Hartman – Wintner type theorem for PDE with $p-$Laplacian

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Abstract

The well known Hartman–Wintner oscillation criterion is extended to the PDE

$$\operatorname{div}(||\nabla u||^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0, \quad p > 1. \tag{E}$$

The condition on the function $c(x)$ under which (E) has no solution positive for large $||x||$, i.e. ∞ belongs to the closure of the set of zeros of every solution defined on the domain $\Omega = \{x \in \mathsf{R}^n : ||x|| > 1\}$, is derived.

Keywords. $p-$Laplacian, positive solution, Riccati equation.

1 Introduction

Let us consider the following partial differential equation with $p-$Laplacian

$$\operatorname{div}(||\nabla u||^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0 \tag{1.1}$$

where $p > 1$, $x = (x_1, x_2, \dots, x_n)$, $||\cdot||$ is the usual Euclidean norm in $\mathbb{R}^n$ and ∇ is the usual nabla operator. Define the sets $\Omega(a) = \{x \in \mathbb{R}^n : a \le ||x||\}$, $\Omega(a, b) = \{x \in \mathbb{R}^n : a \le ||x|| \le b\}$. The function $c(x)$ is assumed to be integrable on every compact subset of $\Omega(1)$. Under solution of the equation (1.1) we understand every absolutely continuous function $u : \Omega(1) \to \mathbb{R}$ such that $||\nabla u||^{p-2}\frac{\partial u}{\partial x_i}$ is absolutely continuous with respect to x_i and u satisfies the equation (1.1) almost everywhere on $\Omega(1)$.

Equation (1.1) appears for example in the study of non-Newtonian fluids, nonlinear elasticity and in glaciology. Special cases of the equation (1.1) are the linear Schrödinger equation

$$\Delta u + c(x)u = 0 \tag{1.2}$$

if $p = 2$, the half–linear ordinary differential equation

$$\left(|u'|^{p-2}u'\right)' + c(x)|u|^{p-2}u = 0 \qquad ' = \frac{\mathrm{d}}{\mathrm{d}x} \tag{1.3}$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

if $n = 1$, and the ordinary differential equation

$$u'' + c(x)u = 0 \tag{1.4}$$

if both $n = 1$ and $p = 2$ holds.

Remark that if $c(x)$ is radial function, i.e., $c(x) = \tilde{c}(||x||)$, then the equation for radial solution $u(x) = \tilde{u}(||x||)$ of the equation (1.1) becomes

$$\left(r^{n-1}|\tilde{u}'|^{p-2}\tilde{u}'\right)' + r^{n-1}\tilde{c}(r)|\tilde{u}|^{p-2}\tilde{u} = 0 \qquad ' = \frac{\mathrm{d}}{\mathrm{d}r}, \tag{1.5}$$

which can be transformed into the equation (1.3).

This paper is motivated by the papers [1, 4] and [5, 6], where the Riccati technique is used to establish oscillation criteria for the equation (1.3) and (1.2), respectively.

The well-known result from the theory of second order ODE is the following theorem.

Theorem (Hartman–Wintner). *If either*

$$-\infty < \liminf_{t\to\infty} \frac{1}{t}\int_1^t \int_1^s c(\xi)\,\mathrm{d}\xi\ \mathrm{d}s < \limsup_{t\to\infty} \frac{1}{t}\int_1^t \int_1^s c(\xi)\,\mathrm{d}\xi\ \mathrm{d}s \le \infty, \tag{1.6}$$

or

$$\lim_{t\to\infty} \frac{1}{t}\int_1^t \int_1^s c(\xi)\,\mathrm{d}\xi\ \mathrm{d}s = \infty, \tag{1.7}$$

then the equation (1.4) *is oscillatory.*

This theorem is proved using Riccati technique in [3, Chap. XI]. The aim of this paper is to extend this statement to the equation (1.1). Another statement of this type was proved in [2, Theorem 3.4] under additional condition $p \ge n+1$. Here we prove a similar criterion, without the restriction on p.

We use the following function $C(t)$:

$$C(t) = \frac{p-1}{t^{p-1}}\int_1^t s^{p-2}\int_{\Omega(1,s)} ||x||^{1-n}c(x)\,\mathrm{d}x\ \mathrm{d}s\,. \tag{1.8}$$

2 Main results

First we introduce main ideas from the Riccati technique.

Suppose that there exists a number $a \in \mathbb{R}^+$ and a solution u of (1.1) which is positive on $\Omega(a)$. The vector function $\boldsymbol{w} = \frac{||\nabla u||^{p-2}\nabla u}{|u|^{p-2}u}$ is defined on $\Omega(a)$ and solves the *Riccati type equation*

$$\operatorname{div}\boldsymbol{w} + c(x) + (p-1)||\boldsymbol{w}||^q = 0, \tag{2.1}$$

where q is the conjugate number to p (i.e. $\frac{1}{p} + \frac{1}{q} = 1$ holds). Really, direct computation shows

$$\begin{aligned}\operatorname{div} \boldsymbol{w} &= \operatorname{div}(||\nabla u||^{p-2}\nabla u)u^{1-p} - (p-1)||\nabla u||^{p-2}u^{-p}\langle \nabla u, \nabla u\rangle \\ &= -c(x) - (p-1)\frac{||\nabla u||^p}{u^p} \\ &= -c(x) - (p-1)||\boldsymbol{w}||^q.\end{aligned}$$

The following Lemma plays a crucial role in our consideration. It is a straighforward generalization of [3, Lemma 7.1, Chap XI.]

Lemma 2.1. *Let* $\boldsymbol{w}$ *be the solution of* (2.1) *defined on* $\Omega(a)$ *for some* $a > 1$. *The following statements are equivalent*:

(i)

$$\int_{\Omega(a)} ||x||^{1-n}||\boldsymbol{w}||^q \,\mathrm{d}x < \infty; \tag{2.2}$$

(ii) *there exists a finite limit*

$$\lim_{t\to\infty} C(t) = C_0; \tag{2.3}$$

(iii)

$$\liminf_{t\to\infty} C(t) > -\infty, \tag{2.4}$$

where the function $C(t)$ *is defined by* (1.8).

Our main theorem now follows from Lemma 2.1.

Theorem 2.2 (Hartman–Wintner type oscillation criterion). *If either*

$$-\infty < \liminf_{t\to\infty} C(t) < \limsup_{t\to\infty} C(t) \le \infty$$

or

$$\lim_{t\to\infty} C(t) = \infty,$$

then the equation (1.1) *has no positive solution positive on* $\Omega(a)$ *for any* $a > 1$.

Proof. It follows from the assumptions of the theorem that $\liminf_{t\to\infty} C(t) > -\infty$. If there would exist a number $a > 1$ such that (1.1) has a solution positive on $\Omega(a)$, then Theorem 2.1 would imply that there exists a finite limit $\lim_{t\to\infty} C(t)$. This contradiction ends the proof. □

Corollary 2.3 (Leighton–Wintner type criterion). *If*

$$\lim_{t\to\infty} \int_{\Omega(1,t)} ||x||^{1-n}c(x)\,\mathrm{d}x = \infty, \tag{2.5}$$

then equation (1.1) *has no positive solution on* $\Omega(a)$ *for any* $a > 1$.

Proof. If (2.5) holds, then $\lim_{t\to\infty} C(t) = \infty$ and the statement follows from Theorem 2.2. $\square$

Remark. For the equation (1.2) were the results from this paper proved in [5]. Criteria analogous to the second part of Theorem 2.2 and Corollary 2.3 were proved in [2] without the term $||x||^{1-n}$ but under additional conditions $p \ge n+1$ and $p \ge n$, respectively.

Proof of Lemma 2.1. First we multiply the Riccati equation (2.1) by $||x||^{1-n}$ and integrate on $\Omega(a,t)$. Application of the identity

$$||x||^{1-n} \operatorname{div} \boldsymbol{w} = \operatorname{div}(||x||^{1-n}\boldsymbol{w}) - (1-n)||x||^{-n}\langle \boldsymbol{w}, \boldsymbol{j}\rangle,$$

and Gauss divergence theorem yields

$$\begin{aligned}\int_{||x||=t} ||x||^{1-n}\langle \boldsymbol{w}, \boldsymbol{j}\rangle \,\mathrm{d}\sigma &- \int_{||x||=a} ||x||^{1-n}\langle \boldsymbol{w}, \boldsymbol{j}\rangle \,\mathrm{d}\sigma \\ &- (1-n)\int_{\Omega(a,t)} ||x||^{-n}\langle \boldsymbol{w}, \boldsymbol{j}\rangle \,\mathrm{d}x + (p-1)\int_{\Omega(a,t)} ||x||^{1-n}||\boldsymbol{w}||^q \,\mathrm{d}x \\ &+ \int_{\Omega(a,t)} ||x||^{1-n} c(x)\,\mathrm{d}x = 0, \end{aligned} \tag{2.6}$$

where $\int \cdot \,\mathrm{d}\sigma$ denotes the surface integral, $\boldsymbol{j}$ is the unit outside normal vector to the sphere in $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the usual scalar product.

"(i)=>(ii)" Suppose that (2.2) holds. The Hölder inequality implies that

$$\begin{aligned}\int_{\Omega(a,t)} ||x||^{1-n}|\langle \boldsymbol{w}, \boldsymbol{j}\rangle| \,\mathrm{d}x &\le \Big(\int_{\Omega(a,t)} ||x||^{1-n}||\boldsymbol{w}||^q \,\mathrm{d}x\Big)^{1/q}\Big(\int_{\Omega(a,t)} ||x||^{1-n-p}\,\mathrm{d}x\Big)^{1/p} \\ &\le \Big(\int_{\Omega(a)} ||x||^{1-n}||\boldsymbol{w}||^q \,\mathrm{d}x\Big)^{1/q}\Big(\omega_n \int_a^t s^{-p}\,\mathrm{d}s\Big)^{1/p},\end{aligned}$$

where ω_n is the measure of surface of the $n-$dimensional unit sphere in $\mathbb{R}^n$. Hence

$$\int_{\Omega(a)} ||x||^{-n}\langle \boldsymbol{w}, \boldsymbol{j}\rangle \,\mathrm{d}x \le \infty. \tag{2.7}$$

Denote

$$\begin{aligned}\widehat{C} = -(p-1)\int_{\Omega(a)} ||x||^{1-n}||\boldsymbol{w}||^q \,\mathrm{d}x &+ \int_{||x||=a} ||x||^{1-n}\langle \boldsymbol{w}, \boldsymbol{j}\rangle \,\mathrm{d}\sigma \\ + (1-n)\int_{\Omega(a)} ||x||^{-n}\langle \boldsymbol{w}, \boldsymbol{j}\rangle \,\mathrm{d}\sigma &+ \int_{\Omega(1,a)} ||x||^{1-n} c(x)\,\mathrm{d}x\,.\end{aligned}$$

Below we will show that $\widehat{C} = C_0$. The equation (2.6) can be written in the form

$$\widehat{C} - \int_{\Omega(1,t)} ||x||^{1-n} c(x)\,\mathrm{d}x = \int_{||x||=t} ||x||^{1-n} \langle \boldsymbol{w}, \boldsymbol{j} \rangle \,\mathrm{d}\sigma$$
$$- (p-1) \int_{\Omega(t)} ||x||^{1-n} ||\boldsymbol{w}||^q \,\mathrm{d}x + (1-n) \int_{\Omega(t)} ||x||^{-n} \langle \boldsymbol{w}, \boldsymbol{j} \rangle \,\mathrm{d}x\,. \quad (2.8)$$

Multiplying (2.8) by t^{p-2}, integrating over $[a,t]$ and multiplying by $\frac{p-1}{t^{p-1}}$ we obtain

$$\widehat{C} - \Big(\frac{a}{t}\Big)^{p-1} [\widehat{C} - C(a)] - C(t) = \frac{p-1}{t^{p-1}} \int_a^t s^{p-2} \int_{||x||=s} ||x||^{1-n} \langle \boldsymbol{w}, \boldsymbol{j} \rangle \,\mathrm{d}\sigma \,\mathrm{d}s$$
$$- \frac{(p-1)^2}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(s)} ||x||^{1-n} ||\boldsymbol{w}||^q \,\mathrm{d}x \,\mathrm{d}s$$
$$+ \frac{(1-n)(p-1)}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(s)} ||x||^{-n} \langle \boldsymbol{w}, \boldsymbol{j} \rangle \,\mathrm{d}x \,\mathrm{d}s\,. \quad (2.9)$$

The second and the third integral on the right hand side tend to zero as t tends to infinity in view of (2.2) and (2.7). The Hölder inequality implies

$$\left| \frac{1}{t^{p-1}} \int_a^t s^{p-2} \int_{||x||=s} ||x||^{1-n} \langle \boldsymbol{w}, \boldsymbol{j} \rangle \,\mathrm{d}\sigma \,\mathrm{d}s \right|$$
$$\le \frac{1}{t^{p-1}} \int_a^t s^{p-2} \Big(\int_{||x||=s} ||x||^{1-n} ||\boldsymbol{w}||^q \,\mathrm{d}\sigma \Big)^{1/q} \omega_n^{1/p} \,\mathrm{d}s$$
$$\le \frac{\omega_n^{1/p}}{t^{p-1}} \Big(\int_{\Omega(a,t)} ||x||^{1-n} ||\boldsymbol{w}||^q \,\mathrm{d}x \Big)^{1/q} \Big(\int_0^t s^{p^2-2p} \,\mathrm{d}s \Big)^{1/p}$$
$$\le \frac{\omega_n^{1/p}}{(p-1)^{2/p}} \Big(\int_{\Omega(a)} ||x||^{1-n} ||\boldsymbol{w}||^q \,\mathrm{d}x \Big)^{1/q} t^{\frac{(p-1)^2}{p} - (p-1)} \quad (2.10)$$

and the first integral in (2.9) tends to zero too. Hence

$$\lim_{t\to\infty} C(t) = \widehat{C} = C_0. \quad (2.11)$$

The implication "(ii)=>(iii)" is trivial.

"(iii)=>(i)" Suppose, by contradiction, that (2.4) holds and

$$\int_{\Omega(a)} ||x||^{1-n} ||\boldsymbol{w}||^q \,\mathrm{d}x = +\infty. \quad (2.12)$$

Multiplication of (2.6) by t^{p-2}, integration over the interval $[a,b]$ and multiplication by t^{1-p} gives

$$\frac{1}{t^{p-1}} \int_a^t s^{p-2} \int_{||x||=s} ||x||^{1-n} \langle \boldsymbol{w}, \boldsymbol{j} \rangle \,\mathrm{d}\sigma \,\mathrm{d}s$$
$$+ \frac{p-1}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(a,s)} ||x||^{1-n} ||\boldsymbol{w}||^q \,\mathrm{d}x \,\mathrm{d}s$$

$$
\begin{aligned}
&- \frac{1-n}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(a,s)} ||x||^{-n} \langle \boldsymbol{w}, \boldsymbol{j} \rangle \, \mathrm{d}x \; \mathrm{d}s \\
&= \frac{1}{t^{p-1}} \int_a^t s^{p-2} \, \mathrm{d}s \int_{||x||=a} ||x||^{1-n} \langle \boldsymbol{w}, \boldsymbol{j} \rangle \, \mathrm{d}\sigma \\
&\qquad - \frac{1}{t^{p-1}} \int_a^t s^{p-2} \int_{\Omega(a,s)} ||x||^{1-n} c(x) \, \mathrm{d}x \; \mathrm{d}s \, . \qquad (2.13)
\end{aligned}
$$

Define the function

$$
v(t) := (p-1) \int_a^t s^{p-2} \int_{\Omega(a,s)} ||x||^{1-n} ||\boldsymbol{w}||^q \, \mathrm{d}x \; \mathrm{d}s \, .
$$

The function v satisfies

$$
\frac{v(t)}{t^{p-1}} \to \infty \text{ for } t \to \infty. \qquad (2.14)
$$

Because of the right hand side of the equality (2.13) is bounded from above, there exists t_a such that the right hand side of (2.13) is less than $\frac{v(t)}{3t^{p-1}}$ for $t \geq t_a$. Now we have from (2.13)

$$
\begin{aligned}
\frac{2}{3} v(t) < &\left| \int_a^t s^{p-2} \int_{||x||=s} ||x||^{1-n} \langle \boldsymbol{w}, \boldsymbol{j} \rangle \, \mathrm{d}\sigma \; \mathrm{d}s \right| \\
&+ \left| (1-n) \int_a^t s^{p-2} \int_{\Omega(a,s)} ||x||^{-n} \langle \boldsymbol{w}, \boldsymbol{j} \rangle \, \mathrm{d}x \; \mathrm{d}s \right| \qquad (2.15)
\end{aligned}
$$

for $t \geq t_a$. The same way as in the inequality (2.10) gives

$$
\begin{aligned}
&\left| \int_a^t s^{p-2} \int_{||x||=s} ||x||^{1-n} \langle \boldsymbol{w}, \boldsymbol{j} \rangle \, \mathrm{d}\sigma \; \mathrm{d}s \right| \\
&\qquad \leq \left(\int_{\Omega(a,t)} ||x||^{1-n} ||\boldsymbol{w}||^q \, \mathrm{d}x \right)^{1/q} \frac{\omega_n^{1/p} t^{\frac{(p-1)^2}{p}}}{(p-1)^{2/p}} = K \big(t v'(t) \big)^{1/q}, \qquad (2.16)
\end{aligned}
$$

where $K = \omega_n^{1/p} (p-1)^{-\frac{2}{p} - \frac{1}{q}}$. The Hölder inequality gives

$$
\begin{aligned}
&\left| (1-n) \int_a^t s^{p-2} \int_{\Omega(a,s)} ||x||^{-n} \langle \boldsymbol{w}, \boldsymbol{j} \rangle \, \mathrm{d}x \; \mathrm{d}s \right| \\
&\qquad \leq (n-1) \int_a^t s^{p-2} \left(\int_{\Omega(a,t)} ||x||^{1-n} ||\boldsymbol{w}||^q \, \mathrm{d}x \right)^{1/q} \left(\int_1^\infty \omega_n \xi^{-p} \, \mathrm{d}\xi \right)^{1/p} \mathrm{d}s \\
&\qquad \leq (n-1) \left(\int_a^t s^{p-2} \int_{\Omega(a,t)} ||x||^{1-n} ||\boldsymbol{w}||^q \, \mathrm{d}x \; \mathrm{d}s \right)^{1/q} \\
&\qquad\qquad \times \left(\int_1^\infty \omega_n s^{-p} \, \mathrm{d}s \right)^{1/p} \left(\int_0^t s^{p-2} \, \mathrm{d}s \right)^{1/p}
\end{aligned}
$$

$$
\begin{aligned}
&= (n-1)\Big(\frac{v(t)}{p-1}\Big)^{1/q} \frac{t^{\frac{p-1}{p}} \omega_n^{1/p}}{(p-1)^{2/p}} \\
&= \frac{(n-1)\omega_n^{1/p}}{(p-1)^{\frac{1}{q}+\frac{2}{p}}} v^{1/q}(t) t^{\frac{p-1}{p}}. \qquad (2.17)
\end{aligned}
$$

In view of the fact (2.14) there exists a number $t_b \geq t_a$ such that

$$
\frac{(n-1)\omega_n^{1/p}}{(p-1)^{\frac{1}{q}+\frac{2}{p}}} t^{\frac{p-1}{p}} \leq \frac{1}{3} v^{1/p}(t) \qquad (2.18)
$$

for $t \geq t_b$. Combining (2.15), (2.16), (2.17) and (2.18) we get

$$
\frac{1}{3} v(t) \leq K\big(tv'(t)\big)^{1/q}
$$

for $t \geq t_b$. From here

$$
\frac{v'(t)}{v^q(t)} \geq \frac{1}{t}\Big(\frac{1}{3\,K}\Big)^q
$$

for $t \geq t_b$. Integration of this inequality from t_b to ∞ gives a convergent integral on the left hand side and divergent integral on the right hand side. This contradiction ends the proof. □

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