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SINGULAR HIGHER ORDER BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

We study singular boundary value problems for differential equations with left focal boundary conditions of the form,

$$(-1)^n u^{(n)}(t) + f(t, u(t), \dots, u^{(n-2)}(t)) = 0, \quad t \in (0, 1),$$

$$u^{(n-1)}(0) = u^{(n-2)}(1) = \cdots = u(1) = 0.$$

We assume that $f(t, x_0, \dots, x_{n-2})$ is continuous on $[0, 1] \times (0, \infty) \times \mathbb{R}^{n-2}$ and f has a singularity at $x_0 = 0$. We prove the existence of a positive solution by means of the lower and upper solutions method, the Brouwer fixed point theorem, and by perturbation methods to approximate regular problems.

Key words and phrases: Boundary value problem, lower and upper solutions, ordinary differential equation, higher order, positive solution, perturbation methods, fixed point.
AMS Subject Classifications: 34B16

1 Introduction

This paper is somewhat of an extension of the recent work done by Kunkel [6]. Kunkel looked at an extension of Rachůnková and Rachůnek's work where they studied a second order singular boundary value problem for the discrete p-Laplacian, $\phi_p(x) = |x|^{p-2}x$ [7]. Kunkel's results extend theirs to the second order differential case, but only for $p = 2$, i.e. $\phi_2(x) = x$. In this paper, we extend Kunkel's work to a higher order boundary value problem that is a generalization of the lower order case.

For our purposes, we will define what it means to be a lower solution and an upper solution and, along with the Brouwer fixed point theorem, create a lower and upper solutions method that can be used in our higher order differential case. We can then apply this method to our higher order boundary value problem to achieve the desired result.

The use of an upper and lower solutions method for proving the existence of a solution to a boundary value problem has been studied by others, including [2], [3], [4], and [5]. Singular boundary value problems for differential and difference equations have been investigated by several authors in recent years and an exhaustive survey of many of these results is found in [1].

2 Preliminaries

In this section we will state some of the definitions that are used throughout the remainder of the paper.

We consider the singular higher order differential equation,

$$(-1)^n u^{(n)}(t) + f(t, u(t), \ldots, u^{(n-2)}(t)) = 0, \quad t \in (0,1), \tag{1}$$

satisfying left focal boundary conditions,

$$u^{(n-1)}(0) = u^{(n-2)}(1) = \cdots = u(1) = 0. \tag{2}$$

Our goal is to prove the existence of a positive solution of problem (1), (2).

Definition 2.1 *By a* solution u *of problem (1), (2) we mean* $u : [0,1] \to \mathbb{R}$ *such that* u *satisfies the differential equation (1) on* $(0,1)$ *and the boundary conditions (2). If* $u(t) > 0$ *for* $t \in (0,1)$, *we say* u *is a* positive solution *of the problem (1), (2).*

We also define what is considered a regular problem and what is a singular problem.

Definition 2.2 *Let* $f : [0,1] \times \mathcal{D} \to \mathbb{R}$ *be continuous. Let* $D \subseteq \mathbb{R}^{n-1}$. *If* $\mathcal{D} = \mathbb{R}^{n-1}$, *problem (1), (2) is called* regular. *If* $\mathcal{D} \neq \mathbb{R}^{n-1}$ *and* f *has singularities on the boundary of* $\mathcal{D}$, *then problem (1), (2) is* singular.

We will assume throughout this paper that the following hold:

A: $\mathcal{D} = (0, \infty) \times \mathbb{R}^{n-2}$.

B: f is continuous on $[0,1] \times \mathcal{D}$.

C: $f(t, x_0, \ldots, x_{n-2})$ has a singularity at $x_0 = 0$, i.e. $\limsup_{x_0 \to 0^+} |f(t, x_0, \ldots, x_{n-2})| = \infty$ for each $t \in (0,1)$ and $(x_1, \ldots, x_{n-2}) \in \mathbb{R}^{n-2}$.

3 Lower and Upper Solutions Method for Regular Problems

Let us first consider the regular differential equation,

$$(-1)^n u^{(n)}(t) + h(t, u(t), \ldots, u^{(n-2)}(t)) = 0, \quad t \in (0,1), \tag{3}$$

where h is continuous on $[0,1] \times \mathbb{R}^{n-1}$. In this section, we establish a lower and upper solutions method for the regular problem (3) satisfying boundary conditions (2). We first define our lower and upper solutions.

Definition 3.1 $\alpha : [0,1] \to \mathbb{R}$ *is called a* lower solution *of (3), (2) if,*

$$(-1)^n \alpha^{(n)}(t) + h(t, \alpha(t), \ldots, \alpha^{(n-2)}(t)) \geq 0, \quad t \in (0,1), \tag{4}$$

$$\begin{array}{rcl} (-1)^{n-1}\alpha^{(n-1)}(0) & \leq & 0, \\ (-1)^{n-2}\alpha^{(n-2)}(1) & \leq & 0, \\ & \vdots & \\ \alpha'(1) & \geq & 0, \\ \alpha(1) & \leq & 0. \end{array} \tag{5}$$

Definition 3.2 $\beta : [0,1] \to \mathbb{R}$ *is called an* upper solution *of (3), (2) if,*

$$(-1)^n \beta^{(n)}(t) + h(t, \beta(t), \ldots, \beta^{(n-2)}(t)) \leq 0, \quad t \in (0,1), \tag{6}$$

$$\begin{array}{rcl} (-1)^{n-1}\beta^{(n-1)}(0) & \geq & 0, \\ (-1)^{n-2}\beta^{(n-2)}(1) & \geq & 0, \\ & \vdots & \\ \beta'(1) & \leq & 0, \\ \beta(1) & \geq & 0. \end{array} \tag{7}$$

Theorem 3.1 (Lower and Upper Solutions Method) *Let α and β be given lower and upper solutions of (3), (2), respectively, and $\alpha \leq \beta$ on $[0,1]$. Let $h(t, x_0, \ldots, x_{n-2})$ be continuous on $[0,1] \times \mathbb{R}^{n-1}$. Then (3), (2) has a solution $u(t)$ satisfying,*

$$\alpha(t) \leq u(t) \leq \beta(t), \quad t \in [0,1]. \tag{8}$$

Proof.

We proceed through a sequence of steps involving modifications of the function h. Step 1. For $t \in (0,1)$ and $(x_0, \ldots, x_{n-2}) \in \mathbb{R}^{n-1}$, define

$$\widetilde{h}(t, x_0, \ldots, x_{n-2}) = \begin{cases} \max\{h(t, x_0, \ldots, x_{n-2}), h(t, \beta(t), \ldots, \beta^{(n-2)}(t))\} - \frac{x-\beta(t)}{x-\beta(t)+1}, \\ \qquad \text{if } x > \beta(t), \\ h(t, x_0, \ldots, x_{n-2}), \quad \text{if } \alpha(t) \leq x \leq \beta(t), \\ \min\{h(t, x_0, \ldots, x_{n-2}), h(t, \alpha(t), \ldots, \alpha^{(n-2)}(t))\} + \frac{\alpha(t)-x}{\alpha(t)-x+1}, \\ \qquad \text{if } x < \alpha(t). \end{cases}$$

Thus, $\widetilde{h}$ is continuous on $(0,1)\times\mathbb{R}^{n-1}$ and there exists $M>0$ so that,

$$|\widetilde{h}(t,x_0,\ldots,x_{n-2})|\le M,\quad t\in(0,1),(x_0,\ldots,x_{n-2})\in\mathbb{R}^{n-1}.$$

We now study the auxiliary equation,

$$(-1)^n u^{(n)}(t)+\widetilde{h}(t,u(t),\ldots,u^{(n-2)}(t))=0,\quad t\in(0,1), \tag{9}$$

satisfying boundary conditions (2). Our immediate goal is to prove the existence of a solution to problem (9), (2).

Step 2. We lay the foundation to use the Brouwer fixed point theorem. To this end, define

$$E=\{u:[0,1]\to\mathbb{R}:u^{(i)}\text{ is continuous, } i=0,\ldots,n-2\}$$

and also define for $u\in E$,

$$\|u\|=\max_{0\le i\le n-2}\{\sup\{|u^{(i)}(t)|:t\in[0,1]\}\}.$$

E is a Banach space. Further, we define an operator $\mathcal{T}:E\to E$ by,

$$(\mathcal{T}u)(t)=\int_t^1\int_{r_{n-1}}^1\cdots\int_{r_2}^1\int_0^{r_1}\widetilde{h}(r_0,u(r_0),\ldots,u^{(n-2)}(r_0))dr_0\cdots dr_{n-1}. \tag{10}$$

$\mathcal{T}$ is a continuous operator. Moreover, from the bounds placed on $\widetilde{h}$ in Step 1 and from (10), if $r>M$, then $\mathcal{T}(\overline{B(r)})\subset\overline{B(r)}$, where $B(r):=\{u\in E:\|u\|<r\}$. Therefore, by the Brouwer fixed point theorem [8], there exists $u\in\overline{B(r)}$ such that $u=\mathcal{T}u$.

Step 3. We now show that u is a fixed point of $\mathcal{T}$ if and only if u is a solution of (9), (2).

First assume $u=\mathcal{T}u$. Then, from (10), we have that

$$\begin{aligned} u(t) &= (\mathcal{T}u)(t)\\ &= \int_t^1\int_{r_{n-1}}^1\cdots\int_{r_2}^1\int_0^{r_1}\widetilde{h}(r_0,u(r_0),\ldots,u^{(n-2)}(r_0))dr_0\cdots dr_{n-1}.\end{aligned}$$

Thus,

$$u(1)=\int_1^1\int_{r_{n-1}}^1\cdots\int_{r_2}^1\int_0^{r_1}\widetilde{h}(r_0,u(r_0),\ldots,u^{(n-2)}(r_0))dr_0\cdots dr_{n-1}=0.$$

Also, we have that

$$u'(t)=-\int_t^1\int_{r_{n-2}}^1\cdots\int_{r_2}^1\int_0^{r_1}\widetilde{h}(r_0,u(r_0),\ldots,u^{(n-2)}(r_0))dr_0\cdots dr_{n-2},$$

making

$$u'(1) = -\int_1^1 \int_{r_{n-2}}^1 \cdots \int_{r_2}^1 \int_0^{r_1} \widetilde{h}(r_0, u(r_0), \ldots, u^{(n-2)}(r_0))dr_0 \cdots dr_{n-2} = 0.$$

Continuing in this manner, we have that

$$u''(1) = \cdots = u^{(n-3)}(1) = 0,$$

and that

$$u^{(n-2)}(t) = (-1)^{n-2} \int_t^1 \int_0^{r_1} \widetilde{h}(r_0, u(r_0), \ldots, u^{(n-2)}(r_0))dr_0 dr_1.$$

Thus making,

$$u^{(n-2)}(1) = (-1)^{n-2} \int_1^1 \int_0^{r_1} \widetilde{h}(r_0, u(r_0), \ldots, u^{(n-2)}(r_0))dr_0 dr_1 = 0.$$

Also, we have that

$$u^{(n-1)}(t) = (-1)^{n-1} \int_0^t \widetilde{h}(r_0, u(r_0), \ldots, u^{(n-2)}(r_0))dr_0,$$

making

$$u^{(n-1)}(0) = (-1)^{n-1} \int_0^0 \widetilde{h}(r_0, u(r_0), \ldots, u^{(n-2)}(r_0))dr_0 = 0.$$

Therefore, (2) is satisfied, and differentiating one more time yields,

$$u^{(n)}(t) = (-1)^{n-1}\widetilde{h}(t, u(t), \ldots, u^{(n-2)}(t)),$$

making (9) satisfied.

Now assume $u(t)$ solves (9), (2). Then clearly,

$$(-1)^{n-1}u^{(n)}(t) = \widetilde{h}(t, u(t), \ldots, u^{(n-2)}(t)).$$

Thus, by integrating back and applying appropriate boundary conditions, we get that

$$\begin{aligned} (-1)^{n-1} \int_0^t u^{(n)}(s)ds &= \int_0^t \widetilde{h}(s, u(s), \ldots, u^{(n-2)}(s))ds \\ &= (-1)^{n-1}(u^{(n-1)}(t) - u^{(n-1)}(0)) \\ &= (-1)^{n-1}u^{(n-1)}(t). \end{aligned}$$

Therefore,

$$(-1)^{n-1}u^{(n-1)}(t) = \int_0^t \widetilde{h}(s, u(s), \ldots, u^{(n-2)}(s))ds.$$

Also,

$$\begin{aligned}\int_t^1 (-1)^{n-1}u^{(n-1)}(r)dr &= \int_t^1 \int_0^r \widetilde{h}(s, u(s), \ldots, u^{(n-2)}(s))dsdr \\ &= (-1)^{n-1}(u^{(n-2)}(1) - u^{(n-2)}(t)) \\ &= (-1)^{n-2}u^{(n-2)}(t).\end{aligned}$$

Therefore,

$$(-1)^{n-2}u^{(n-2)}(t) = \int_t^1 \int_0^r \widetilde{h}(s, u(s), \ldots, u^{(n-2)}(s))dsdr.$$

Continuing in this process of integrating and applying appropriate boundary conditions, we get that

$$u(t) = \int_t^1 \int_{r_{n-1}}^1 \cdots \int_{r_2}^1 \int_0^{r_1} \widetilde{h}(r_0, u(r_0), \ldots, u^{(n-2)}(r_0))dr_0 \cdots dr_{n-1},$$

i.e. $u = \mathcal{T}u$.

Step 4. We now show that solutions $u(t)$ of (9), (2) satisfy,

$$\alpha(t) \leq u(t) \leq \beta(t), \quad t \in [0, 1]. \tag{11}$$

Consider the case of obtaining $u(t) \leq \beta(t)$. Let $v(t) = u(t) - \beta(t)$. For the sake of establishing a contradiction, assume that

$$\sup\{v(t) : t \in [0, 1]\} := v(l) > 0.$$

Conditions (2) and (7) imply that $(-1)^n v^{(n)}(l) \geq 0$. Therefore,

$$(-1)^n u^{(n)}(l) \geq (-1)^n \beta^{(n)}(l). \tag{12}$$

On the other hand,

$$\begin{aligned}(-1)^n v^{(n)}(l) &= (-1)^n u^{(n)}(l) - (-1)^n \beta^{(n)}(l) \\ &= \widetilde{h}(l, u(l), \ldots, u^{(n-2)}(l)) - (-1)^n \beta^{(n)}(l) \\ &\leq h(l, \beta(l), \ldots, \beta^{(n-2)}(l)) - \frac{u(l) - \beta(l)}{u(l) - \beta(l) + 1} - (-1)^n \beta^{(n)}(l)\end{aligned}$$

$$
\begin{aligned}
&\le (-1)^n\beta^{(n)}(l) - \frac{v(l)}{v(l)+1} - (-1)^n\beta^{(n)}(l)\\
&= -\frac{v(l)}{v(l)+1}\\
&< 0.
\end{aligned}
$$

Hence, $(-1)^n u^{(n)}(l) < (-1)^n\beta^{(n)}(l)$, but this contradicts (12). Therefore, $v(l) \le 0$, for all $l \in [0,1]$. This implies that

$$u(t) \le \beta(t), \text{ for } t \in [0,1].$$

A similar argument shows that $\alpha(t) \le u(t)$.
Thus, the conclusion of the theorem holds and our proof is complete.

4 Main Result

In this section, we make use of Theorem 3.1 to obtain positive solutions of the singular problem (1), (2).

Theorem 4.1 *Assume conditions (A), (B), and (C) hold, along with the following:*

D: There exists $c \in (0,\infty)$ so that $(-1)^n f(t,c,x_1,\dots,x_{n-2}) \le 0$ for all $t \in (0,1)$ and $(x_1,\dots,x_{n-2}) \in \mathbb{R}^{n-2}$.

E: There exists $\delta > 0$ so that $f(t,x_0,\dots,x_{n-2}) > 0$ for all $t \in (1-\delta,1), x_0 \in (0,\frac{c}{2})$.

F: $\lim_{x_0\to 0+} f(t,x_0,\dots,x_{n-2}) = \infty$ for $t \in (0,1)$ and $(x_1,\dots,x_{n-2}) \in \mathbb{R}^{n-2}$.

Then (1), (2) has a solution u satisfying,

$$0 < u(t) \le c, \quad t \in [0,1). \tag{13}$$

Proof.
We proceed through a sequence of steps.
Step 1. For $k \in \mathbb{N}, t \in (0,1)$, define

$$f_k(t,x_0,\dots,x_{n-2}) = \begin{cases} f(t,|x_0|,x_1,\dots,x_{n-2}), & \text{if } |x_0| \ge \frac{1}{k},\\ f\left(t,\frac{1}{k},x_1,\dots,x_{n-2}\right), & \text{if } |x_0| < \frac{1}{k}. \end{cases} \tag{14}$$

Then f_k is continuous on $(0,1)\times\mathbb{R}^{n-1}$.
Assumption (F) implies that there exists k_0, such that, for all $k \ge k_0$,

$$f_k(t,0,x_1,\dots,x_{n-2}) = f\left(t,\frac{1}{k},x_1,\dots,x_{n-2}\right) > 0, \quad t \in (0,1), (x_1,\dots,x_{n-2}) \in \mathbb{R}^{n-2}.$$

Consider,

$$(-1)^n u^{(n)}(t) + f_k(t, u(t), \dots, u^{(n-2)}(t)) = 0, \quad t \in (0,1), \tag{15}$$

and let $\alpha(t) = 0$ and $\beta(t) = c$. Then α and β are lower and upper solutions for (15), (2) and $\alpha(t) \le \beta(t)$ on $[0,1]$. Thus, by Theorem 3.1, there exists $u_k(t)$ a solution of (15), (2) satisfying $0 \le u_k(t) \le c, t \in [0,1], k \ge k_0$.
Step 2. Let $k \in \mathbb{N}, k \ge k_0$. Then, there exists $\varepsilon \in (0, \frac{c}{2})$ such that if $k_\varepsilon \ge k_0$,

$$f_{k_\varepsilon}(t, x_0, \dots, x_{n-2}) > c, \quad t \in (0, 1-\delta), x_0 \in (0, \varepsilon], (x_1, \dots, x_{n-2}) \in \mathbb{R}^{n-2}. \tag{17}$$

For the sake of establishing a contradiction, assume that for $k_\varepsilon \ge k_0$, we have that $(-1)^{n-2} u_{k_\varepsilon}^{(n-2)}(t) < (-1)^{n-2} \varepsilon_2^{(n-2)}(t)$, where

$$\varepsilon_2(t) = \int_t^1 \int_{r_{n-3}}^1 \cdots \int_{r_2}^1 \int_0^{r_1} \varepsilon_1(r_0) dr_0 \cdots dr_{n-3}, \tag{18}$$

and

$$\varepsilon_1(t) = \begin{cases} \varepsilon, & t \in (0, 1-\delta), \\ \frac{\varepsilon}{\delta}(1-t), & t \in (1-\delta, 1). \end{cases} \tag{19}$$

We must now impose the convention that $\varepsilon_2(t)$ is a continuous function on [0,1] and therefore can be pieced together via integration similar to what was done in Step 2 of Theorem 3.1. Again, we impose additional conditions upon our choice of ε. Namely,

$$u_{k_\varepsilon}(0) \ge \varepsilon_2(0),$$

and

$$u_{k_\varepsilon}(1-\delta) \ge \varepsilon_2(1-\delta).$$

Therefore, $u_{k_\varepsilon}(t)$ solves (15), (2) and

$$\varepsilon_2(t) \le u_{k_\varepsilon}(t) \le c.$$

For the sake of establishing a contradiction, assume for $k_\varepsilon \ge k_0$, we have that

$$(-1)^{n-2} u_{k_\varepsilon}^{(n-2)}(t) < (-1)^{n-2} \varepsilon_2^{(n-2)}(t) = \varepsilon_1(t),$$

for some $t \in [0,1]$. Clearly,

$$u_{k_\varepsilon}(0) \ge \varepsilon_2(0),$$
$$u_{k_\varepsilon}(1) \ge \varepsilon_2(1) = 0,$$

and

$$u_{k_\varepsilon}(1-\delta) \ge \varepsilon_2(1-\delta).$$

We therefore, need only consider two cases: $0 < t < 1-\delta$ and $1-\delta < t < 1$.

First, let's consider $0 < t < 1-\delta$. Then, $(-1)^{n-2}u_{k_\varepsilon}^{(n-2)}(t) > \varepsilon$. However,

$$\begin{aligned}(-1)^{n-2}u_{k_\varepsilon}^{(n-2)}(t) &= \int_t^{1-\delta}\int_0^r f_{k_\varepsilon}(s, u_{k_\varepsilon}(s), \ldots, u_{k_\varepsilon}^{(n-2)}(s))dsdr \\ &> \int_t^{1-\delta}\int_0^r cdsdr \\ &= \frac{c}{2}((1-\delta)^2 - t^2).\end{aligned}$$

This implies that

$$(-1)^{n-1}u_{k_\varepsilon}^{(n)}(t) < c,$$

and that

$$f_{k_\varepsilon}(t, u_{k_\varepsilon}(t), \ldots, u_{k_\varepsilon}^{(n-2)}(t)) < c.$$

But this is a contradiction. Hence, for $0 < t < 1-\delta$, we have that

$$(-1)^{n-2}u_{k_\varepsilon}^{(n-2)}(t) \geq (-1)^{n-2}\varepsilon_2^{(n-2)}(t).$$

Now, let's consider the case where $1-\delta < t < 1$. Clearly, from the underlying assumptions placed upon our function in this interval,

$$(-1)^{n-1}u_{k_\varepsilon}^{(n)}(t) = f_{k_\varepsilon}(t, u_{k_\varepsilon}(t), \ldots, u_{k_\varepsilon}^{(n-2)}(t)) > 0.$$

We also have that $(-1)^{n-2}u_{k_\varepsilon}^{(n-2)}(t) < \frac{c}{2}((1-\delta)^2 - t^2)$, which implies that

$$(-1)^{n-1}u_{k_\varepsilon}^{(n)}(t) < c.$$

Hence,

$$0 < (-1)^{n-1}u_{k_\varepsilon}^{(n)}(t) < c,$$

which implies that $(-1)^{n-1}u_{k_\varepsilon}^{(n-2)}(t)$ is convex in this interval. We also know, by continuity, that

$$(-1)^{n-1}u_{k_\varepsilon}^{(n-2)}(1-\delta) \geq \varepsilon,$$

and

$$(-1)^{n-1}u_{k_\varepsilon}^{(n-2)}(1) = 0.$$

Therefore, by continuity,

$$(-1)^{n-1}u_{k_\varepsilon}^{(n-2)}(t) > (-1)^{n-1}\varepsilon_2^{(n-2)}(t) = \frac{\varepsilon}{\delta}(1-t).$$

Thus, $0 < (-1)^{n-1}\varepsilon_2^{(n-2)}(t) < (-1)^{n-1}u_{k_\varepsilon}^{(n-2)}(t)$. This forces, via integrating back and applying the appropriate boundary conditions that

$$0 < \varepsilon_2(t) < u_{k_\varepsilon}(t).$$

It is clear that $\{u_{k_\varepsilon}^{(i)}\}$ is uniformly bounded and uniformly equicontinuous on $[0,1]$, for $i = 0, \ldots, n-2$.

Hence, we can choose a subsequence $\{u_{k_m}\}_{m=1}^{\infty} \subset \{u_{k_\varepsilon}\}$, such that $\lim_{m\to\infty} u_{k_m}(t) = u(t)$ uniformly on $t \in [0,1], u \in E$, where E is the Banach space defined in Step 2 of Theorem 3.1. Moreover, for sufficiently large m,

$$u_{k_m}(t) = \int_t^1 \int_{r_{n-1}}^1 \cdots \int_{r_2}^1 \int_0^{r_1} f(r_0, u_{k_m}(r_0), \ldots, u_{k_m}^{(n-2)}(r_0)) dr_0 \cdots dr_{n-1}.$$

And from the continuity of f, as we let $m \to \infty$, we get

$$u(t) = \int_t^1 \int_{r_{n-1}}^1 \cdots \int_{r_2}^1 \int_0^{r_1} f(r_0, u(r_0), \ldots, u^{(n-2)}(r_0)) dr_0 \cdots dr_{n-1}.$$

Hence,

$$(-1)^{n-1} u^{(n)}(t) = f(t, u(t), \ldots, u^{(n-2)}(t)),$$

and $0 < u(t) < c$ on $[0,1)$.

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