\big(\{\Phi_3(u_n)\}\big) &= \sup_{t\in[0,T]} \chi\big(\{\Phi_3(u_n(t))\}\big)\\
&\le 2\sup_{t\in[0,T]} \int_0^t \chi\left(R(t-\tau)f(\tau,u_n(\tau))\right)d\tau\\
&\le 2\sup_{t\in[0,T]} \int_0^t \eta_f(t,\tau)\chi(u_n(\tau))\,d\tau.
\end{aligned}$$

It is inferred that

$$\chi_T\big(\{\Phi_3(u_n)\}\big) \le 2\sup_{t\in[0,T]} \|\eta(t,\cdot)\|\chi_T(u_n) \tag{3.15}$$

From (3.14) and (3.15), we obtain

$$\chi_T\big(\Phi_3(D)\big) \le 4 \sup_{t\in[0,T]} \|\eta(t,\cdot)\|\chi_T(D). \tag{3.16}$$

Combining (3.11), (3.12) and (3.16) yields

$$\chi_T\big(\Phi(D)\big) \le \big[\eta_g + \sup_{t\in[0,T]}(\eta_h(t) + 4\|\eta_f(t,.)\|)\big]\chi_T(D). \tag{3.17}$$

The proof is completed. □

Theorem 3.5. *Let the assumptions of Proposition 2.2 and the hypothesis* **(G)**, **(H)**, **(F)** *be satisfied. Then problem (1.1)–(1.2) has at least one mild solution on* $[0,T]$, *provided that*

$$\lim_{n\to\infty} \frac{1}{n}\left[\theta_g(n) + \frac{1}{\sqrt{1+b_0}}(\theta_h(n) + \|m\|\theta_f(n))\right] < 1, \tag{3.18}$$

$$l := \eta_g + \sup_{t\in[0,T]} (\eta_h(t) + 4\|\eta_f(t,\cdot)\|) < 1. \tag{3.19}$$

Proof. By the inequality (3.19), the solution operator Φ is χ_T-condensing. Indeed, if $D \subset \mathcal{M}$ is a bounded set such that $\chi_T(D) \le \chi_T\big(\Phi(D)\big)$, applying Lemma 3.4, we obtain

$$\chi_T(D) \le \chi_T\big(\Phi(D)\big) \le l\chi_T(D).$$

Therefore $\chi_T(D) = 0$, and D is relatively compact.

By assumption (3.18), applying Lemma 3.3, we have $\Phi(\mathcal{M}) \subset \mathcal{M}$. Using Theorem 2.7, the χ_T-condensing map Φ defined by (3.8) has a nonempty and compact fixed point set $\mathrm{Fix}(\Phi) \subset \mathcal{M}$. It implies that the problem (1.1)–(1.2) has at least a mild solution $u(t),\ t \in [0,T]$ described by (3.7). □

4 Existence of decay mild solutions

In this section, we consider solution operator Φ on the following space:

$$BC(\mathbb{R}_+; X_{\frac{1}{2}}) = \left\{u \in C\Big(\mathbb{R}_+; X_{\frac{1}{2}}\Big) : \sup_{t\in\mathbb{R}^+} \|u(t)\|_{X_{\frac{1}{2}}} < \infty\right\},$$

with the supremum norm

$$\|u\|_{BC} = \sup_{t\in\mathbb{R}^+} \|u(t)\|_{X_{\frac{1}{2}}}$$

and its closed subspace

$$\mathcal{M}_\infty = \left\{u \in BC\Big(\mathbb{R}_+; X_{\frac{1}{2}}\Big) :\ u(t) \to 0 \text{ as } t \to +\infty\right\} \subset C_0\Big(\mathbb{R}_+; X_{\frac{1}{2}}\Big).$$

We are going to prove $\Phi(\mathcal{M}_\infty) \subset \mathcal{M}_\infty$ and using the MNC χ^* defined by (2.7) to prove that Φ is a χ^*-condensing map on $\mathcal{M}_\infty$. In the hypothesis **(G), (H), (F)**, we consider the conditions of g, h, f for any $T > 0$. The norm $\|\cdot\|_C$ is replaced by the norm $\|\cdot\|_{BC}$. The conditions $m, \eta_f \in L_1(0,T)$ are replaced by the $m, \eta_f \in L_1(\mathbb{R}_+)$.

We recall from [9, Lemma 6.1] the following result.

Lemma 4.1. *Let X, Y be Banach spaces, and let $\{U(t)\}_{t\geq 0} \subset \mathcal{B}(X,Y)$ be a strongly continuous operator family which is such that $U(\cdot)$ as well as its adjoint operator $U^*(\cdot)$ are strongly integrable. Then the convolution T defined by*

$$(Tf)(t) := \int_0^t U(t-\tau)f(\tau)d\tau, \qquad t \geq 0,$$

is well-defined and bounded from $L_p(\mathbb{R}_+; X)$ into $L_p(\mathbb{R}_+; Y)$, for each $p \in [1, \infty)$. T is also bounded from $C_0(\mathbb{R}_+; X)$ into $C_0(\mathbb{R}_+; Y)$, provided $U(t) \to 0$ strongly as $t \to \infty$.

We also definite the map $\Phi : \mathcal{M}_\infty \to C_0(\mathbb{R}_+, X_{1/2})$ by

$$\Phi(u)(t) = S(t)[x_0 - g(u)] + R(t)[y_0 - h(u)] + \int_0^t R(t-\tau)f(\tau, u(\tau))d\tau, \qquad t \geq 0,\ u \in \mathcal{M}_\infty.$$

Since $R(t)$ is selfadjoint in X, by Proposition 2.2 and Lemma 4.1, Φ is well-defined.

Lemma 4.2. *Let A be a selfadjoint, positive definite operator in the Hilbert space X and let $\beta \in L_1(\mathbb{R}_+)$ be a locally absolutely continuous, nonnegative, nonincreasing map such that $b_0 = \int_0^\infty \beta(t)dt < 1$. Furthermore, the hypotheses* **(G), (H), (F)** *are satisfied. Then we have $\Phi(\mathcal{M}_\infty) \subset \mathcal{M}_\infty$.*

Proof. Let $u \in \mathcal{M}_\infty$ with $\|u\|_\infty = R < \infty$. For every $\varepsilon > 0$, there exists $T > 0$ such that for any $t > T$, we get

$$\|u(t)\| < \varepsilon, \qquad \|S(t)\| < \varepsilon, \qquad \|A^{1/2}R(t)\| < \varepsilon, \qquad \left\|\int_0^t A^{1/2}R(t-\tau)f(\tau, u(\tau))d\tau\right\| < \varepsilon.$$

We find that for every $t \in \mathbb{R}_+$

$$\begin{aligned}\|\Phi(u)(t)\|_{1/2} &\leq \|A^{1/2}S(t)[x_0 - g(u)]\| + \|A^{1/2}R(t)[y_0 - h(u)]\| \\ &\quad + \left\|\int_0^t A^{1/2}R(t-\tau)f(\tau, u(\tau))d\tau\right\| \\ &=: P + Q + K. \end{aligned} \tag{4.1}$$

Then for any $t > T$, we have

$$P < \varepsilon\big(\|x_0\|_{1/2} + \theta_g(R)\big), \qquad Q < \varepsilon\big(\|y_0\| + \theta_h(R)\big), \qquad K < \varepsilon. \tag{4.2}$$

From (4.1), (4.2), we obtain $\Phi(u)(t) \to 0$ as $t \to \infty$. The proof is completed. □

Lemma 4.3. *Let the assumptions of Lemma 4.2 be satisfied. Then we have*

$$\chi^*\big(\Phi(D)\big) \leq \big[\eta_g + \sup_{t\in\mathbb{R}_+}(\eta_h(t) + 4\|\eta_f(t,.)\|)\big]\chi^*(D), \tag{4.3}$$

for all bounded sets $D \subset M_\infty$.

Proof. Let $D \subset M_\infty$ be a bounded set. We have

$$\chi^*\big(\Phi(D)\big) = \chi_\infty\big(\Phi(D)\big) + d_\infty\big(\Phi(D)\big). \tag{4.4}$$

1. Thanks to the Lemma 3.4, we obtain the following estimates:

$$\chi_\infty(\Phi(D)) \leq \chi_\infty(\Phi_1(D)) + \chi_\infty(\Phi_2(D)) + \chi_\infty(\Phi_3(D)), \tag{4.5}$$

$$\chi_\infty(\Phi_1(D)) \leq \eta_g \chi_\infty(D), \tag{4.6}$$

$$\chi_\infty(\Phi_2(D)) \leq \sup_{t\in\mathbb{R}_+} \eta_h(t)\chi_\infty(D), \tag{4.7}$$

and

$$\chi_\infty(\Phi_3(D)) \leq 4 \sup_{t\in\mathbb{R}_+} \|\eta_f(t,.)\| \chi_\infty(D). \tag{4.8}$$

From (4.5)–(4.8), we have

$$\chi_\infty(\Phi(D)) \leq \left[\eta_g + \sup_{t\in\mathbb{R}_+}(\eta_h(t) + 4\|\eta_f(t,.)\|)\right]\chi_\infty(D) \tag{4.9}$$

2. Next, we find that

$$d_\infty(\Phi(D)) = \lim_{T\to\infty} d_T(\Phi(D)), d_T(\Phi(D)) = \sup_{u\in D}\sup_{t\geq T} \|\Phi(u)(t)\|_{1/2}.$$

Applying Lemma 4.2, we obtain

$$d_\infty(\Phi(D)) = 0. \tag{4.10}$$

From (4.4), (4.9) and (4.10), we complete the proof of Lemma 4.3. □

Theorem 4.4. *Let the assumptions of Lemma 4.2 be satisfied. Then problem (1.1)–(1.2) has at least one mild solution $u \in \mathcal{M}_\infty$ provided that*

$$l_\infty := \eta_g + \sup_{t\in\mathbb{R}_+}(\eta_h(t) + 4\|\eta_f(t,\cdot)\|) < 1, \tag{4.11}$$

$$\lim_{n\to\infty} \frac{1}{n}\left[\theta_g(n) + \frac{1}{\sqrt{1+b_0}}(\theta_h(n) + \|m\|\theta_f(n))\right] < 1. \tag{4.12}$$

Proof. By the inequality (4.11), the solution operator Φ is a χ^*-condensing. Indeed, if $D \subset \mathcal{M}_\infty$ is a bounded set such that $\chi^*(D) \leq \chi^*(\Phi(D))$. Applying Lemma 4.3, we obtain

$$\chi^*(D) \leq \chi^*(\Phi(D)) \leq l_\infty \chi^*(D).$$

Therefore, $\chi^*(D) = 0$, and so D is relatively compact. On the other hand, by condition (4.12) and the arguments in the proof of Lemma 3.3, one can find $R > 0$ such that $\Phi(B_R) \subset B_R$ where B_R is the ball in $\mathcal{M}_\infty$ with center at origin and radius R. Applying Theorem 2.7, the χ^*-condensing map Φ defined by (3.8) has a nonempty and compact fixed point set $\mathrm{Fix}(\Phi) \subset \mathcal{M}_\infty$. Hence, the problem (1.1)–(1.2) has at least a mild solution $u(t)$, $t \in \mathbb{R}_+$ described by (3.7) which satisfies $\lim_{t\to\infty} u(t) = 0$. □

5 An example

Let Ω be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$ is smooth enough. Considering the operator L given by $Lu(x) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}u(x)\right)$, $x \in \Omega$, where $a_{ij} : \overline{\Omega} \to \mathbb{R}$, and $a_{ij} = a_{ji} \in C^1(\Omega)$. Moreover, there exists a constant $c > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2, \qquad \forall \xi \in \mathbb{R}^n,\ x \in \Omega.$$

Let $\beta \in L_1(\mathbb{R}_+)$ be the scalar memory kernel in previous section and let $x_0 \in H_0^1(\Omega)$, $y_0 \in L_2(\Omega)$. We consider the following problem:

$$\begin{cases} u_{tt}(t,x) - Lu(t,x) + \int_0^t \beta(t-s)Lu(s,x)ds = F(t,x,u(t,x)), \\ u(0,x) + \int_\Omega k(x,y)u(0,y)dy = x_0(x), \quad x \in \Omega, \\ u_t(0,x) + \sum_{i=1}^p c_i u(t_i,x) = y_0(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega, \ t > 0. \end{cases} \tag{5.1}$$

Here, $0 \le t_1 < t_2 < \cdots < t_p < +\infty$, c_i are positive constants and the function $k: \ \Omega \times \Omega \longrightarrow \mathbb{R}$ is such that $k \in L_2(\Omega \times \Omega), k(\cdot, y) \in X_{1/2}, y \in \Omega$, and

$$\sup_{y\in\Omega} \int_\Omega |k(x,y)|^2 + |\nabla_x k(x,y)|^2 dx = C < +\infty.$$

Let $X = L^2(\Omega), A = -L$ with $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. It is well known that A is a selfadjoint, positive definite operator in X. Moreover, the fractional power $\sqrt{A}$ of A is well defined and $X_{1/2} = D(\sqrt{A}) = H_0^1(\Omega)$. Then problem (5.1) is in the form of the abstract model (1.1)–(1.2) with

$$f(t,u(t))(x) = F(t,x,u(t,x)) \quad \text{(will be defined later), and}$$

$$g(u)(x) = \int_\Omega k(x,y)u(0,y)dy, \qquad h(u)(x) = \sum_{i=1}^p c_i u(t_i,x).$$

Now we give the description for the functions g, h and f.

(G) $g : BC(\mathbb{R}_+, X_{1/2}) \to X_{1/2}$ is continuous.

(i) $\|g(u)\|_X \le \sqrt{C}\|u(0)\|_X \le \sqrt{C}\|u(0)\|_{X_{1/2}} \le \sqrt{C}\|u\|_{BC}$.

(ii) By Theorem 8.83 in [10], g is a compact operator, so $\chi\big(g(D)\big) = 0$, for all bounded sets $D \subset BC(\mathbb{R}_+; X_{1/2})$.

Therefore, g fulfills **(G)** with $\theta_g(t) = t\sqrt{C}, t \ge 0$, and $\eta_g = 0$.

(H) $h : BC(\mathbb{R}_+, X_{1/2}) \to X$ is continuous.

(i)

$$\begin{aligned} \|h(u)\|_X = \left\| \sum_{i=1}^p c_i u(t_i,x) \right\|_X &\le \sum_{i=1}^p c_i \|u(t_i,x)\|_{X_{1/2}} \\ &\le \sum_{i=1}^p c_i \|u\|_{BC}, \ \forall u \in BC(\mathbb{R}_+; X_{1/2}), \end{aligned}$$

(ii) Next, for every $u_1, u_2 \in BC(\mathbb{R}_+, X_{1/2})$, we get

$$\begin{aligned} \|R(t)(h(u_1) - h(u_2))\|_{1/2} &= \left\| \sum_{i=1}^p c_i A^{1/2} R(t) \big[u_1(t_i,x) - u_2(t_i,x)\big] \right\|_X \\ &\le \frac{1}{\sqrt{1+b_0}} \sum_{i=1}^p c_i \|u_1(t_i,x) - u_2(t_i,x)\|_{X_{1/2}} \\ &\le \frac{1}{\sqrt{1+b_0}} \sum_{i=1}^p c_i \|u_1 - u_2\|_{BC}. \end{aligned}$$

Therefore

$$\chi\big(R(t)h(D)\big) \leq \frac{1}{\sqrt{1+b_0}} \sum_{i=1}^{p} c_i \chi_\infty(D),$$

for every bounded set $D \subset BC(\mathbb{R}^+; X_{1/2})$.

Hence, h fulfills **(H)** with $\theta_h(t) = t\sum_{i=1}^{p} c_i, t \geq 0$ and $\eta_h = \frac{1}{\sqrt{1+b_0}} \sum_{i=1}^{p} c_i$.

(F) $F_1 : \mathbb{R}_+ \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}, \mu : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, and $F_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

(a) F_1 is a continuous function such that $F_1(t, x, 0) = 0$ and

$$|F_1(t, x, z_1) - F_1(t, x, z_2)| \leq m_1(t)|z_1 - z_2|$$

for all $x \in \Omega, z_1, z_2 \in \mathbb{R}$, here $m_1 \in L_2(\mathbb{R}_+)$;

(b) $\mu \in BC(\mathbb{R}_+, L_2(\Omega))$;

(c) F_1 is a continuous function and $|F_2(x, z)| \leq l(x)|z|$ for $l \in L_2(\Omega)$;

Let $f : \mathbb{R}_+ \times X_{1/2} \to X$ such that

$$f(t, v)(x) = f_1(t, v)(x) + f_2(t, v)(x)$$

with

$$\begin{aligned} f_1(t, v)(x) &= F_1(t, x, v(x)), \\ f_2(t, v)(x) &= \mu(t, x) \int_\Omega F_2(x, v(x)) dx. \end{aligned}$$

Considering f_1, we have

$$\|R(t-s)(f_1(s, v_1) - f_1(s, v_2))\|_{1/2} \leq \frac{m_1(s)}{\sqrt{1+b_0}} \|v_1 - v_2\|_X \leq \frac{m_1(s)}{\sqrt{1+b_0}} \|v_1 - v_2\|_{X_{1/2}}.$$

This implies

$$\chi(R(t-s)f_1(s, V)) \leq \frac{m_1(s)}{\sqrt{1+b_0}} \chi(V), \quad \text{for all bounded sets } V \subset X_{1/2}. \tag{5.2}$$

Regarding f_2, using Hölder inequality, we get

$$\begin{aligned} \|f_2(t, v)\|_X^2 &\leq \|\mu(t)\|_X^2 \left(\int_\Omega F_2(x, v(x)) dx \right)^2 \\ &\leq \|\mu(t)\|_X^2 \left(\int_\Omega |l(x)||v(x)| dx \right)^2 \\ &\leq \|\mu(t)\|_X^2 \|l\|_X^2 \|v\|_{1/2}^2. \end{aligned}$$

On the other hand, for any bounded set $V \subset X_{1/2}$, we see that

$$R(t-s)f_2(s, V) \subset \{\lambda R(t-s)\mu(s, \cdot) : \lambda \in \mathbb{R}\},$$

that is, $R(t-s)f_2(s, V)$ lies in an one dimensional subspace of $X_{1/2}$. Hence

$$\chi(R(t-s)f_2(s, V)) = 0, \tag{5.3}$$

for a.e. $t, s \in \mathbb{R}_+, s \leq t$. From (5.2) and (5.3), we obtain

$$\chi(R(t-s)f_2(s,V))) \leq \chi(R(t-s)f_2(s,V))) + \chi(R(t-s)f_2(s,V))) \leq m_1(s)\chi(V).$$

Thus f fulfills **(F)** with $m(t) = \max\{m_1(t), \|\mu(t)\|_X\|l\|_X\}, \theta_f(t) = t$ and

$$\eta_f(t,s) = \frac{m_1(s)}{\sqrt{1+b_0}}, \qquad s \geq 0.$$

Under the above settings, applying Theorem 4.4, one can state that problem (5.1) has at least one decay mild solution in $\mathcal{M}_\infty$, provided that

$$\left(\sum_{i=1}^{p} c_i + 4\|m_1\|\right) \Big/ \sqrt{1+b_0} < 1,$$

$$\sqrt{C} + \left(\sum_{i=1}^{p} c_i + \|m\|\right) \Big/ \sqrt{1+b_0} < 1.$$

References

[1] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina, B. N. Sadovskii, *Measures of noncompactness and condensing operators*, Birkhäuser, Boston–Basel–Berlin, 1992. MR1153247

[2] F. Alabau-Boussouira, P. Cannarsa, D. Sforza, Decay estimates for second order evolution equations with memory, *J. Funct. Anal.* **254**(2008), No. 5, 1342–1372. MR2386941; url

[3] N. T. Anh, T. D. Ke, Decay integral solutions for neutral fractional differential equations with infinite delays, *Math. Meth. Appl. Sci.* **38**(2015), No. 8, 1601–1622. MR3343575

[4] P. Cannarsa, D. Sforza, Integro-differential equations of hyperbolic type with positive definite kernels, *J. Differential Equations* **250**(2011), No. 12, 4289–4335. MR2793256

[5] E. M. Hernández, Existence of solutions to a second order partial differential equation with nonlocal conditions, *Electron. J. Differential Equations* **2003**, No. 51, 1–10. MR1971117

[6] M. Kamenskii, V. Obukhovskii, P. Zecca, Condensing multivalued maps and semilinear differential inclusions in Banach spaces, Walter de Gruyter & Co., Berlin, 2001. MR1831201

[7] N. M. Chuong, T. D. Ke, N. N.Quan, Stability for a class of fractional partial integro-differential equations, *J. Integral Equations Appl.* **26**(2014), No. 2, 145–170. MR3233516

[8] J. Prüss, *Evolutionary integral equations and applications*, Monographs in Mathematics, Vol. 87, Birkhäuser, 1993. MR1238939; url

[9] J. Prüss, Decay properties for the solutions of a partial differential equation with memory, *Arch. Math.* **92**(2009), No. 2, 158–173. MR2481511

[10] M. Renardy, R. S. Rogers, *Introduction to partial differential equations*, Springer-Verlag, New York, 2004 MR2028503

[11] S. Xie, Existence of solutions for nonlinear mixed type integro-differential functional evolution equations with nonlocal conditions, *Bound. Value Probl.* **2012**, 2012:100, 8 pp. MR3165987