Global analysis of a multi-group SIR epidemic model with nonlinear incidence rates and distributed moving delays between patches

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Abstract. In this paper, applying Lyapunov functional approach, we establish sufficient conditions under which each equilibrium is globally asymptotically stable for a class of multi-group SIR epidemic models. The incidence rate is given by nonlinear incidence rates and distributed delays incorporating not only an exchange of individuals between patches through migration but also cross patch infection between different groups. We show that nonlinear incidence rates and distributed delays have no influence on the global stability, but patch structure has. Moreover, the present results generalize known results on the global stability of a heroin model with two delays considered in the recent literatures. We also offer new techniques to prove the boundedness of the solutions, the existence of the endemic equilibrium and permanence of the model.

Keywords: heroin model, multi-group SIR epidemic model, patch, global stability, Lyapunov function.

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1 Introduction

Due to the recent development of qualitative and quantitative analysis for disease transmission models, mathematical models have widely been applied to investigate the spread of habituation of getting addicted to drugs such as heroin (see e.g., [7,12,17,22,28,34,35]).

Dividing the host population into three compartments; susceptible individuals $S(t)$, heroin users $U_1(t)$ and heroin users undergoing treatment $U_2(t)$, White and Comiskey [35] investi-

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gated the following heroin model:

$$\begin{cases} S'(t) = b - \dfrac{\beta S(t) U_1(t)}{N(t)} - dS(t), \\ U_1'(t) = \dfrac{\beta S(t) U_1(t)}{N(t)} - \gamma U_1(t) + \dfrac{\kappa U_1(t) U_2(t)}{N(t)} - (d + \varepsilon) U_1(t), \\ U_2'(t) = \gamma U_1(t) - \dfrac{\kappa U_1(t) U_2(t)}{N(t)} - (d + \delta) U_2(t). \end{cases} \quad (1.1)$$

Here b denotes the birth rate at which individuals in the general population enter the susceptible population. d denotes the death rate from natural causes. It is assumed that new infection of a drug user arises through a standard incidence function, $\beta S(t) U_1(t)/N(t)$ (resp. $\kappa U_1(t) U_2(t)/N(t)$) with β (resp. κ) denoting the probability for susceptible individuals (resp. drug users in treatment relapsing to untreated use) to be drug users not in treatment. Here drug users in treatment are not assumed to be susceptible again after they quit using drugs. γ denotes the rate at which drug users undertake treatment and ε represents a removal rate of drug users not in treatment, a sum of drug-related deaths rate and a spontaneous recovery rate. δ is a removal rate of drug users in treatment, a sum of drug-related death rate and a rate of successful "care" for drug user to be in a drug-free recovery. All the parameters are assumed to be positive.

In addition to stability analysis for a drug-free equilibrium [35, Section 3], the stability analysis for a unique endemic equilibrium is achieved by Mulone and Straughan [22] when $\kappa < \beta$. Later, Wang et al. [34] has formulated the model incorporating the mass action incidence rate and established the global stability of the drug-free equilibrium and the unique endemic equilibrium by means of the second compound matrix and under some conditions. These systems are extended to a non-autonomous model by Samanta [28], proving that there exists a unique positive periodic solution which is globally asymptotically stable by a direct Lyapunov method.

Recently, Liu and Zhang [17] introduced distributed delays in the relapse term into a heroin epidemic model without delays. Constructing a proper Lyapunov function, Huang and Liu [12] established the global stability for the heroin epidemic model with a distributed delay $\int_0^h \tilde{f}(\tau) \gamma U_1(t - \tau) e^{-(d+\delta)s} ds$ in place of the term $\kappa U_1(t) U_2(t)/N(t)$ in the third equation of the model (1.1).

Compared to the above "heavy drug" epidemic model with varying total population size incorporating drug-related deaths as the model (1.1), Muroya et al. [25] considered the following disease "light drug" epidemic model with "not varying total population size" eventually and such that there are no drug-related deaths of the light drug users who are not in treatment and in treatment.

$$\begin{cases} S'(t) = b - \dfrac{\beta S(t) U_1(t)}{N(t)} - dS(t) + \varepsilon U_1(t) + \delta U_2(t), \\ U_1'(t) = \dfrac{\beta S(t) U_1(t)}{N(t)} - \gamma U_1(t) + \sigma U_2(t) - (d + \varepsilon) U_1(t), \\ U_2'(t) = \gamma U_1(t) - \sigma U_2(t) - (d + \delta) U_2(t). \end{cases}$$

In this case, we have $\lim_{t \to +\infty} (S(t) + U_1(t) + U_2(t)) = b/d$. This also implies that the total population size does not eventually vary oscillatory but converges to a positive constant b/d.

On the other hand, Guo et al. [8] have first succeeded in the proof of global stability for a multi-group SIR epidemic model by making use of the theory of non-negative matrices, Lyapunov functions and a subtle grouping technique in estimating the derivatives of Lyapunov

functions guided by graph theory. To analyze the global stability of various multi-group epidemic models, many authors literature on multi-group models follow to use this graph theoretic approach (see for example, [4, 10, 15, 16, 29, 30, 36, 37]).

Recently, there are some interesting papers on construction techniques of Lyapunov functions to prove the global stability of equilibria (see, for e.g., Li *et al.* [14], Kajiwara *et al.* [20] and Vargas-De-León [32]). Guo *et al.* [9,11] considered the stage-progression models for HIV/AIDS with amelioration. Li *et al.* [14] established the global stability of a class of epidemic models by using quite interesting approach, and Muroya *et al.* [24] generalized their method.

Multi-group epidemic models have played a crucial role to clarify one of the important problems; transportation affects on the spreading pattern of the global pandemic of diseases such as heroin (see for example, Arino [1], Bartlett [2] for a population movement among different groups, Liu and Takeuchi [19], Liu and Zhou [18] and Nakata [26] for the effect of transport-related infection with entry screening, Wang and Zhao [33] for an epidemic model in a patchy environment). In particular, Muroya *et al.* [24] established general sufficient conditions of the global stability for a multi-group SIR epidemic model with patch structure which takes into account not only an exchange of individuals between patches through migration but also cross patch infection between different groups.

Recently, Fang *et al.* [7] presented the following heroin epidemic model with two distributed delays and establish the global asymptotic stability.

$$S'(t) = \Lambda - \beta S(t) \int_0^{h_1} \tilde{f}(\tau) U_1(t-\tau) e^{-(\mu+\delta_1+p)\tau} d\tau - \mu S(t),$$

$$U_1'(t) = \beta S(t) \int_0^{h_1} \tilde{f}(\tau) U_1(t-\tau) e^{-(\mu+\delta_1+p)\tau} d\tau + p \int_0^{h_2} \tilde{g}(\tau) U_1(t-\tau) e^{-(\mu+\delta_1+p)\tau} d\tau,$$

$$U_2'(t) = p U_1(t) - (\mu+\delta_2) U_2(t) - p \int_0^{h_2} \tilde{g}(\tau) U_1(t-\tau) e^{-(\mu+\delta_1+p)\mu} d\tau,$$

where $\int_0^{h_1} \tilde{f}(\tau) d\tau = 1$ and $\int_0^{h_2} \tilde{g}(\tau) d\tau = 1$. Motivated by Fang *et al.* [7], in this paper, we aim to investigate the global dynamics of a multi-group epidemic model related to heroin model with nonlinear incidence rates and distributed delays. In the formulation of the model, we divide each population into $n \in \mathbb{N}$ groups and use the following notations (in what follows, k and j belong to $\{1, 2, \ldots, n\}$):

- $S_k(t)$: the number of susceptible individuals in city k at time t;

- $I_k(t)$: the number of infected individuals (heroin users) in city k at time t;

- $R_k(t)$: the number of recovered individuals (heroin users under treatment) in city k at time t;

- b_k: the recruitment rate of the population in city k;

- μ_{ki}: the natural death rates of susceptible ($i = 1$), infected ($i = 2$) and recovered ($i = 3$) individuals in city k, respectively, satisfying

$$\mu_{k1} \leq \min(\mu_{k2}, \mu_{k3}) \quad \text{for any} \quad k = 1, 2, \ldots, n; \tag{1.2}$$

- β_{kj}: the transmission parameter between the susceptible individuals in city k and the infected individuals in city j;

- γ_k: the recovery rate of the infected individuals in city k;

- l_{kj}: the per capita rate at which the susceptible individuals in city j leave toward city k ($l_{kk} = 0$);

- m_{kj}: the per capita rate at which the infected individuals in city j leave toward city k ($m_{kk} = 0$);

- n_{kj}: the per capita rate at which the recovered individuals in city j leave toward city k;

- $f_{kj}(\tau)$ ($0 \leq \tau \leq h_1$): the distribution kernel for the time delay of infection such that $\int_0^{h_1} f_{kj}(\tau)d\tau = 1$;

- $g_{kj}(\sigma)$ ($0 \leq \sigma \leq h_2$): the distribution kernel for the time delay of movement such that $\int_0^{h_2} g_{kj}(\sigma)d\sigma = 1$;

- $p_k(\sigma)$ ($0 \leq \sigma \leq h_3$): the distribution kernel for the time delay for which a heroin user under treatment returns to untreated user after cessation of a drug treatment program such that $\int_0^{h_3} p_k(\sigma)d\sigma = 1$.

In what follows, we assume that $g_{kj}(\sigma) = 0$ and $p_k(\sigma) = 0$ outside of each domain. Then, putting $\max(h_2, h_3)$ as h_2 again, we have that $\int_0^{h_2} g_{kj}(\sigma)d\sigma = \int_0^{h_2} p_k(\sigma)d\sigma = 1$. Moreover, we use a locally Lipschitz continuous function $G(I)$ on $[0, +\infty)$ such that

$$\begin{cases} \text{there exists some sufficiently large positive constant } b \text{ such that } G(I) \text{ is monotone} \\ \text{increasing on } [0, b] \text{ and } I/G(I) \text{ is monotone increasing on } (0, +\infty) \text{ and} \\ \lim_{I \to +0} (I/G(I)) = 1. \end{cases} \tag{1.3}$$

Notice that from the condition (1.3), one can see that $G(I) \leq I$ for $I > 0$ and $G(0) = 0$. Using these parameters, we formulate the following multi-group SIR epidemic model with nonlinear incidence rates and distributed delays, which is related to the heroin model:

$$\begin{cases} \dfrac{dS_k(t)}{dt} = b_k - \mu_{k1} S_k(t) - S_k(t) \sum_{j=1}^n \beta_{kj} \int_0^{h_1} f_{kj}(\tau) G(I_j(t-\tau))d\tau \\ \qquad\qquad + \sum_{j=1}^n \left(l_{kj} \int_0^{h_2} g_{kj}(\sigma) S_j(t-\sigma)d\sigma - l_{jk} S_k(t) \right), \\ \dfrac{dI_k(t)}{dt} = S_k(t) \left(\sum_{j=1}^n \beta_{kj} \int_0^{h_1} f_{kj}(\tau) G(I_j(t-\tau))d\tau \right) - (\mu_{k2} + \gamma_k) I_k(t) \\ \qquad\qquad + \gamma_k \int_0^{h_2} p_k(\sigma) e^{-\mu_{k3}\sigma} I_k(t-\sigma)d\sigma \\ \qquad\qquad + \sum_{j=1}^n \left(m_{kj} \int_0^{h_2} g_{kj}(\sigma) I_j(t-\sigma)d\sigma - m_{jk} I_k(t) \right), \\ \dfrac{dR_k(t)}{dt} = \gamma_k I_k(t) - \gamma_k \int_0^{h_2} p_k(\sigma) e^{-\mu_{k3}\sigma} I_k(t-\sigma)d\sigma - \mu_{k3} R_k(t) \\ \qquad\qquad + \sum_{j=1}^n \left(n_{kj} \int_0^{h_2} g_{kj}(\sigma) R_j(t-\sigma)d\sigma - n_{jk} R_k(t) \right), \qquad k = 1, 2, \ldots, n. \end{cases} \tag{1.4}$$

In the model (1.4), the number of newly infected individuals in city k is given by a sum of nonlinear incidence rates $\beta_{kj} S_k(t) \int_0^{h_1} f_{kj}(\tau) G(I_j(t-\tau))d\tau$ for $k, j = 1, 2, \ldots, n$. One can

see that the term $\beta_{kj}S_k(t)\int_0^{h_1}f_{kj}(\tau)G(I_j(t-\tau))d\tau$ with $k \neq j$ describes the effect of cross patch infection between groups k and j, $j \neq k$ who travel shortly from other city j into city k with a time delay $\tau \in [0, h_1]$. On the other hand, the term $\sum_{j=1}^n l_{kj}\int_0^{h_2}g_{kj}(\sigma)S_j(t-\sigma)d\sigma$ (resp. $\sum_{j=1}^n m_{kj}\int_0^{h_2}g_{kj}(\sigma)I_j(t-\sigma)d\sigma$) describes the inflow of susceptible individuals (resp. infected individuals) from all other cities j into city k at time t. The term $\sum_{j=1}^n l_{jk}S_k(t)$ (resp. $\sum_{j=1}^n m_{jk}I_k(t)$) is the outflow of susceptible individuals (resp. infected individuals) from city k towards all other cities j. Once an individual in patch j moves to patch k, then the individual homogeneously mixes with individuals in patch k and is counted as an individual in patch k since there is no track for each individual. By regarding I_k as the number of heroin users and R_k as that of heroin users under treatment, the model (1.4) can be interpreted as the heroin model. In the model (1.4), as in Fang *et al.* [7], we assume that the heroin users under treatment can return to untreated users depending on their different characters and external influences. Such difference can be taken into account by the time delay modulated by the distribution kernel $p_k(\sigma)$.

Since the first two equations in system (1.4) do not contain the variable R_k, $k = 1, 2, \ldots, n$, it is equivalent to

$$
\begin{cases}
\dfrac{dS_k(t)}{dt} = b_k - \mu_{k1}S_k(t) - S_k(t)\sum_{j=1}^n \beta_{kj}\int_0^{h_1}f_{kj}(\tau)G(I_j(t-\tau))d\tau \\
\qquad + \sum_{j=1}^n\left(l_{kj}\int_0^{h_2}g_{kj}(\sigma)S_j(t-\sigma)d\sigma - l_{jk}S_k(t)\right), \\
\dfrac{dI_k(t)}{dt} = S_k(t)\left(\sum_{j=1}^n \beta_{kj}\int_0^{h_1}f_{kj}(\tau)G(I_j(t-\tau))d\tau\right) - (\mu_{k2} + \gamma_k)I_k(t) \\
\qquad + \gamma_k\int_0^{h_2}p_k(\sigma)e^{-\mu_{k3}\sigma}I_k(t-\sigma)d\sigma \\
\qquad + \sum_{j=1}^n\left(m_{kj}\int_0^{h_2}g_{kj}(\sigma)I_j(t-\sigma)d\sigma - m_{jk}I_k(t)\right), \qquad k = 1, 2, \ldots, n.
\end{cases} \tag{1.5}
$$

Let

$$
\kappa_k = \gamma_k\int_0^{h_2}p_k(\sigma)e^{-\mu_{k3}\sigma}d\sigma \ (< \gamma_k), \qquad \tilde{g}_{kk}(\sigma) = \frac{p_k(\sigma)e^{-\mu_{k3}\sigma}}{\int_0^{h_2}p_k(\sigma)e^{-\mu_{k3}\sigma}d\sigma}, \qquad \tilde{\gamma}_k = \gamma_k - \kappa_k \ (> 0).
$$

Then, the second equation in (1.5) becomes

$$
\begin{aligned}
\frac{dI_k(t)}{dt} &= S_k(t)\left(\sum_{j=1}^n \beta_{kj}\int_0^{h_1}f_{kj}(\tau)G(I_j(t-\tau))d\tau\right) - (\mu_{k2} + \tilde{\gamma}_k)I_k(t) \\
&\quad - \kappa_k I_k(t) + \kappa_k\int_0^{h_2}\tilde{g}_{kk}(\sigma)I_k(t-\sigma)d\sigma \\
&\quad + \sum_{j=1, j\neq k}^n\left(m_{kj}\int_0^{h_2}g_{kj}(\sigma)I_j(t-\sigma)d\sigma - m_{jk}I_k(t)\right), \qquad k = 1, 2, \ldots, n.
\end{aligned}
$$

Without loss of generality, we can regard κ_k as m_{kk}, $\tilde{g}_{kk}(\sigma)$ as $g_{kk}(\sigma)$ and since $m_{kk} = 0$ in original and $\int_0^{h_2}\tilde{g}_{kk}(\sigma)d\sigma = 1$. Furthermore, for simplicity, we omit the notation $\sim$ from $\tilde{\gamma}_k$.

Then, we arrive at the following main form of our model:

$$
\begin{cases}
\dfrac{dS_k(t)}{dt} = b_k - \mu_{k1}S_k(t) - S_k(t)\sum_{j=1}^{n}\beta_{kj}\int_0^{h_1}f_{kj}(\tau)G(I_j(t-\tau))d\tau \\[2mm]
\qquad\qquad + \sum_{j=1}^{n}\left(l_{kj}\int_0^{h_2}g_{kj}(\sigma)S_j(t-\sigma)d\sigma - l_{jk}S_k(t)\right), \\[2mm]
\dfrac{dI_k(t)}{dt} = S_k(t)\left(\sum_{j=1}^{n}\beta_{kj}\int_0^{h_1}f_{kj}(\tau)G(I_j(t-\tau))d\tau\right) - (\mu_{k2}+\gamma_k)I_k(t) \\[2mm]
\qquad\qquad + \sum_{j=1}^{n}\left(m_{kj}\int_0^{h_2}g_{kj}(\sigma)I_j(t-\sigma)d\sigma - m_{jk}I_k(t)\right), \qquad k=1,2,\dots,n.
\end{cases}
\tag{1.6}
$$

Note that (1.6) is an extended model of that in Muroya *et al.* [24] in that nonlinear incidence rates with delays are incorporated.

The initial conditions of system (1.6) take the form

$$
S_{k,0} = \phi_1^k \in C\left([-h_2,0],\mathbb{R}_+\right), \quad I_{k,0} = \phi_2^k \in C\left([-h,0],\mathbb{R}_+\right), \quad h = \max\left(h_1,h_2\right), \quad k=1,2,\dots,n.
\tag{1.7}
$$

Moreover, we assume that

$$
\text{the } n\times n \text{ matrix } \mathbf{B} = \left[\beta_{kj}\right]_{n\times n} \text{ is irreducible,}
\tag{1.8}
$$

and there exists a positive vector $(c_1, c_2, \dots, c_n)$ such that

$$
c_k\mu_{k1} + \sum_{j=1}^{n}(l_{jk}c_k - l_{kj}c_j) > 0, \qquad k=1,2,\dots,n.
\tag{1.9}
$$

The last condition (1.9) is used to guarantee the boundedness of the solutions of (1.6) for the delayed terms of patch structure in the proof of Lemma 2.1 (cf. Muroya *et al.* [24]). For example, if $\mu_{k1} + \sum_{j=1}^{n}(l_{jk}-l_{kj}) > 0$, $k=1,2,\dots,n$, then (1.9) is satisfied.

Put

$$
\tilde{l}_{kk} = \sum_{j=1}^{n}(1-\delta_{jk})l_{jk}, \quad \tilde{m}_{kk} = \sum_{j=1}^{n}(1-\delta_{jk})m_{jk}, \quad \delta_{kj} = \begin{cases}1, & \text{if } k=j, \\ 0, & \text{if } k\neq j,\end{cases} \quad k=1,2,\dots,n.
\tag{1.10}
$$

From (1.8), one can see that $\sum_{j=1}^{n}(1-\delta_{kj})\beta_{kj} + \tilde{m}_{kk} > 0$, $k=1,2,\dots,n$. Let $\mathbf{H}$ and $\mathbf{b}$ be an $n\times n$ matrix and a positive n-column vector defined by

$$
\mathbf{H} = \begin{bmatrix}
\mu_{11}+\tilde{l}_{11} & -l_{12} & \cdots & -l_{1n} \\
-l_{21} & \mu_{21}+\tilde{l}_{22} & \cdots & -l_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-l_{n1} & -l_{n2} & \cdots & \mu_{n1}+\tilde{l}_{nn}
\end{bmatrix}
\quad \text{and} \quad
\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},
\tag{1.11}
$$

and $\mathbf{S}^0 = (S_1^0, S_2^0, \dots, S_n^0)^T$ be the positive n-column vector such that

$$
\mathbf{S}^0 = \mathbf{H}^{-1}\mathbf{b}.
\tag{1.12}
$$

By (1.10), $\mathbf{H}$ is an M-matrix (see, e.g., Berman and Plemmons [3] or Varga [31]), and $\mathbf{S}^0$ depends on l_{kj}, $k,j=1,2,\dots,n$.

For $\mathbf{S} = (S_1, S_2, \cdots, S_n)^T$ and $\mathbf{S}^0 = (S_1^0, S_2^0, \cdots, S_n^0)^T$ defined by (1.12), let $\tilde{\mathbf{V}}$ be an $n \times n$ matrix such that

$$\tilde{\mathbf{V}} = \begin{bmatrix} \tilde{V}_1 & 0 & \cdots & 0 \\ 0 & \tilde{V}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{V}_n \end{bmatrix}, \qquad \tilde{V}_k = \mu_{k2} + \gamma_k + \tilde{m}_{kk}, \ k = 1, 2, \ldots, n,$$

and $\tilde{\mathbf{F}}(\mathbf{S})$ be an $n \times n$ matrix such that

$$\tilde{\mathbf{F}}(\mathbf{S}) = \begin{bmatrix} \tilde{F}_{11}(S_1) & \tilde{F}_{12}(S_1) & \cdots & \tilde{F}_{1n}(S_1) \\ \tilde{F}_{21}(S_2) & \tilde{F}_{22}(S_2) & \cdots & \tilde{F}_{2n}(S_2) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{F}_{n1}(S_n) & \tilde{F}_{n2}(S_n) & \cdots & \tilde{F}_{nn}(S_n) \end{bmatrix} = \left[\tilde{F}_{kj}(S_k) \right]_{n \times n},$$

$$\tilde{F}_{kj}(S_k) = \begin{cases} S_k \beta_{kj}, & k = j \\ S_k \beta_{kj} + m_{kj}, & k \neq j. \end{cases}$$

Let us also define $\tilde{\mathbf{M}}(\mathbf{S})$ to be an $n \times n$ matrix such that

$$\tilde{\mathbf{M}}(\mathbf{S}) = \tilde{\mathbf{V}}^{-1} \tilde{\mathbf{F}}(\mathbf{S}) = \left[\tilde{M}_{kj} \right]_{n \times n}, \qquad \tilde{M}_{kj} = \frac{S_k \beta_{kj} + (1 - \delta_{kj}) m_{kj}}{\mu_{k2} + \gamma_k + \tilde{m}_{kk}}, \qquad k, j = 1, 2, \ldots, n.$$

Let the threshold parameter $\tilde{R}_0$ be defined by

$$\tilde{R}_0 = \rho(\tilde{\mathbf{M}}(\mathbf{S}^0)). \tag{1.13}$$

It is easy to see that $\tilde{R}_0$ corresponds to the well-known basic reproduction number R_0 (see for example, Diekmann and Heesterbeek [5]). We now consider the following set $\mathbf{\Gamma}$ defined by

$$\mathbf{\Gamma} = \left\{ (S_1, I_1, S_2, I_2, \ldots, S_n, I_n) \in \mathbb{R}_+^{2n} \mid S_k \leq S_k^0, \ S_k + I_k \leq \bar{N}_k^*, \ k = 1, 2, \ldots, n \right\}, \tag{1.14}$$

where $\bar{N}_k^*, \ k = 1, 2, \ldots, n$ are the positive solutions of the following system:

$$(\mu_{k2} + \tilde{m}_{kk} + m_{kk}) \bar{N}_k^* - \sum_{j=1}^{n} m_{kj} \bar{N}_j^* = \bar{\bar{b}}_k, \qquad k = 1, 2, \ldots, n,$$

and

$$\bar{\bar{b}}_k := b_k + \max\{(\mu_{k2} + \gamma_k + \tilde{m}_{kk}) - (\mu_{k1} + \tilde{l}_{kk}), 0\} S_k^0 + \sum_{j=1}^{n} \max\{l_{kj} - m_{kj}, 0\} S_j^0.$$

Since $\mathbf{\Gamma}$ is a positive invariant set (see Lemma 2.1) for the solutions of (1.6), to choose $\mathbf{\Gamma}$ as the feasible region of (1.6), we need the last part of (1.14) (see the proof of the first part of Theorem 1.1 for $\tilde{R}_0 < 1$ in Section 3). Let $\mathbf{\Gamma}^0$ be the interior of $\mathbf{\Gamma}$.

By (1.8), we have that

$$\tilde{\mathbf{M}}(\mathbf{S}) \text{ is irreducible in } \mathbf{\Gamma}. \tag{1.15}$$

In this paper, we establish the global stability for the multi-group SIR model (1.6) with patch structure. This implies that we extend not only the result of Fang *et al.* [7] for the multi-group heroin model with patch structure, but also the result of Muroya *et al.* [24] for the model with delays and nonlinear incidence rates. Moreover, we offer new techniques to prove the boundedness of the solutions of (1.6), the existence of the endemic equilibrium and permanence of (1.6).

The main theorem in this paper is as follows.

Theorem 1.1.

(i) *For $\tilde{R}_0 < 1$, if there exists a positive n-column vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$ such that*

$$\begin{cases} u_k(\mu_{k1} + \tilde{l}_{kk}) - \displaystyle\sum_{j=1}^{n} u_j(1 - \delta_{jk})l_{jk} \geq 0, \\ u_k(\mu_{k2} + \gamma_k + \tilde{m}_{kk}) - \displaystyle\sum_{j=1}^{n} u_j\{\beta_{jk}S_j^0 + (1 - \delta_{jk})m_{jk}\} > 0, \quad \text{for any } k = 1, 2, \ldots, n, \end{cases} \tag{1.16}$$

then the disease-free equilibrium $\mathbf{E}^0 = (S_1^0, 0, S_2^0, 0, \ldots, S_n^0, 0)$ of (1.6) is globally asymptotically stable in Γ.

(ii) *For $\tilde{R}_0 > 1$, system (1.6) is uniformly persistent in Γ^0 and there exists at least one endemic equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$ in Γ^0. Moreover, if there exists a positive n-column vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ such that*

$$\begin{cases} v_k(\mu_{k1} + \tilde{l}_{kk}) - \displaystyle\sum_{j=1}^{n} v_j(1 - \delta_{jk})l_{jk} \geq 0, \\ v_k(\mu_{k2} + \gamma_k + \tilde{m}_{kk}) - \displaystyle\sum_{j=1}^{n} v_j\left\{\beta_{jk}S_j^* \frac{G(I_k^*)}{I_k^*} + (1 - \delta_{jk})m_{jk}\right\} \geq 0, \quad \text{for any } k = 1, 2, \ldots, n, \end{cases}$$

$$\tag{1.17}$$

then $\mathbf{E}^*$ is globally asymptotically stable in Γ^0.

We note that if $S_k^0 = b_k/\mu_{k1}$ of (1.12), for example, $l_{kj} = 0$ for any $k \neq j$, then for $\tilde{R}_0 = 1$, we can conclude that the disease-free equilibrium $\mathbf{E}^0 = (S_1^0, 0, S_2^0, 0, \ldots, S_n^0, 0)$ of (1.6) is globally asymptotically stable in Γ, because for $S_k^0 = b_k/\mu_{k1}$ of (1.12), we have that by (1.8), $\rho(\tilde{\mathbf{M}}(\mathbf{S})) < \rho(\tilde{\mathbf{M}}(\mathbf{S}^0))$ for any $\mathbf{S} = (S_1, S_2, \ldots, S_n)$ such that $0 < S_k < S_k^0$, $k = 1, 2, \ldots, n$. Otherwise, we can not prove the fact that for $\tilde{R}_0 = 1$, the disease-free equilibrium $\mathbf{E}^0 = (S_1^0, 0, S_2^0, 0, \ldots, S_n^0, 0)$ of (1.6) is globally asymptotically stable in Γ (see Proof of the first part of Theorem 1.1 for $\tilde{R}_0 < 1$ in Section 3).

Corollary 1.2. *Assume (1.15) and $\tilde{R}_0 > 1$. Then, there exists positive n-column vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ such that*

$$\sum_{j=1}^{n} v_j\left\{\beta_{jk}S_j^* \frac{G(I_k^*)}{I_k^*} + (1 - \delta_{jk})m_{jk}\right\} = v_k(\mu_{k2} + \gamma_k + \tilde{m}_{kk}), \qquad k = 1, 2, \ldots, n, \tag{1.18}$$

and for this $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$, if

$$v_k(\mu_{k1} + \tilde{l}_{kk}) - \sum_{j=1}^{n} v_j(1 - \delta_{jk})l_{jk} \geq 0, \quad \text{for any } k = 1, 2, \ldots, n, \tag{1.19}$$

then $\mathbf{E}^*$ is globally asymptotically stable in Γ^0.

If there exists a positive n-column vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ such that

$$\sum_{j=1}^{n} w_j(1 - \delta_{jk})l_{jk} = w_k(\mu_{k1} + \tilde{l}_{kk}), \qquad k = 1, 2, \ldots, n, \tag{1.20}$$

and

$$w_k(\mu_{k2} + \gamma_k + \tilde{m}_{kk}) - \sum_{j=1}^{n} w_j\left\{\beta_{jk}S_j^* \frac{G(I_k^*)}{I_k^*} + (1 - \delta_{jk})m_{jk}\right\} \geq 0, \qquad k = 1, 2, \ldots, n, \tag{1.21}$$

then $\mathbf{E}^*$ is globally asymptotically stable in Γ^0.

The results generalize the known results of a heroin model with two delays considered in the recent literature and imply that nonlinear incidence rates and distributed delays have no influence on the global stability of the model but it depends on patch structure.

The rest of the present paper is organized as follows. In Section 2, we show eventual boundedness of solutions for system (1.6). In Section 3, we prove the global asymptotic stability of the disease-free equilibrium for $\tilde{R}_0 < 1$ (see Theorem 3.1). In Section 4, for $\tilde{R}_0 > 1$, we investigate the existence of the endemic equilibrium $\mathbf{E}^*$ of system (1.6) for $\tilde{R}_0 > 1$ and the permanence of system (1.6). In Section 5, by means of a direct Lyapunov method, under the condition (1.17), we establish the global asymptotic stability of the endemic equilibrium to complete the proof of Theorem 1.1 and Corollary 1.2.

2 Positivity and eventual boundedness of solutions

Adding the first and second equations of (1.6), we have that

$$\frac{d}{dt}\{S_k(t) + I_k(t)\} = \tilde{b}_k(t) - (\mu_{k2} + \tilde{m}_{kk} + m_{kk})\{S_k(t) + I_k(t)\}$$
$$+ \sum_{j=1}^{n} m_{kj} \int_0^{h_2} g_{kj}(\sigma)\{S_j(t-\sigma) + I_j(t-\sigma)\}d\sigma$$

where

$$\tilde{b}_k(t) = b_k + \{((\mu_{k2} + \gamma_k + \tilde{m}_{kk}) - (\mu_{k1} + \tilde{l}_{kk})\}S_k(t) + \sum_{j=1}^{n} (l_{kj} - m_{kj}) \int_0^{h_2} g_{kj}(\sigma)S_j(t-\sigma)d\sigma,$$

$k = 1, 2, \ldots, n$. Thus, for $N_k(t) = S_k(t) + I_k(t)$, $k = 1, 2, \ldots, n$, we have

$$\frac{dN_k(t)}{dt} = \tilde{b}_k(t) - (\mu_{k2} + \tilde{m}_{kk} + m_{kk})N_k(t) + \sum_{j=1}^{n} m_{kj} \int_0^{h_2} g_{kj}(\sigma)N_j(t-\sigma)d\sigma$$

and $\tilde{b}_k(t) \leq \bar{\bar{b}}_k$, $k = 1, 2, \ldots, n$.

The following lemma shows the positivity and eventual boundedness of S_k and I_k, $k = 1, 2, \ldots, n$ of (1.6) (see Muroya *et al.* [24, Lemma 2.1]).

Lemma 2.1. *For system* (1.6), *it holds that*

$$S_k(t) > 0, \ I_k(t) > 0, \quad \text{for any } k = 1, 2, \ldots, n \text{ and } t > 0,$$

and

$$\begin{cases} \limsup_{t \to +\infty} S_k(t) \leq S_k^0, & k = 1, 2, \ldots, n, \\ \limsup_{t \to +\infty} N_k(t) \leq \bar{N}_k^*, & k = 1, 2, \ldots, n, \end{cases} \tag{2.1}$$

and $\mathbf{\Gamma}$ *is a positive invariant set.*

Proof. Suppose that there exist a positive constant t_1 and a positive integer $k_1 \in \{1, 2, \ldots, n\}$ such that $S_{k_1}(t_1) = 0$ and $S_k(t) > 0$ for any $0 < t < t_1$ and $k \in \{1, 2, \cdots, n\}$. On the other hand, by (1.6), we have that $S'_{k_1}(t_1) \geq b_{k_1} > 0$. This is a contradiction to the fact that

$S_{k_1}(t) > 0 = S_{k_1}(t_1)$ for any $0 < t < t_1$. Hence, we obtain that $S_k(t) > 0$ for any $0 < t < +\infty$ and $k = 1, 2, \ldots, n$. By (1.6) and (1.10),

$$
I_k(t) = e^{-(\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk})t} I_k(0) + e^{-(\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk})t} \int_0^t e^{(\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk})u}
$$
$$
\times \left\{ S_k(u) \left(\sum_{j=1}^n \beta_{kj} \int_0^{h_2} g_{kj}(\sigma) G(I_j(u - \sigma)) d\sigma \right) + \sum_{j=1}^n (1 - \delta_{kj}) m_{kj} \int_0^{h_2} g_{kj}(\sigma) I_j(u - \sigma) d\sigma \right\} du
$$

for $k = 1, 2, \ldots, n$ and $t > 0$. This implies that $I_k(t) > 0$ for any $k = 1, 2, \ldots, n$ and $t > 0$.

On the other hand, for $\bar{S}_k := \limsup_{t \to +\infty} S_k(t)$, $k = 1, 2, \ldots, n$, by the equation of S_k', $k = 1, 2, \ldots, n$ of (1.6), one can see that $\bar{S}_k < +\infty$, $k = 1, 2, \ldots, n$. Because otherwise, for the positive constants c_k, $k = 1, 2, \ldots, n$ of (1.9), there exists an integer $\bar{k} \in \{1, 2, \ldots, n\}$ and a sequence $\{t_p\}_{p=1}^{+\infty}$ such that

$$
\lim_{p \to +\infty} \frac{S_{\bar{k}}(t_p)}{c_{\bar{k}}} = +\infty, \qquad \lim_{p \to +\infty} \frac{S_{\bar{k}}'(t_p)}{c_{\bar{k}}} \geq 0, \quad \text{and}
$$
$$
\frac{S_k(t)}{c_k} \leq \frac{S_{\bar{k}}(t_p)}{c_{\bar{k}}}, \quad \text{for any } -h_2 \leq t \leq t_p, \ k = 1, 2 \ldots, n. \tag{2.2}
$$

By the first part of (1.6) and the fluctuation lemma, we have

$$
0 \leq \frac{S_{\bar{k}}'(t_p)}{c_{\bar{k}}} \leq \frac{b_{\bar{k}}}{c_{\bar{k}}} - \mu_{\bar{k}1} \frac{S_{\bar{k}}(t_p)}{c_{\bar{k}}} + \sum_{j=1}^n \left(l_{\bar{k}j} \frac{c_j}{c_{\bar{k}}} \int_0^{h_2} g_{\bar{k}j}(\sigma) \frac{S_j(u - \sigma)}{c_j} d\sigma - l_{j\bar{k}} \frac{S_{\bar{k}}(t_p)}{c_{\bar{k}}} \right)
$$
$$
\leq \frac{b_{\bar{k}}}{c_{\bar{k}}} - \mu_{\bar{k}1} \frac{S_{\bar{k}}(t_p)}{c_{\bar{k}}} + \sum_{j=1}^n l_{\bar{k}j} \frac{c_j}{c_{\bar{k}}} \cdot \frac{S_{\bar{k}}(t_p)}{c_{\bar{k}}} - \left(\sum_{j=1}^n l_{j\bar{k}} \right) \frac{S_{\bar{k}}(t_p)}{c_{\bar{k}}}
$$
$$
= \frac{1}{c_{\bar{k}}} \left[b_{\bar{k}} - \left\{ c_{\bar{k}} \mu_{\bar{k}1} + \sum_{j=1}^n (l_{j\bar{k}} c_{\bar{k}} - l_{\bar{k}j} c_j) \right\} \frac{S_{\bar{k}}(t_p)}{c_{\bar{k}}} \right],
$$

from which we obtain

$$
\frac{S_{\bar{k}}(t_p)}{c_{\bar{k}}} \leq \frac{b_{\bar{k}}}{c_{\bar{k}} \mu_{\bar{k}1} + \sum_{j=1}^n (l_{j\bar{k}} c_{\bar{k}} - l_{\bar{k}j} c_j)} < +\infty, \qquad p = 1, 2, \ldots
$$

This is a contradiction to (2.2).

By (1.12) and the fact that $\mathbf{H}$ defined by (1.11) is an M-matrix, we obtain $\bar{S}_k \leq S_k^0$, $k = 1, 2, \ldots, n$. Thus,

$$
\limsup_{t \to +\infty} S_k(t) \leq S_k^0, \qquad k = 1, 2, \ldots, n,
$$

which is the first part of (2.1), and $\bar{\bar{b}} < +\infty$, $k = 1, 2, \ldots, n$.

Next, for the solution $\bar{N}_k(t)$ ($k = 1, 2, \ldots, n$) of the following system:

$$
\frac{d\bar{N}_k(t)}{dt} = \bar{\bar{b}}_k - (\mu_{k2} + \tilde{m}_{kk} + m_{kk}) \bar{N}_k(t) + \sum_{j=1}^n m_{kj} \int_0^{h_2} g_{kj}(\sigma) \bar{N}_j(t - \sigma) d\sigma, \quad k = 1, 2, \ldots, n, \tag{2.3}
$$

let us consider the following Lyapunov functional:

$$
\begin{cases}
U_{\bar{N}}(t) := \sum_{k=1}^n \left\{ \bar{N}_k^* g \left(\frac{\bar{N}_k(t)}{\bar{N}_k^*} \right) + \sum_{j=1}^n m_{kj} \bar{N}_j^* \int_0^{h_2} g_{kj}(\sigma) \int_{t-\sigma}^t g(n_j(u)) du d\sigma \right\}, \\
g(x) := x - 1 - \ln x \geq g(1) = 0, \quad \text{for } x > 0.
\end{cases}
$$

For the usage of the function $g(x)$ in Lyapunov functions, see McCluskey [21] for instance. We show that

$$\frac{d\bar{U}_N(t)}{dt} \leq -\sum_{k=1}^{n}\left\{\mu_{k2}\bar{N}_k^* g(n_k(t)) + \bar{\bar{b}}_k g\left(\frac{1}{n_k(t)}\right)\right\} \leq 0, \ n_k(t) := \frac{\bar{N}_k(t)}{\bar{N}_k^*}, \ k = 1,2,\ldots,n \quad (2.4)$$

and

$$\lim_{t\to+\infty}\bar{N}_k(t) = \bar{N}_k^*, \quad k = 1,2,\ldots,n. \quad (2.5)$$

Differentiating $\bar{U}_N$ along the solution of (2.3) and using the equilibrium condition $\bar{\bar{b}}_k = (\mu_{k2} + \tilde{m}_{kk} + m_{kk})\bar{N}_k^* - \sum_{j=1}^{n}m_{kj}\bar{N}_j^*$, $k = 1,2,\ldots,n$, we obtain

$$\frac{d\bar{U}_N(t)}{dt} = \sum_{k=1}^{n}\left\{\left(1 - \frac{\bar{N}_k^*}{\bar{N}_k(t)}\right)\frac{d\bar{N}_k(t)}{dt} + \sum_{j=1}^{n}m_{kj}\bar{N}_j^*\int_0^{h_2}g_{kj}(\sigma)\{g(n_j(t)) - g(n_j(t-\sigma))\}d\sigma\right\},$$

and

$$\left(1 - \frac{\bar{N}_k^*}{\bar{N}_k(t)}\right)\frac{d\bar{N}_k(t)}{dt}$$

$$= \left(1 - \frac{\bar{N}_k^*}{\bar{N}_k(t)}\right)\left\{\bar{\bar{b}}_k - (\mu_{k2} + \tilde{m}_{kk} + m_{kk})\bar{N}_k(t) + \sum_{j=1}^{n}m_{kj}\int_0^{h_2}g_{kj}(\sigma)\bar{N}_j(t-\sigma)d\sigma\right\}$$

$$= \left(1 - \frac{\bar{N}_k^*}{\bar{N}_k(t)}\right)\left\{-(\mu_{k2} + \tilde{m}_{kk} + m_{kk})\{\bar{N}_k(t) - \bar{N}_k^*\} + \sum_{j=1}^{n}m_{kj}\int_0^{h_2}g_{kj}(\sigma)\{\bar{N}_j(t-\sigma) - N_j^*\}d\sigma\right\}$$

$$= \left(1 - \frac{1}{n_k(t)}\right)\left\{-(\mu_{k2} + \tilde{m}_{kk} + m_{kk})\bar{N}_k^*\{n_k(t) - 1\} + \sum_{j=1}^{n}m_{kj}\bar{N}_j^*\int_0^{h_2}g_{kj}(\sigma)\{n_j(t-\sigma) - 1\}d\sigma\right\}.$$

It is easy to check that the following equalities hold:

$$\left(1 - \frac{1}{n_k(t)}\right)\{n_k(t) - 1\} = g(n_k(t)) + g\left(\frac{1}{n_k(t)}\right),$$

and

$$\left(1 - \frac{1}{n_k(t)}\right)\{n_j(t-\sigma) - 1\} = g(n_j(t-\sigma)) - g\left(\frac{n_j(t-\sigma)}{n_k(t)}\right) + g\left(\frac{1}{n_k(t)}\right).$$

It follows that

$$\left(1 - \frac{\bar{N}_k^*}{\bar{N}_k(t)}\right)\frac{d\bar{N}_k(t)}{dt}$$

$$= -(\mu_{k2} + \tilde{m}_{kk} + m_{kk})\bar{N}_k^*\left\{g(n_k(t)) + g\left(\frac{1}{n_k(t)}\right)\right\}$$

$$+ \sum_{j=1}^{n}m_{kj}\bar{N}_j^*\int_0^{h_2}g_{kj}(\sigma)\left\{g(n_j(t-\sigma)) - g\left(\frac{n_j(t-\sigma)}{n_k(t)}\right) + g\left(\frac{1}{n_k(t)}\right)\right\}d\sigma.$$

Thus, we have

$$
\begin{aligned}
\frac{d\bar{U}_N(t)}{dt} &= \sum_{k=1}^{n}\Bigg[-(\mu_{k2}+\tilde{m}_{kk}+m_{kk})\bar{N}_k^*\Big\{g(n_k(t))+g\Big(\frac{1}{n_k(t)}\Big)\Big\} \\
&\quad + \sum_{j=1}^{n} m_{kj}\bar{N}_j^* \int_0^{h_2} g_{kj}(\sigma)\Big\{g(n_j(t-\sigma))-g\Big(\frac{n_j(t-\sigma)}{n_k(t)}\Big)+g\Big(\frac{1}{n_k(t)}\Big)\Big\}d\sigma \\
&\quad + \sum_{j=1}^{n} m_{kj}\bar{N}_j^* \int_0^{h_2} g_{kj}(\sigma)\{g(n_j(t))-g(n_j(t-\sigma))\}d\sigma\Bigg] \\
&= \sum_{k=1}^{n}\Bigg[-(\mu_{k2}+\tilde{m}_{kk}+m_{kk})\bar{N}_k^*\Big\{g(n_k(t))+g\Big(\frac{1}{n_k(t)}\Big)\Big\} \\
&\quad + \sum_{j=1}^{n} m_{kj}\bar{N}_j^*\Big\{g(n_j(t))-\int_0^{h_2} g_{kj}(\sigma)g\Big(\frac{n_j(t-\sigma)}{n_k(t)}\Big)d\sigma+g\Big(\frac{1}{n_k(t)}\Big)\Big\}\Bigg].
\end{aligned}
$$

Since $\sum_{j=1}^{n} m_{kj}\bar{N}_j^* = (\mu_{k2}+\tilde{m}_{kk}+m_{kk})\bar{N}_k^* - \bar{\bar{b}}_k$, $k=1,2,\ldots,n$, we obtain

$$
\begin{aligned}
\sum_{k=1}^{n}\sum_{j=1}^{n} m_{kj}\bar{N}_j^*\Big\{g(n_j(t))&+g\Big(\frac{1}{n_k(t)}\Big)\Big\} \\
&= \sum_{k=1}^{n}\Big(\sum_{j=1}^{n} m_{jk}\Big)N_k^* g(n_k(t)) + \sum_{k=1}^{n}\Big(\sum_{j=1}^{n} m_{kj}\bar{N}_j^*\Big)g\Big(\frac{1}{n_k(t)}\Big) \\
&= \sum_{k=1}^{n}(\tilde{m}_{kk}+m_{kk})\bar{N}_k^* g(n_k(t)) + \sum_{k=1}^{n}\Big(\sum_{j=1}^{n} m_{kj}\bar{N}_j^*\Big)g\Big(\frac{1}{n_k(t)}\Big) \\
&= \sum_{k=1}^{n}(\tilde{m}_{kk}+m_{kk})\bar{N}_k^* g(n_k(t)) + \sum_{k=1}^{n}\{(\mu_{k2}+\tilde{m}_{kk}+m_{kk})\bar{N}_k^* - \bar{\bar{b}}_k\}g\Big(\frac{1}{n_k(t)}\Big).
\end{aligned}
$$

Hence, we obtain (2.4), which implies (2.5). By the comparison principle, we obtain the second part of (2.1).

By the first and second part of (2.1), it is evident that Γ is a positive invariant set. Thus, we obtain the last part of (2.1). This completes the proof. □

Lemma 2.2. *For any solution of system* (1.6) *with initial condition* (1.7), *it holds that*

$$
\liminf_{t\to+\infty} S_k(t) \geq \hat{\underline{S}}_k := \frac{b_k}{\mu_{k1}+\tilde{l}_{kk}+\sum_{j=1}^{n}\beta_{kj}G(\bar{N}_j^*)}, \qquad k=1,2,\ldots,n.
$$

Proof. Let $(S_1(t),I_1(t),R_1(t),S_2(t),I_2(t),R_2(t),\ldots,S_n(t),I_n(t),R_n(t))$ be any solution of system (1.6) with initial condition (1.7). By (2.1), it holds that $\limsup_{t\to+\infty} I_k(t) \leq \bar{N}_k^*$, $k=1,2,\ldots,n$. This implies that $\varepsilon > 0$ sufficiently small, there is a $T_1 > 0$ such that $I_k(t) < \bar{N}_k^* + \varepsilon$ for $t > T_1$, $k=1,2,\ldots,n$. Therefore, from the first part of the hypothesis (1.3) that there exists some sufficiently large positive constant b such that $G(I)$ is monotone increasing on $[0,b]$, we derive

$$
\frac{dS_k(t)}{dt} \geq b_k - \Big\{\mu_{k1}+\tilde{l}_{kk}+\sum_{j=1}^{n}\beta_{kj}G\big(\bar{N}_j^*+\varepsilon\big)\Big\}S_k(t), \qquad k=1,2,\ldots,n,
$$

which implies that

$$
\liminf_{t\to+\infty} S_k(t) \geq \frac{b_k}{\mu_{k1}+\tilde{l}_{kk}+\sum_{j=1}^{n}\beta_{kj}G(\bar{N}_j^*+\varepsilon)}, \qquad k=1,2,\ldots,n.
$$

Since the above inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, it immediately follows that

$$\liminf_{t \to +\infty} S_k(t) \geq \hat{\underline{S}}_k, \qquad k = 1, 2, \ldots, n,$$

which completes the proof. □

By Lemma 2.2, for any small positive constant $\varepsilon < \hat{\underline{S}}_k$, there exists a positive constant T_1 such that

$$S_k(t) > \hat{\underline{S}}_k - \varepsilon > 0, \qquad k = 1, 2, \ldots, n, \quad \text{for any } t \geq T_1. \tag{2.6}$$

3 Global stability of the disease-free equilibrium $\mathbf{E}^0$ for $\tilde{R}_0 < 1$

In this section, for $\tilde{R}_0 < 1$, we prove the first part of Theorem 1.1 by means of Lyapunov functional method. We note that there always exists the disease-free equilibrium $\mathbf{E}^0 = (S_1^0, 0, S_2^0, 0, \ldots, S_n^0, 0)$ of (1.6) in Γ such that

$$b_k = (\mu_{k1} + \tilde{l}_{kk})S_k^0 - \sum_{j=1}^n (1 - \delta_{kj})l_{kj}S_j^0, \qquad k = 1, 2, \ldots, n \tag{3.1}$$

holds. We rewrite (1.6) as

$$\begin{cases} \dfrac{dS_k(t)}{dt} = b_k - (\mu_{k1} + \tilde{l}_{kk} + l_{kk})S_k(t) \\ \qquad - \displaystyle\sum_{j=1}^n \left(\beta_{kj}S_k(t) \int_0^{h_1} f_{kj}(\tau)G(I_j(t-\tau))d\tau - l_{kj} \int_0^{h_2} g_{kj}(\sigma)S_j(t-\sigma)d\sigma \right), \\ \dfrac{dI_k(t)}{dt} = \displaystyle\sum_{j=1}^n \left(\beta_{kj}S_k(t) \int_0^{h_1} f_{kj}(\tau)G(I_j(t-\tau))d\tau + m_{kj} \int_0^{h_2} g_{kj}(\sigma)I_j(t-\sigma)d\sigma \right) \\ \qquad - (\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk})I_k(t), \qquad k = 1, 2 \ldots, n. \end{cases} \tag{3.2}$$

Proof of the first part of Theorem 1.1 for $\tilde{R}_0 < 1$. It is sufficient to show that the disease-free equilibrium $\mathbf{E}^0 = (S_1^0, 0, S_2^0, 0, \ldots, S_n^0, 0)$ of (1.6) is globally asymptotically stable.

Assume that (1.16) holds and for this positive vector $(u_1, u_2, \ldots, u_n)$, consider the following Lyapunov functional:

$$W(t) = \sum_{k=1}^n u_k \left[S_k^0 g\left(\frac{S_k(t)}{S_k^0} \right) + I_k(t) + \sum_{j=1}^n \left\{ \beta_{kj}S_k^0 \int_0^{h_1} f_{kj}(\tau) \int_{t-\tau}^t G(I_j(u))du\,d\tau \right. \right.$$

$$\left. \left. + l_{kj}S_j^0 \int_0^{h_2} g_{kj}(\sigma) \int_{t-\sigma}^t g\left(\frac{S_j(u)}{S_j^0} \right)du\,d\sigma + m_{kj} \int_0^{h_2} g_{kj}(\sigma) \int_{t-\sigma}^t I_j(u)du\,d\sigma \right\} \right],$$

$$g(x) = x - 1 - \ln x \geq g(1) = 0, \quad \text{for any } x > 0.$$

Differentiating $W(t)$ along the solution of (1.6), we have

$$\frac{dW(t)}{dt} = \sum_{k=1}^{n} u_k \left\{ \left(1 - \frac{S_k^0}{S_k(t)} \right) \frac{dS_k(t)}{dt} + \frac{dI_k(t)}{dt} \right\}$$

$$+ \sum_{k=1}^{n} u_k \sum_{j=1}^{n} \left[\beta_{kj} S_k^0 \int_0^{h_1} f_{kj}(\tau) \{ G(I_j(t)) - G(I_j(t-\tau)) \} d\tau \right.$$

$$+ l_{kj} S_j^0 \int_0^{h_2} g_{kj}(\sigma) \left\{ g\left(\frac{S_j(t)}{S_j^0} \right) - g\left(\frac{S_j(t-\sigma)}{S_j^0} \right) \right\} d\sigma$$

$$\left. + m_{kj} \int_0^{h_2} g_{kj}(\sigma) \{ I_j(t) - I_j(t-\sigma) \} d\sigma \right].$$

Put

$$x_k^0(t) = \frac{S_k(t)}{S_k^0}, \qquad k = 1, 2, \ldots, n.$$

By (3.1) and (3.2), we have that

$$\frac{dS_k(t)}{dt} = b_k - (\mu_{k1} + \tilde{l}_{kk} + l_{kk}) S_k(t)$$

$$- \sum_{j=1}^{n} \left(\beta_{kj} S_k(t) \int_0^{h_1} f_{kj}(\tau) G(I_j(t-\tau)) d\tau - l_{kj} \int_0^{h_2} g_{kj}(\sigma) S_j(t-\sigma) d\sigma \right)$$

$$= -(\mu_{k1} + \tilde{l}_{kk} + l_{kk})(S_k(t) - S_k^0)$$

$$- \sum_{j=1}^{n} \left(\beta_{kj} S_k(t) \int_0^{h_1} f_{kj}(\tau) G(I_j(t-\tau)) d\tau - l_{kj} \int_0^{h_2} g_{kj}(\sigma)(S_j(t-\sigma) - S_j^0) d\sigma \right),$$

and

$$\frac{dI_k(t)}{dt} = \sum_{j=1}^{n} \left(\beta_{kj} S_k(t) \int_0^{h_1} f_{kj}(\tau) G(I_j(t-\tau)) d\tau + m_{kj} \int_0^{h_2} g_{kj}(\sigma) I_j(t-\sigma) d\sigma \right)$$

$$- (\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk}) I_k(t).$$

Then,

$$\frac{dW(t)}{dt} = \sum_{k=1}^{n} u_k \left[\left(1 - \frac{S_k^0}{S_k(t)} \right) \left\{ -(\mu_{k1} + \tilde{l}_{kk} + l_{kk})(S_k(t) - S_k^0) \right. \right.$$

$$- \sum_{j=1}^{n} \left(\beta_{kj} S_k(t) \int_0^{h_1} f_{kj}(\tau) G(I_j(t-\tau)) d\tau \right.$$

$$\left. \left. - l_{kj} \int_0^{h_2} g_{kj}(\sigma)(S_j(t-\sigma) - S_j^0) d\sigma \right) \right\}$$

$$+ \left\{ \sum_{j=1}^{n} \left(\beta_{kj} S_k(t) \int_0^{h_1} f_{kj}(\tau) G(I_j(t-\tau)) d\tau + m_{kj} \int_0^{h_2} g_{kj}(\sigma) I_j(t-\sigma) d\sigma \right) \right.$$

$$\left. - (\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk}) I_k(t) \right\}$$

$$+ \sum_{j=1}^{n} \left(\beta_{kj} S_k^0 \int_0^{h_1} f_{kj}(\tau) \{ G(I_j(t)) - G(I_j(t-\tau)) \} d\tau \right.$$

$$+ l_{kj} S_j^0 \int_0^{h_2} g_{kj}(\sigma) \{ g(x_j^0(t)) - g(x_j^0(t - \sigma)) \} d\sigma$$

$$+ m_{kj} \int_0^{h_2} g_{kj}(\sigma) \{ I_j(t) - I_j(t - \sigma) \} d\sigma \Bigg) \Bigg]$$

$$= \sum_{k=1}^n u_k \Bigg[\left(1 - \frac{S_k^0}{S_k(t)} \right) \Bigg\{ -(\mu_{k1} + \tilde{l}_{kk} + l_{kk}) S_k^0 \left(\frac{S_k(t)}{S_k^0} - 1 \right)$$

$$+ \sum_{j=1}^n l_{kj} S_j^0 \int_0^{h_2} g_{kj}(\sigma) \left(\frac{S_j(t - \sigma)}{S_j^0} - 1 \right) d\sigma \Bigg\}$$

$$- \sum_{j=1}^n \left(\beta_{kj} S_k(t) \int_0^{h_1} f_{kj}(\tau) G(I_j(t - \tau)) d\tau - \beta_{kj} S_k^0 \int_0^{h_1} f_{kj}(\tau) G(I_j(t - \tau)) d\tau \right)$$

$$+ \Bigg\{ \sum_{j=1}^n \left(\beta_{kj} S_k(t) \int_0^{h_1} f_{kj}(\tau) G(I_j(t - \tau)) d\tau + m_{kj} \int_0^{h_2} g_{kj}(\sigma) I_j(t - \sigma) d\sigma \right)$$

$$- (\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk}) I_k(t) \Bigg\}$$

$$+ \sum_{j=1}^n \left(\beta_{kj} S_k^0 \int_0^{h_1} f_{kj}(\tau) \{ G(I_j(t)) - G(I_j(t - \tau)) \} d\tau \right.$$

$$+ l_{kj} S_j^0 \int_0^{h_2} g_{kj}(\sigma) \{ g(x_j^0(t)) - g(x_j^0(t - \sigma)) \} d\sigma$$

$$+ m_{kj} \int_0^{h_2} g_{kj}(\sigma) \{ I_j(t) - I_j(t - \sigma) \} d\sigma \Bigg) \Bigg]$$

$$= -\sum_{k=1}^n u_k \Bigg[(\mu_{k1} + \tilde{l}_{kk} + l_{kk}) S_k^0 \left(1 - \frac{1}{x_k^0(t)} \right) (x_k^0(t) - 1)$$

$$- \sum_{j=1}^n l_{kj} S_j^0 \int_0^{h_2} g_{kj}(\sigma) \left(1 - \frac{1}{x_k^0(t)} \right) \left(x_j^0(t - \sigma) - 1 \right) d\sigma \Bigg]$$

$$+ \sum_{k=1}^n u_k \Bigg\{ \sum_{j=1}^n \left(\beta_{kj} S_k^0 G(I_j(t)) + (1 - \delta_{kj}) m_{kj} I_j(t) \right) - (\mu_{k2} + \gamma_k + \tilde{m}_{kk}) I_k(t) \Bigg\}$$

$$+ \sum_{k=1}^n u_k \sum_{j=1}^n l_{kj} S_j^0 \int_0^{h_2} g_{kj}(\sigma) \{ g(x_j^0(t)) - g(x_j^0(t - \sigma)) \} d\sigma. \tag{3.3}$$

Let us now consider the first part of the last equation in (3.3). Since

$$\left(1 - \frac{1}{x_k^0(t)} \right) (x_k^0(t) - 1) = x_k^0(t) + \frac{1}{x_k^0(t)} - 2 = g(x_k^0(t)) + g\left(\frac{1}{x_k^0(t)} \right)$$

and

$$\left(1 - \frac{1}{x_k^0(t)} \right) (x_j^0(t - \sigma) - 1) = x_j^0(t - \sigma) - \frac{x_j^0(t - \sigma)}{x_k^0(t)} + \frac{1}{x_j^0(t)} - 1$$

$$= g(x_j^0(t - \sigma)) - g\left(\frac{x_j^0(t - \sigma)}{x_k^0(t)} \right) + g\left(\frac{1}{x_k^0(t)} \right),$$

and by (1.10), it holds that

$$\sum_{k=1}^{n} u_k(\mu_{k1} + \tilde{l}_{kk} + l_{kk})S_k^0\left(1 - \frac{1}{x_k^0(t)}\right)(x_k^0(t) - 1)$$

$$= \sum_{k=1}^{n} u_k(\mu_{k1} + \tilde{l}_{kk} + l_{kk})S_k^0\left\{g(x_k^0(t)) + g\left(\frac{1}{x_k^0(t)}\right)\right\}, \tag{3.4}$$

and

$$\sum_{k=1}^{n} u_k \sum_{j=1}^{n} l_{kj}S_j^0 \int_0^{h_2} g_{kj}(\sigma)\left(1 - \frac{1}{x_k^0(t)}\right)(x_j^0(t - \sigma) - 1)d\sigma$$

$$= \sum_{k=1}^{n} u_k \sum_{j=1}^{n} l_{kj}S_j^0 \int_0^{h_2} g_{kj}(\sigma)\left\{g(x_j^0(t - \sigma)) - g\left(\frac{x_j^0(t - \sigma)}{x_k^0(t)}\right) + g\left(\frac{1}{x_k^0(t)}\right)\right\}d\sigma. \tag{3.5}$$

It follows from (3.3), (3.4) and (3.5) that

$$\frac{dW(t)}{dt} = -\sum_{k=1}^{n} u_k(\mu_{k1} + \tilde{l}_{kk} + l_{kk})S_k^0\left\{g(x_k^0(t)) + g\left(\frac{1}{x_k^0(t)}\right)\right\}$$

$$+ \sum_{k=1}^{n} u_k \sum_{j=1}^{n} l_{kj}S_j^0 \int_0^{h_2} g_{kj}(\sigma)\left\{g(x_j^0(t - \sigma)) - g\left(\frac{x_j^0(t - \sigma)}{x_k^0(t)}\right) + g\left(\frac{1}{x_k^0(t)}\right)\right\}d\sigma$$

$$+ \sum_{k=1}^{n} u_k\left\{\sum_{j=1}^{n}\left(\beta_{kj}S_k^0 G(I_j(t)) + (1 - \delta_{kj})m_{kj}I_j(t)\right) - (\mu_{k2} + \gamma_k + \tilde{m}_{kk})I_k(t)\right\}$$

$$+ \sum_{k=1}^{n} u_k \sum_{j=1}^{n} l_{kj}S_j^0 \int_0^{h_2} g_{kj}(\sigma)\{g(x_j^0(t)) - g(x_j^0(t - \sigma))\}d\sigma$$

$$= -\sum_{k=1}^{n}\left(u_k(\mu_{k1} + \tilde{l}_{kk}) - \sum_{j=1}^{n} u_j(1 - \delta_{jk})l_{jk}\right)S_k^0 g(x_k^0(t))$$

$$- \sum_{k=1}^{n} u_k\left\{(\mu_{k1} + \tilde{l}_{kk})S_k^0 - \sum_{j=1}^{n}(1 - \delta_{kj})l_{kj}S_j^0\right\}g\left(\frac{1}{x_k^0(t)}\right)$$

$$- \sum_{k=1}^{n} u_k \sum_{j=1}^{n}(1 - \delta_{kj})l_{kj}S_j^0 g\left(\frac{x_j^0(t - \sigma)}{x_k^0(t)}\right)$$

$$+ \sum_{k=1}^{n} u_k\left\{\sum_{j=1}^{n}\left(\beta_{kj}S_k^0 + (1 - \delta_{kj})m_{kj}\right)I_j(t) - (\mu_{k2} + \gamma_k + \tilde{m}_{kk})I_k(t)\right\}$$

$$+ \sum_{k=1}^{n} u_k \sum_{j=1}^{n} \beta_{kj}S_k^0\{G(I_j(t)) - I_j(t)\}$$

$$= -\sum_{k=1}^{n}\left(u_k(\mu_{k1} + \tilde{l}_{kk}) - \sum_{j=1}^{n} u_j(1 - \delta_{jk})l_{jk}\right)S_k^0 g(x_k^0(t)) - \sum_{k=1}^{n} u_k b_k g\left(\frac{1}{x_k^0(t)}\right)$$

$$- \sum_{k=1}^{n} u_k \sum_{j=1}^{n}(1 - \delta_{kj})l_{kj}S_j^0 g\left(\frac{x_j^0(t - \sigma)}{x_k^0(t)}\right)$$

$$+ \sum_{k=1}^{n}\left\{\sum_{j=1}^{n} u_j\left(\beta_{jk}S_j^0 + (1 - \delta_{jk})m_{jk}\right) - u_k(\mu_{k2} + \gamma_k + \tilde{m}_{kk})\right\}I_k(t)$$

$$+ \sum_{k=1}^{n} u_k \sum_{j=1}^{n} \beta_{kj}S_k^0\{G(I_j(t)) - I_j(t)\} \leq 0. \tag{3.6}$$

Let us note that we make use of assumption (1.2), (1.16) and $G(I_j(t)) \leq I_j(t)$.

It is obvious from (3.6) that $\mathcal{R}_0 < 1$ if and only if $S_k(t) = S_k^0$ and $I_k(t) \equiv 0$, $k = 1, 2, \ldots, n$. Thus, it follows from the classical LaSalle's invariance principle (see [13]) that the disease-free equilibrium $\mathbf{E}^0$ of the system (1.6) is globally asymptotically stable. Hence, we obtain the first part $\tilde{R}_0 < 1$ of Theorem 1.1.   □

It should also be noted that if $\mathcal{R}_0 = 1$, then from (3.6), we can not show $S_k(t) = S_k^0$ and $I_k(t) \equiv 0$, $k = 1, 2, \ldots, n$.

4   Existence of the endemic equilibrium and permanence for $\tilde{R}_0 > 1$

In this section, our attention is focused on the case $\tilde{R}_0 > 1$. First, we show the existence of an endemic equilibrium $\mathbf{E}^*$ and the second, we show the permanence of system (1.6). Finally, we prove the global asymptotic stability of it by applying Lyapunov functional method.

By (1.13), it follows from the Perron–Frobenius theorem that $\tilde{R}_0 = \rho(\tilde{\mathbf{M}}(\mathbf{S}^0))$ is an eigenvalue of matrix $\tilde{\mathbf{M}}(\mathbf{S}^0)$ associated with a positive (right) eigenvector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ such that $\omega_k > 0$, $k = 1, 2, \ldots, n$. For such ω, we have

$$\tilde{\mathbf{M}}(\mathbf{S}^0)\omega = \rho(\tilde{\mathbf{M}}(\mathbf{S}^0))\omega,$$

which is equivalent to

$$\sum_{j=1}^{n}\{S_k^0\beta_{kj} + (1 - \delta_{kj})m_{kj}\}\omega_j - \rho(\tilde{\mathbf{M}}(\mathbf{S}^0))(\mu_{k2} + \gamma_k + m_{kk})\omega_k = 0.$$

Thus, by $\rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) = \tilde{R}_0 > 1$, it holds that

$$\sum_{j=1}^{n}\{\beta_{kj}S_k^0 + (1 - \delta_{kj})m_{kj}\}\omega_j - (\mu_{k2} + \gamma_k + \tilde{m}_{kk})\omega_k > 0, \qquad k = 1, 2, \ldots, n.$$

4.1   Existence of the endemic equilibrium

First, we prove the existence of an endemic equilibrium $\mathbf{E}^*$ of system (1.6) when $\tilde{R}_0 > 1$.

Lemma 4.1. *Let $\tilde{R}_0$ and Γ be defined by (1.13) and (1.14), respectively. If $\tilde{R}_0 > 1$, then system (1.6) has an endemic equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*) \in \Gamma^0$ such that*

$$\begin{cases} b_k = (\mu_{k1} + \tilde{l}_{kk})S_k^* + \displaystyle\sum_{j=1}^{n}\beta_{kj}S_k^*G(I_j^*) - \sum_{j=1}^{n}(1 - \delta_{kj})l_{kj}S_j^*, \\[2ex] (\mu_{k2} + \gamma_k + \tilde{m}_{kk})I_k^* = \displaystyle\sum_{j=1}^{n}\beta_{kj}S_k^*G(I_j^*) + \sum_{j=1}^{n}(1 - \delta_{kj})m_{kj}I_j^*, \qquad k = 1, 2, \ldots, n. \end{cases} \tag{4.1}$$

Proof. Since $\tilde{R}_0 > 1$ implies that for $\rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) > 1$, there exists a positive eigenvector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\tilde{\mathbf{M}}(\mathbf{S}^0)\omega = \rho(\tilde{\mathbf{M}}(\mathbf{S}^0))\omega$, where $S_k^0 \leq \bar{N}_k^*$, $k = 1, 2, \ldots, n$ and $\tilde{F}(\mathbf{S}^0) = \left[S_k^0\beta_{kj} + (1 - \delta_{kj})m_{kj}\right]_{n \times n}$ and $\tilde{V} = \left[\delta_{kj}(\mu_{k2} + \gamma_k + \tilde{m}_{kk})\right]_{n \times n}$ are $n \times n$ matrices.

Since by Lemma 2.1, $S_k^* + I_k^* = N_k^* = S_k^0$, $k = 1, 2, \ldots, n$ if the endemic equilibrium exists, we may consider the existence of solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of the system $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, where

$$\begin{cases} \mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))^T = \left[f_k(\mathbf{x})\right]_{n \times 1}, \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)^T, \\[2ex] f_k(\mathbf{x}) = -\left(S_k(\mathbf{x})\displaystyle\sum_{j=1}^{n}\beta_{kj}G(x_j) - (\mu_{k2} + \gamma_k + \tilde{m}_{kk})x_k + \sum_{j=1}^{n}(1 - \delta_{kj})m_{kj}x_j\right) \end{cases}$$

under the conditions $0 < x_k < \bar{N}_k^*$ and

$$\left(\sum_{j=1}^{n} \beta_{kj} G(x_j) + (\mu_{k1} + \tilde{l}_{kk}) \right) S_k(\mathbf{x}) - \sum_{j=1}^{n} (1 - \delta_{kj}) l_{kj} S_j(\mathbf{x}) - b_k = 0$$

for $k = 1, 2, \ldots, n$. Let us also note that x_k $(k = 1, 2, \ldots, n)$ correspond to the number of infected individual for patch k $(k = 1, 2, \ldots, n)$ appeared in the endemic equilibrium $\mathbf{E}^*$. One can see that

$$\mathbf{F}(\mathbf{x}) = - \left[S_k(\mathbf{x}) \sum_{j=1}^{n} \beta_{kj} G(x_j) - (\mu_{k2} + \gamma_k + \tilde{m}_{kk}) x_k + \sum_{j=1}^{n} (1 - \delta_{kj}) m_{kj} x_j \right]_{n \times 1}$$

$$= - (\tilde{F}(\mathbf{S}^0) \mathbf{x} - \tilde{V} \mathbf{x}) - \left[S_k(\mathbf{x}) \beta_{kj} \frac{G(x_j)}{x_j} - S_k^0 \beta_{kj} \right]_{n \times n} \mathbf{x},$$

and hence,

$$\begin{cases} \mathbf{F}(\omega) = - (\tilde{F}(\mathbf{S}^0) \omega - \tilde{V} \omega) - \left[S_k(\omega) \beta_{kj} \dfrac{G(\omega_j)}{\omega_j} - S_k^0 \beta_{kj} \right]_{n \times n} \omega, \quad \text{and} \\ - (\tilde{F}(\mathbf{S}^0) \omega - \tilde{V} \omega) = - \tilde{V} \{ \tilde{\mathbf{M}}(\mathbf{S}^0) \omega - \omega \} = - \tilde{V} \{ \rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) - 1 \} \omega < \mathbf{0}, \end{cases}$$

where the ordering of vectors in $\mathbb{R}^n$ is the usual element-wise one in $\mathbb{R}^n$.

For any $\alpha > 0$, we have

$$\mathbf{F}(\alpha \omega) = \alpha \left\{ - (\tilde{F}(\mathbf{S}^0) \omega - \tilde{V} \omega) - \left[S_k(\alpha \omega) \beta_{kj} \frac{G(\alpha \omega_j)}{\alpha \omega_j} - S_k^0 \beta_{kj} \right]_{n \times n} \omega \right\},$$

and by assumption (1.3) and (2.1), $\lim_{I \to +0} I / G(I) = 1$ and by (1.12) and the last part of (4.1), one can see that $\lim_{\alpha \to 0} S_k(\alpha \omega) = S_k^0$, $k = 1, 2, \ldots, n$. Thus, there exists a small positive constant $\underline{\alpha} < 1$ such that

$$\mathbf{F}(\underline{\alpha} \omega) \leq \mathbf{0}.$$

On the other hand, from the second and the last part of (4.1), we have

$$(\mu_{k1} + \tilde{l}_{kk}) S_k(\mathbf{x}) - \sum_{j=1}^{n} (1 - \delta_{kj}) l_{kj} S_j(\mathbf{x}) = b_k + \sum_{j=1}^{n} (1 - \delta_{kj}) m_{kj} x_j - (\mu_{k2} + \gamma_k + \tilde{m}_{kk}) x_k. \quad (4.2)$$

Let $A = [a_{kj}]_{n \times n}$ be a $n \times n$ matrix defined by

$$a_{kk} = \mu_{k1} + \tilde{l}_{kk}, \qquad a_{kj} = - l_{kj}, \qquad j \neq k, \; k = 1, 2, \ldots, n.$$

Then, by the theory of M-matrices, $A = [a_{kj}]_{n \times n}$ is a nonsingular M-matrix and there exists its nonnegative positive inverse matrix $A^{(-1)} = [a_{kj}^{(-1)}]_{n \times n}$, $a_{kj} > 0$, $k, j = 1, 2, \ldots, n$ and from (4.2),

$$S_k(\mathbf{x}) = \sum_{j=1}^{n} a_{kj}^{(-1)} \left(b_j + \sum_{l=1}^{n} (1 - \delta_{jl}) m_{jl} x_l - (\mu_{j2} + \gamma_j + \tilde{m}_{jj}) x_j \right)$$

$$\leq \left(\max_{k, j = 1, 2, \ldots, n} a_{kj}^{(-1)} \right) \sum_{j=1}^{n} \left(b_j + \sum_{l=1}^{n} (1 - \delta_{jl}) m_{jl} x_l - (\mu_{j2} + \gamma_j + \tilde{m}_{jj}) x_j \right)$$

$$= \left(\max_{k, j = 1, 2, \ldots, n} a_{kj}^{(-1)} \right) \sum_{j=1}^{n} \left(b_j - (\mu_{j2} + \gamma_j) x_j \right),$$

where we use the relation $\sum_{j=1}^{n}\sum_{l=1}^{n}(1-\delta_{jl})m_{jl}x_l = \sum_{j=1}^{n}\sum_{l=1}^{n}(1-\delta_{jl})m_{lj}x_j = \sum_{j=1}^{n}\tilde{m}_{jj}x_j$. Therefore, for $\bar{\mathbf{x}} = (\bar{x}, \bar{x}, \bar{x}, \ldots, \bar{x})$ with $\bar{x} > 0$, we have

$$
\begin{aligned}
f_k(\bar{\mathbf{x}}) &= -\left(S_k(\bar{\mathbf{x}}) \sum_{j=1}^{n} \beta_{kj} G(\bar{x}) - (\mu_{k2} + \gamma_k + \tilde{m}_{kk})\bar{x} + \sum_{j=1}^{n}(1-\delta_{kj})m_{kj}\bar{x} \right) \\
&\geq -\left[\left\{ \left(\max_{k,j=1,2,\ldots,n} a_{kj}^{(-1)} \right) \sum_{j=1}^{n} \left(b_j - (\mu_{j2} + \gamma_j)\bar{x} \right) \right\} \sum_{j=1}^{n} \beta_{kj}\bar{x} \right. \\
&\qquad \left. - (\mu_{k2} + \gamma_k + \tilde{m}_{kk})\bar{x} + \sum_{j=1}^{n}(1-\delta_{kj})m_{kj}\bar{x} \right] \\
&\geq -\left[\left\{ \left(\max_{k,j=1,2,\ldots,n} a_{kj}^{(-1)} \right) \sum_{j=1}^{n} \left(b_j - (\mu_{j2} + \gamma_j)\bar{x} \right) \right\} \sum_{j=1}^{n} \beta_{kj} \right. \\
&\qquad \left. - (\mu_{k2} + \gamma_k + \tilde{m}_{kk}) + \sum_{j=1}^{n}(1-\delta_{kj})m_{kj} \right] \bar{x},
\end{aligned}
$$

and hence, for a sufficiently large positive constant $\bar{x}$, we obtain $\mathbf{F}(\bar{\mathbf{x}}) \geq \mathbf{0}$.

Moreover, for system (4.1), the elements of the Fréchet derivative $\mathbf{F}'(\mathbf{x}) = [\partial f_k / \partial x_j]$ are

$$
\frac{\partial f_k(x_1, x_2, \ldots, x_n)}{\partial x_j}
$$
$$
= \begin{cases}
-\dfrac{\partial S_k(\bar{\mathbf{x}})}{\partial x_k} \displaystyle\sum_{j=1}^{n} \beta_{kj} G(x_j) - S_k(\bar{\mathbf{x}})\beta_{kk}G'(x_k) + (\mu_{k2} + \gamma_k + \tilde{m}_{kk}), & \text{for } j = k, \; k = 1,2,\ldots,n, \\[3mm]
-\dfrac{\partial S_k(\bar{\mathbf{x}})}{\partial x_j} \displaystyle\sum_{j=1}^{n} \beta_{kj} G(x_j) - S_k(\bar{\mathbf{x}})\beta_{kj}G'(x_j) - m_{kj} \leq 0, & \text{for } j \neq k, \; k = 1,2,\ldots,n,
\end{cases}
$$

where by the last part of (4.1), $\frac{\partial S_k(\bar{\mathbf{x}})}{\partial x_l}$, $l = 1, 2, \ldots, n$ satisfy

$$
\left(\sum_{j=1}^{n} \beta_{kj} G(x_j) + (\mu_{k1} + \tilde{l}_{kk}) \right) \frac{\partial S_k(\bar{\mathbf{x}})}{\partial x_l} - \sum_{j=1}^{n}(1-\delta_{kj})l_{kj} \frac{\partial S_k(\bar{\mathbf{x}})}{\partial x_l} = -\beta_{kj}G'(x_j), \quad k = 1,2,\ldots,n,
$$

from which by the monotone property on the argument of function G and the theory of M-matrices (see, for example, Berman and Plemmons [3] or Varga [31]), one can easily see that

$$
\frac{\partial S_k(\bar{\mathbf{x}})}{\partial x_l} \leq 0, \qquad j \neq k, \quad k = 1,2,\ldots,n.
$$

Then, for a sufficiently large positive constant $\bar{l} > 0$ such that

$$
\begin{aligned}
\bar{l} > \max \bigg\{ & \max_{1 \leq k \leq n} \left(-\frac{\partial S_k(\bar{\mathbf{x}})}{\partial x_k} \sum_{j=1}^{n} \beta_{kj} G(x_j) - S_k(\bar{\mathbf{x}})\beta_{kk}G'(x_k) + (\mu_{k2} + \gamma_k + \tilde{m}_{kk}) \right), \\
& \frac{\partial S_k(\bar{\mathbf{x}})}{\partial x_j} \sum_{j=1}^{n} \beta_{kj} G(x_j) + S_k(\bar{\mathbf{x}})\beta_{kj}G'(x_j) + m_{kj} \bigg\},
\end{aligned}
$$

$B = [\delta_{kj}\bar{l}^{-1}]$ becomes a non-negative non-singular matrix and *subinverse* (see Ortega and Rheinboldt [27] or Muroya [23]) of $\mathbf{F}'(\mathbf{x})$, i.e., $B\mathbf{F}'(\mathbf{x}) \leq I$ and $\mathbf{F}'(\mathbf{x})B \leq I$ for the unit matrix I and any $\mathbf{x}$ with $0 < x_k < \bar{x}_k$, $k = 1,2,\ldots,n$. Thus, to the equation $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ in $\mathbb{R}^n$, we can apply the monotone iteration such that $\mathbf{x}_0 = \underline{\alpha}\omega$, $\mathbf{y}_0 = \bar{\mathbf{x}}$ with $\mathbf{F}(\mathbf{x}_0) \leq \mathbf{0} \leq \mathbf{F}(\mathbf{y}_0)$

and $\mathbf{x}_{p+1} = \mathbf{x}_p - B\mathbf{F}(\mathbf{x}_p)$, $p = 1, 2, \ldots, n$ of Ortega and Rheinboldt [27, Theorem 4.1 and Corollary 4.1] or Muroya [23, Theorem 3.1], we have a monotone increasing vector sequence $\mathbf{x}_0 \leq \mathbf{x}_1 \leq \ldots \leq \mathbf{x}_p \leq \mathbf{x}_{p+1} \leq \mathbf{y}_0$ and a limit $\lim_{p \to +\infty} \mathbf{x}_p = \mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ which is a solution of $\mathbf{F}(\mathbf{x}) = \mathbf{0}$. Hence, we conclude that there exists an endemic equilibrium of (1.6). □

4.2 Permanence

Now, we show the permanence of system (1.6). Let $\underline{I} := \min_{1 \leq j \leq n} \liminf_{t \to +\infty} I_j(t)/I_j^*$. Without loss of generality, we can assume that $k \in \{1, 2, \ldots, n\}$ is an integer such that $\liminf_{t \to +\infty} I_k(t)/I_k^* \equiv \underline{I}$. Let us set

$$\mathbf{j}_k = \{j \in \{1, 2, \ldots, n\} \mid \beta_{kj} > 0\}.$$

By (1.8), we have $\mathbf{j}_k \neq \emptyset$.

Extending the result of Enatsu et al. [6, Lemma 4.1] on permanence to the model (1.6), we introduce the following lemmas, which play a key role to obtain the permanence.

Lemma 4.2. *For $R_0 > 1$, if*

$$\begin{cases} S_k(t) \geq S_k^*, \quad \dfrac{G(I_j(t-\tau))}{G(I_j^*)} \geq \dfrac{I_k(t)}{I_k^*}, & \text{for any} \quad 0 \leq \tau \leq h_1, \quad \text{and} \\ \dfrac{I_j(t)}{I_j^*} \geq \dfrac{I_k(t-\sigma)}{I_k^*}, & \text{for any} \quad 0 \leq \sigma \leq h_2, \quad j = 1, 2, \ldots, n, \end{cases}$$

then $I_k'(t) \geq 0$. If $I_k'(t) \leq 0$, then

$$\begin{cases} S_k(t) \leq S_k^*, \quad \text{or} \quad \dfrac{G(I_j(t-\tau))}{G(I_j^*)} \leq \dfrac{I_k(t)}{I_k^*}, & \text{for any} \quad 0 \leq \tau \leq h_1, \quad \text{or} \\ \dfrac{I_j(t-\sigma)}{I_j^*} \leq \dfrac{I_k(t-\sigma)}{I_k^*}, & \text{for any} \quad 0 \leq \sigma \leq h_2, \quad j = 1, 2, \ldots, n. \end{cases}$$

Proof. Assume that $R_0 > 1$. By the second equations of (1.6) and (4.1), we have that

$$\begin{aligned} I_k'(t) &= \sum_{j=1}^n \beta_{kj} S_k(t) G(I_j^*) \int_0^{h_1} f_{kj}(\tau) \frac{G(I_j(t-\tau))}{G(I_j^*)} d\tau - (\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk}) I_k^* \frac{I_k(t)}{I_k^*} \\ &\quad + \sum_{j=1}^n m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma) \frac{I_j(t-\sigma)}{I_j^*} d\sigma \\ &= \sum_{j=1}^n \beta_{kj} S_k(t) G(I_j^*) \int_0^{h_1} f_{kj}(\tau) \frac{G(I_j(t-\tau))}{G(I_j^*)} d\tau - \left(\sum_{j=1}^n \beta_{kj} S_k^* G(I_j^*) + \sum_{j=1}^n m_{kj} I_j^* \right) \frac{I_k(t)}{I_k^*} \\ &\quad + \sum_{j=1}^n m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma) \frac{I_j(t-\sigma)}{I_j^*} d\sigma \\ &= \sum_{j=1}^n \beta_{kj} \{S_k(t) - S_k^*\} \int_0^{h_1} f_{kj}(\tau) G(I_j(t-\tau)) d\tau \\ &\quad + \sum_{j=1}^n \beta_{kj} S_k^* G(I_j^*) \int_0^{h_1} f_{kj}(\tau) \left(\frac{G(I_j(t-\tau))}{G(I_j^*)} - \frac{I_k(t)}{I_k^*} \right) d\tau \\ &\quad + \sum_{j=1}^n m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma) \left(\frac{I_j(t-\sigma)}{I_j^*} - \frac{I_k(t)}{I_k^*} \right) d\sigma \end{aligned}$$

from which we obtain the conclusion of this lemma.   □

For a positive constant T_1, the following lemma holds true.

Lemma 4.3. *Assume that $0 < q < 1$ and there exist a $t \geq T_1$ and some $k \in \{1, 2, \ldots, n\}$ such that $I'_k(t) \leq 0$. If there exists an integer $\bar{j} \in \mathbf{j}_k$ such that*

$$\frac{I_{\bar{j}}(t - \tau)}{I^*_{\bar{j}}} > q, \quad \text{for any} \quad 0 \leq \tau \leq h_1, \tag{4.3}$$

then

$$I_k(t) > \underline{I}_k(\bar{j}; q) := \frac{(\hat{\underline{S}}_k - \varepsilon)\beta_{k\bar{j}}G(qI^*_{\bar{j}})}{\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk}} > 0. \tag{4.4}$$

Proof. Assume that $0 < q < 1$. If there exist a $t \geq T_1$, some $k \in \{1, 2, \ldots, n\}$ and an integer $\bar{j} \in \mathbf{j}_k$ such that (4.3) holds. Then, by (2.6) and (1.8), $S_k(t) > \hat{\underline{S}}_k - \varepsilon > 0$ and $\beta_{k\bar{j}} > 0$, and by the second equation of (1.6),

$$0 \geq I'_k(t) = S_k(t) \sum_{j=1}^{n} \beta_{kj} \int_0^{h_1} f_{kj}(\tau)G(I_j(t - \tau))d\tau - (\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk})I_k(t)$$

$$+ \sum_{j=1}^{n} m_{kj} \int_0^{h_2} g_{kj}(\sigma)I_j(t - \sigma)d\sigma$$

$$> (\hat{\underline{S}}_k - \varepsilon)\beta_{k\bar{j}}G(qI^*_{\bar{j}}) - (\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk})I_k(t),$$

from which we have

$$I_k(t) > \frac{(\hat{\underline{S}}_k - \varepsilon)\beta_{k\bar{j}}G(qI^*_{\bar{j}})}{\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk}} > 0.$$

Thus, (4.4) holds. This completes the proof. □

Lemma 4.4. *If $R_0 > 1$, then for any solution of system (1.6) with initial condition (1.7), it holds that*

$$\liminf_{t \to +\infty} I_k(t) \geq \hat{\underline{I}}_q := \hat{\underline{I}}(q) \exp\left\{-\max_{1 \leq k \leq n}(\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk})\rho(q)\right\}, \quad k = 1, 2, \ldots, n,$$

where for $0 < q < 1$, put

$$\begin{cases} \hat{\underline{I}}(q) := \min_{1 \leq k \leq n, \bar{j} \in \mathbf{j}_k} \underline{I}_k(\bar{j}; q) > 0, \\ q_G = \max_{1 \leq j \leq n} \frac{G(qI^*_j)}{G(I^*_j)} < 1, \quad \text{and} \quad r_k = \mu_{k1} + \tilde{l}_{kk} + l_{kk} + q_G \sum_{j=1}^{n} \beta_{kj}G(I^*_j), \end{cases} \tag{4.5}$$

and q is chosen sufficiently small and $\rho(q) > 0$ is chosen sufficiently large to satisfy the following inequalities:

$$S^*_k < \frac{b_k}{r_k}, \quad \text{and} \quad S^*_k < S^\triangle_k := \frac{b_k}{r_k}(1 - e^{-r_k\rho(q)}), \quad k = 1, 2, \ldots, n. \tag{4.6}$$

Proof. Assume that $\tilde{R}_0 > 1$ and let $(S_1(t), I_1(t), S_2(t), I_2(t), \ldots, S_n(t), I_n(t))$ be any solution of system (1.6) with initial condition (1.7), and $0 < q < 1$ and $\rho(q) > 0$ are a fixed sufficiently small constant and a large constant, respectively, such that (4.6) holds.

For $t \geq T_1$, let $k(t) \in \{1, 2, \ldots, n\}$ be defined by

$$\frac{I_{k(t)}(t)}{I^*_{k(t)}} = \min_{1 \leq j \leq n} \frac{I_j(t)}{I^*_j}, \quad \text{for any } t \geq T_1, \tag{4.7}$$

and put

$$\underline{I}(t) := I_{k(t)}(t), \quad \underline{S}(t) = S_{k(t)}(t), \quad \text{and} \quad \underline{\beta}_j = \beta_{k(t)j}, \quad \text{for any } j \in \mathbf{j}_{k(t)}, \text{ and } t \geq T_1.$$

Hereafter, for simplicity without loss of generality and for the readers' convenience, it is better to express $k(t)$ temporally as k for each $t \geq T_1$.

Put

$$D_k(q) = \left\{ t \geq T_1 \mid \frac{I_j(t-\tau)}{I_j^*} \leq q, \text{ for any } j \in \mathbf{j}_k, \text{ and } 0 \leq \tau \leq h_1 \right\}, \qquad k = 1, 2, \ldots, n,$$

and for $t \geq T_1$,

$$\underline{I}(t) = \frac{I_k(t)}{I_k^*},$$

where $k = k(t) \in \{1, 2, \ldots, n\}$ is an integer such that (4.7) holds.

Because by Lemma 4.3, for $t \geq T_1$,

$$I_k(t) > \hat{I}_k(q) := \min_{\bar{j} \in \mathbf{j}_k} \underline{I}_k(\bar{j}; q) > 0, \quad \text{if } t \notin D_k(q), \tag{4.8}$$

to prove that there exists a positive constant $\hat{I}$ independent from the initial conditions (1.7) such that

$$\underline{I} = \min_{1 \leq j \leq n} \liminf_{t \to +\infty} \frac{I_j(t)}{I_j^*} \geq \hat{I},$$

we may hereafter restrict our attention to the case that there exists a sufficiently large $t \in D_k(q)$.

We now prove that it is impossible that there exists a nonnegative constant $t_0 \geq T_1$ such that $t \in D_q(k)$ for all $t \geq t_0$. Suppose on the contrary that $t \in D_q(k)$ for all $t \geq t_0$. Consider the following functional:

$$V(t) = I_k(t) + \sum_{j=1}^{n} \left(\beta_{kj} \int_0^{h_1} f_{kj}(\tau) \int_{t-\tau}^{t} S_k(u+\tau)G(I_j(u)) du\, d\tau + m_{kj} \int_0^{h_2} g_{kj}(\sigma) \int_{t-\sigma}^{t} I_j(u) du\, d\sigma \right).$$

By (4.5) and the inequality $G(I_j(t-\tau)) \leq G(qI_j^*) \leq q_G G(I_j^*)$, $j = 1, 2, \ldots, n$ for $t \geq t_0 + h_1$, one can obtain that

$$\frac{dS_k(t)}{dt} \geq b_k - \left(\mu_k + \tilde{l}_{kk} + l_{kk} + q_G \sum_{j=1}^{n} \beta_{kj}G(I_j^*) \right) S_k(t) = b_k - r_k S_k(t), \quad \text{for } t \geq t_0 + h_1,$$

which yields

$$S_k(t) \geq e^{-r_k(t-t_0)} \left(S_k(t_0) + b_k \int_{t_0}^{t} e^{r_k(s-t_0)} ds \right)$$

$$> \frac{b_k}{r_k}(1 - e^{-r_k(t-t_0)}), \quad \text{for } t \geq t_0 + h_1. \tag{4.9}$$

Hence, if we choose $\rho(q) > 0$ sufficiently large to satisfy $S_k^* < S_k^{\triangle} := b_k(1 - e^{-r_k\rho(q)})/r_k$ of (4.6), then it follows from (4.9) that

$$S_k(t) > \frac{b_k}{r_k}(1 - e^{-r_k\rho(q)}) = S_k^{\triangle} > S_k^*, \quad \text{for } t \geq t_0 + h_1 + \rho(q). \tag{4.10}$$

Since $t \in D_k(q)$ implies $I_j(t - \tau)/I_j^* \le q < 1$ for $j \in \mathbf{j}_k$, $0 \le \tau \le h_1$ and $t \ge T_1$, we have $I_j(t) \le I_j^*$ for $j \in \mathbf{j}_k$ and $t \ge t_0 + h_1$. By the second part of the hypothesis (1.3), it holds that $G(I_j(t))/I_j(t) \ge G(I_j^*)/I_j^*$, which is equivalent to $I_j(t)/I_j^* \le G(I_j(t))/G(I_j^*)$, $j \in \mathbf{j}_k$ for $t \ge t_0 + h_1$. Thus, by $I_k(t)/I_k^* \le I_j(t)/I_j^*$, $j = 1, 2, \ldots, n$, we obtain $I_k(t)/I_k^* \le G(I_j(t))/G(I_j^*)$, $j = 1, 2, \ldots, n$ for $t \ge t_0 + h_1$.

By the above discussion and Lemma 4.2, we obtain $I_k'(t) \ge 0$ for any $t \ge t_0 + h_1 + \rho(q)$, and there exists a $j \in \{1, 2, \ldots, n\}$ and an $s_t \in [t - h_1, t)$ such that $I_k(t) \ge I_j(s_t)$. For a positive constant $\hat{I} \equiv \min_{t_0 + \rho(q) \le s \le t_0 + h_1 + \rho(q)} I_j(s) > 0$, we thus obtain that

$$I_k(t) \ge \hat{I} \quad \text{for any } t \ge t_0 + h_1 + \rho(q). \tag{4.11}$$

Calculating the derivative of $V(t)$ along solutions of system (1.6) gives as follows.

$$
\begin{aligned}
\frac{dV(t)}{dt} &= \sum_{j=1}^{n} \beta_{kj} S_k(t) \int_0^{h_1} f_{kj}(\tau) G(I_j(t - \tau)) d\tau - (\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk}) I_k(t) \\
&\quad + \sum_{j=1}^{n} m_{kj} \int_0^{h_2} g_{kj}(\sigma) I_j(t - \sigma) d\sigma \\
&\quad + \sum_{j=1}^{n} \left(\beta_{kj} \int_0^{h_1} f_{kj}(\tau) \{ S_k(t + \tau) G(I_j(t)) - S_k(t) G(I_j(t - \tau)) \} d\tau \right. \\
&\qquad \left. + m_{kj} \int_0^{h_2} g_{kj}(\sigma) \{ I_j(t) - I_j(t - \sigma) \} d\sigma \right) \\
&= \sum_{j=1}^{n} \beta_{kj} \int_0^{h_1} f_{kj}(\tau) S_k(t + \tau) G(I_j(t)) d\tau - (\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk}) I_k^* \frac{I_k(t)}{I_k^*} + \sum_{j=1}^{n} m_{kj} I_j^* \frac{I_j(t)}{I_j^*} \\
&\ge \sum_{j=1}^{n} \beta_{kj} S_k^{\triangle} G(I_j^*) \frac{G(I_j(t))}{G(I_j^*)} - (\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk}) I_k^* \frac{I_k(t)}{I_k^*} + \sum_{j=1}^{n} m_{kj} I_j^* \frac{I_j(t)}{I_j^*} \\
&\ge \sum_{j=1}^{n} \beta_{kj} S_k^{\triangle} G(I_j^*) \frac{I_j(t)}{I_j^*} - \sum_{j=1}^{n} \beta_{kj} S_k^* G(I_j^*) \frac{I_k(t)}{I_k^*} + \sum_{j=1}^{n} m_{kj} I_j^* \left(\frac{I_j(t)}{I_j^*} - \frac{I_k(t)}{I_k^*} \right) \\
&\ge (S_k^{\triangle} - S_k^*) \sum_{j=1}^{n} \beta_{kj} G(I_j^*) \frac{I_k(t)}{I_k^*}, \quad \text{for } t \ge t_0 + h_1 + \rho(q),
\end{aligned}
$$

because by (4.10), $S_k(t) \ge S_k^{\triangle} > S_k^*$ for $t \ge t_0 + h_1 + \rho(q)$ and by $t \in D_k(q)$ and the definition of k and the above note, we have that $G(I_j(t))/G(I_j^*) \ge I_j(t)/I_j^* \ge I_k(t)/I_k^*$ for $t \ge t_0 + h_1$.

Thus, $\lim_{t \to +\infty} V(t) = +\infty$. However, from (2.1) and the first part of the hypothesis (1.3), it holds that

$$\limsup_{t \to +\infty} V(t) \le \bar{N}_k^* + \sum_{j=1}^{n} \{ \beta_{kj} S_k^0 h_1 G(\bar{N}_j^*) + m_{kj} h_2 \bar{N}_j^* \} < +\infty.$$

This leads to a contradiction. Hence the claim is proved.

By the claim and (4.8), there exist sufficiently large constants $t_1 < t_2$ and $t_1, t_2 \in D_k(q)$ such that for any $t_1 < t < t_2$, $t \in D_k(q)$, but there exists a $0 \le t_3 < t_1$ and $t_4 > t_2$ such that $t_3, t_4 \notin D_k(q)$.

We now show that $I(t) \ge \hat{I}_q$ for all t sufficiently large $t \in D_k(q)$. Let $t_1 < t_2$ be sufficiently large constants $t_1, t_2 \in D_k(q)$ such that

$$I_k(t_1) \ge \hat{I}_k(q), \ I_k(t_2) \ge \hat{I}_k(q), \ t \in D_k(q), \quad \text{for any } t_1 < t < t_2.$$

Since from the second equation of system (1.6), we have

$$\frac{dI_k(t)}{dt} \geq -(\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk})I_k(t), \quad \text{for } t \geq t_1.$$

Then,

$$
\begin{aligned}
I_k(t) &\geq I_k(t_1)e^{-(\mu_{k1} + \gamma_k + \tilde{m}_{kk} + m_{kk})(t-t_1)} \\
&\geq \hat{\underline{I}}_k(q)e^{-(\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk})\rho(q)} \\
&\geq \hat{\underline{I}}_q, \quad \text{for } t_1 \leq t \leq t_1 + \rho(q).
\end{aligned}
$$

If $t_2 > t_1 + \rho(q)$, then by applying the similar discussion to (4.10) and (4.11) in place of t_0 by t_1, we obtain $I_k(t) \geq \hat{\underline{I}}_q$ for $t_1 + \rho(q) < t \leq t_2$. Hence, we prove that $I(t) \geq \hat{\underline{I}}_q$ for $t_1 \leq t \leq t_2$.

Since the interval $t_1 \leq t \leq t_2$ in $D_k(q)$ is arbitrarily chosen, we can easily conclude that $I(t) \geq \hat{\underline{I}}_q$ holds for all t sufficiently large $t \in D_k(q)$. Thus, we obtain that $\liminf_{t \to +\infty} I(t) \geq \hat{\underline{I}}_q$. This completes the proof. $\square$

By (2.1) and Lemmas 2.2, 4.2–4.4, we obtain the permanence of system (1.6) for $R_0 > 1$.

5 Global stability of the endemic equilibrium E* for $\tilde{R}_0 > 1$

In this section, we assume $\tilde{R}_0 > 1$. By Lemma 4.1, there exists an endemic equilibrium

$$\mathbf{E}^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$$

of (1.6) in $\mathbf{\Gamma}^0$ such that (4.1) holds. We rewrite (1.6) as

$$
\begin{cases}
\dfrac{dS_k(t)}{dt} = b_k - (\mu_{k1} + \tilde{l}_{kk} + l_{kk})S_k(t) \\
\qquad - \displaystyle\sum_{j=1}^{n}\left(\beta_{kj}S_k(t)\int_0^{h_1} f_{kj}(\tau)G(I_j(t-\tau))d\tau - l_{kj}\int_0^{h_2} g_{kj}(\sigma)S_j(t-\sigma)d\sigma\right), \\
\dfrac{dI_k(t)}{dt} = \displaystyle\sum_{j=1}^{n}\left(\beta_{kj}S_k(t)\int_0^{h_1} f_{kj}(\tau)G(I_j(t-\tau))d\tau + m_{kj}\int_0^{h_2} g_{kj}(\sigma)I_j(t-\sigma)d\sigma\right) \\
\qquad - (\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk})I_k(t), \qquad k = 1, 2 \ldots, n.
\end{cases}
\tag{5.1}
$$

Let us set

$$
\begin{aligned}
U(t) := \sum_{k=1}^{n} v_k &\left[S_k^* g\left(\frac{S_k(t)}{S_k^*}\right) + I_k^* g\left(\frac{I_k(t)}{I_k^*}\right) \right. \\
&+ \sum_{j=1}^{n}\left\{ \beta_{kj}S_k^*G(I_j^*)\int_0^{h_1} f_{kj}(\tau)\int_{t-\tau}^{t} g\left(\frac{G(I_j(u))}{G_j(I_j^*)}\right)du\,d\tau \right. \\
&\qquad + l_{kj}S_j^*\int_0^{h_2} g_{kj}(\sigma)\int_{t-\sigma}^{t} g\left(\frac{S_j(u)}{S_j^*}\right)du\,d\sigma \\
&\qquad \left.\left. + m_{kj}I_j^*\int_0^{h_2} g_{kj}(\sigma)\int_{t-\sigma}^{t} g\left(\frac{I_j(u)}{I_j^*}\right)du\,d\sigma \right\} \right],
\end{aligned}
\tag{5.2}
$$

where $v_1, v_2, \ldots, v_n$ will be appropriately chosen later (see (1.17)). Differentiating U along the solutions of (1.6), we have

$$\frac{dU(t)}{dt} = \sum_{k=1}^{n} v_k \left\{ \left(1 - \frac{S_k^*}{S_k(t)}\right) \frac{dS_k(t)}{dt} + \left(1 - \frac{I_k^*}{I_k(t)}\right) \frac{dI_k(t)}{dt} \right\}$$

$$+ \sum_{k=1}^{n} v_k \left[\sum_{j=1}^{n} \beta_{kj} S_k^* G(I_j^*) \int_0^{h_1} f_{kj}(\tau) \left\{ g\left(\frac{G(I_j(t))}{G(I_j^*)}\right) - g\left(\frac{G(I_j(t-\tau))}{G(I_j^*)}\right) \right\} d\tau \right.$$

$$+ \sum_{j=1}^{n} l_{kj} S_j^* \int_0^{h_2} g_{kj}(\sigma) \left\{ g\left(\frac{S_j(t)}{S_j^*}\right) - g\left(\frac{S_j(t-\sigma)}{S_j^*}\right) \right\} d\sigma$$

$$+ \sum_{j=1}^{n} m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma) \left\{ g\left(\frac{I_j(t)}{I_j^*}\right) - g\left(\frac{I_j(t-\sigma)}{I_j^*}\right) \right\} d\sigma \right].$$

For simplicity, we put

$$x_k(t) = \frac{S_k(t)}{S_k^*}, \qquad y_k(t) = \frac{I_k(t)}{I_k^*}, \qquad \tilde{y}_j(t) = \frac{G(I_j(t))}{G(I_j^*)}, \qquad k, j = 1, 2, \ldots, n.$$

From (4.1) and (5.1), we derive that for $k = 1, 2, \ldots, n$,

$$\frac{dS_k(t)}{dt} = b_k - (\mu_{k1} + \tilde{l}_{kk} + l_{kk}) S_k(t)$$

$$- \sum_{j=1}^{n} \left(\beta_{kj} S_k(t) \int_0^{h_1} f_{kj}(\tau) G(I_j(t-\tau)) d\tau - l_{kj} \int_0^{h_2} g_{kj}(\sigma) S_j(t-\sigma) d\sigma \right)$$

$$= -(\mu_{k1} + \tilde{l}_{kk} + \tilde{l}_{kk} + l_{kk})(S_k(t) - S_k^*)$$

$$- \sum_{j=1}^{n} \left\{ \beta_{kj} \left(S_k(t) \int_0^{h_1} f_{kj}(\tau) G(I_j(t-\tau)) d\tau - S_k^* G(I_j^*) \right) \right.$$

$$\left. - l_{kj} \int_0^{h_2} g_{kj}(\sigma)(S_j(t-\sigma) - S_j^*) d\sigma \right\}$$

$$= -(\mu_{k1} + \tilde{l}_{kk} + l_{kk}) S_k^*(x_k(t) - 1)$$

$$- \sum_{j=1}^{n} \left(\beta_{kj} S_k^* G(I_j^*) \int_0^{h_1} f_{kj}(\tau)(x_k(t)\tilde{y}_j(t-\tau) - 1) d\tau \right.$$

$$\left. - l_{kj} S_j^* \int_0^{h_2} g_{kj}(\sigma)(x_j(t-\sigma) - 1) d\sigma \right),$$

and

$$\frac{dI_k(t)}{dt} = \sum_{j=1}^{n} \left(\beta_{kj} S_k(t) \int_0^{h_1} f_{kj}(\tau) G(I_j(t-\tau)) d\tau + m_{kj} \int_0^{h_2} g_{kj}(\sigma) I_j(t-\sigma) d\sigma \right)$$

$$- (\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk}) I_k(t)$$

$$= \sum_{j=1}^{n} \left(\beta_{kj} S_k^* G(I_j^*) \int_0^{h_1} f_{kj}(\tau) x_k(t)\tilde{y}_j(t-\tau) d\tau + m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma) y_j(t-\sigma) d\sigma \right)$$

$$- (\mu_{k2} + \gamma_k + \tilde{m}_{kk} + m_{kk}) I_k^* y_k(t)$$

$$= \sum_{j=1}^{n} \left(\beta_{kj} S_k^* G(I_j^*) \int_0^{h_1} f_{kj}(\tau)(x_k(t)\tilde{y}_j(t-\tau) - y_k(t)) d\tau \right.$$

$$\left. + m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma)(y_j(t-\sigma) - y_k(t)) d\sigma \right).$$

Then, we have

$$\sum_{k=1}^{n} v_k \left\{ \left(1 - \frac{S_k^*}{S_k(t)}\right) \frac{dS_k(t)}{dt} + \left(1 - \frac{I_k^*}{I_k(t)}\right) \frac{dI_k(t)}{dt} \right\}$$

$$= \sum_{k=1}^{n} v_k \left[\left(1 - \frac{1}{x_k(t)}\right) \left\{ -(\mu_{k1} + \tilde{l}_{kk} + l_{kk}) S_k^*(x_k(t) - 1) \right.\right.$$

$$- \sum_{j=1}^{n} \left(\beta_{kj} S_k^* G(I_j^*) \int_0^{h_1} f_{kj}(\tau)(x_k(t)\tilde{y}_j(t-\tau) - 1)d\tau \right.$$

$$\left.- l_{kj} S_j^* \int_0^{h_2} g_{kj}(\sigma)(x_j(t-\sigma) - 1)d\sigma \right) \right\}$$

$$+ \left(1 - \frac{1}{y_k(t)}\right) \sum_{j=1}^{n} \left(\beta_{kj} S_k^* G(I_j^*) \int_0^{h_1} f_{kj}(\tau)(x_k(t)\tilde{y}_j(t-\tau) - y_k(t))d\tau \right.$$

$$\left.\left.+ m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma)(y_j(t-\sigma) - y_k(t))d\sigma \right) \right]$$

$$= - \sum_{k=1}^{n} v_k \left[(\mu_{k1} + \tilde{l}_{kk} + l_{kk}) S_k^* \left(1 - \frac{1}{x_k(t)}\right)(x_k(t) - 1) \right.$$

$$\left.- \sum_{j=1}^{n} l_{kj} S_j^* \int_0^{h_2} g_{kj}(\sigma) \left(1 - \frac{1}{x_k(t)}\right)(x_j(t-\sigma) - 1)d\sigma \right]$$

$$+ \sum_{k=1}^{n} v_k \left[\sum_{j=1}^{n} \beta_{kj} S_k^* G(I_j^*) \int_0^{h_1} f_{kj}(\tau) \left\{ \left(1 - \frac{1}{x_k(t)}\right)(1 - x_k(t)\tilde{y}_j(t-\tau)) \right.\right.$$

$$\left.+ \left(1 - \frac{1}{y_k(t)}\right)(x_k(t)\tilde{y}_j(t-\tau) - y_k(t)) \right\} d\tau$$

$$\left.+ \sum_{j=1}^{n} m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma) \left(1 - \frac{1}{y_k(t)}\right)(y_j(t-\sigma) - y_k(t))d\sigma \right]. \tag{5.3}$$

For the first part of the last equation in (5.3), we have

$$\left(1 - \frac{1}{x_k(t)}\right)(x_k(t) - 1) = x_k(t) + \frac{1}{x_k(t)} - 2$$

$$= g(x_k(t)) + g\left(\frac{1}{x_k(t)}\right)$$

and

$$\left(1 - \frac{1}{x_k(t)}\right)(x_j(t-\sigma) - 1) = x_j(t-\sigma) - \frac{x_j(t-\sigma)}{x_k(t)} + \frac{1}{x_k(t)} - 1$$

$$= g(x_j(t-\sigma)) - g\left(\frac{x_j(t-\sigma)}{x_k(t)}\right) + g\left(\frac{1}{x_k(t)}\right),$$

and hence, by (1.10),

$$\sum_{k=1}^{n} v_k(\mu_{k1} + \tilde{l}_{kk} + l_{kk}) S_k^* \left(1 - \frac{1}{x_k(t)}\right)(x_k(t) - 1)$$

$$= \sum_{k=1}^{n} v_k(\mu_{k1} + \tilde{l}_{kk} + l_{kk}) S_k^* \left\{ g(x_k(t)) + g\left(\frac{1}{x_k(t)}\right) \right\},$$

and

$$\sum_{k=1}^{n} v_k \sum_{j=1}^{n} l_{kj} S_j^* \left(1 - \frac{1}{x_k(t)} \right) (x_j(t-\sigma) - 1)$$

$$= \sum_{k=1}^{n} v_k \sum_{j=1}^{n} l_{kj} S_j^* \left\{ g(x_j(t-\sigma)) - g\left(\frac{x_j(t-\sigma)}{x_k(t)} \right) + g\left(\frac{1}{x_k(t)} \right) \right\}.$$

For the remaining parts of the last equation in (5.3), we have

$$\left(1 - \frac{1}{x_k(t)} \right) (1 - x_k(t)\tilde{y}_j(t-\tau)) + \left(1 - \frac{1}{y_k(t)} \right) (x_k(t)\tilde{y}_j(t-\tau) - y_k(t))$$

$$= \left(1 - \frac{1}{x_k(t)} - x_k(t)\tilde{y}_j(t-\tau) + \tilde{y}_j(t-\tau) \right)$$

$$+ \left(x_k(t)\tilde{y}_j(t-\tau) - \frac{x_k(t)\tilde{y}_j(t-\tau)}{y_k(t)} - y_k(t) + 1 \right)$$

$$= 2 - \frac{1}{x_k(t)} + \tilde{y}_j(t-\tau) - \frac{x_k(t)\tilde{y}_j(t-\tau)}{y_k(t)} - y_k(t)$$

$$= -g\left(\frac{1}{x_k(t)} \right) - g\left(\frac{x_k(t)\tilde{y}_j(t-\tau)}{y_k(t)} \right) + \{g(\tilde{y}_j(t-\tau)) - g(y_k(t))\},$$

and

$$\left(1 - \frac{1}{y_k(t)} \right) (y_j(t-\sigma) - y_k(t)) = y_j(t-\sigma) - \frac{y_j(t-\sigma)}{y_k(t)} - y_k(t) + 1$$

$$= -g\left(\frac{y_j(t-\sigma)}{y_k(t)} \right) + \{g(y_j(t-\sigma)) - g(y_k(t))\}.$$

Then,

$$\sum_{k=1}^{n} v_k \left[\sum_{j=1}^{n} \beta_{kj} S_k^* G(I_j^*) \int_0^{h_1} f_{kj}(\tau) \left\{ \left(1 - \frac{1}{x_k(t)} \right) (1 - x_k(t)\tilde{y}_j(t-\tau)) \right. \right.$$

$$+ \left(1 - \frac{1}{y_k(t)} \right) (x_k(t)\tilde{y}_j(t-\tau) - y_k(t)) \right\} d\tau$$

$$\left. + \left(1 - \frac{1}{y_k(t)} \right) \sum_{j=1}^{n} m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma)(y_j(t-\sigma) - y_k(t)) d\sigma \right]$$

$$= -\sum_{k=1}^{n} v_k \sum_{j=1}^{n} \left[\beta_{kj} S_k^* G(I_j^*) \left\{ g\left(\frac{1}{x_k(t)} \right) + \int_0^{h_1} f_{kj}(\tau) g\left(\frac{x_k(t)\tilde{y}_j(t-\tau)}{y_k(t)} \right) d\tau \right\} \right.$$

$$\left. + m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma) g\left(\frac{y_j(t-\sigma)}{y_k(t)} \right) d\sigma \right]$$

$$+ \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \left(\beta_{kj} S_k^* G(I_j^*) \int_0^{h_1} f_{kj}(\tau) \{g(\tilde{y}_j(t-\tau)) - g(y_k(t))\} d\tau \right.$$

$$\left. + m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma) \{g(y_j(t-\sigma)) - g(y_k(t))\} d\sigma \right).$$

We finally obtain that

$$
\begin{aligned}
\frac{dU(t)}{dt} =\ & -\sum_{k=1}^{n} v_k(\mu_{k1} + \tilde{l}_{kk} + l_{kk})S_k^* \left\{ g(x_k(t)) + g\left(\frac{1}{x_k(t)}\right) \right\} \\
& + \sum_{k=1}^{n} v_k \sum_{j=1}^{n} l_{kj} S_j^* \int_0^{h_2} g_{kj}(\sigma) \left\{ g(x_j(t-\sigma)) - g\left(\frac{x_j(t-\sigma)}{x_k(t)}\right) + g\left(\frac{1}{x_k(t)}\right) \right\} d\sigma \\
& - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \left[\beta_{kj} S_k^* G(I_j^*) \left\{ g\left(\frac{1}{x_k(t)}\right) + \int_0^{h_1} f_{kj}(\tau) g\left(\frac{x_k(t)\tilde{y}_j(t-\tau)}{y_k(t)}\right) d\tau \right\} \right. \\
& \qquad\qquad \left. + m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma) g\left(\frac{y_j(t-\sigma)}{y_k(t)}\right) d\sigma \right] \\
& + \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \left(\beta_{kj} S_k^* G(I_j^*) \int_0^{h_1} f_{kj}(\tau)\{g(\tilde{y}_j(t-\tau)) - g(y_k(t))\} d\tau \right. \\
& \qquad\qquad \left. + m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma)\{g(y_j(t-\sigma)) - g(y_k(t))\} d\sigma \right) \\
& + \sum_{k=1}^{n} v_k \left(\sum_{j=1}^{n} \beta_{kj} S_k^* G(I_j^*) \int_0^{h_1} f_{kj}(\tau)\{g(\tilde{y}_j(t)) - g(\tilde{y}_j(t-\tau))\} d\tau \right. \\
& \qquad\qquad + \sum_{j=1}^{n} l_{kj} S_j^* \int_0^{h_2} g_{kj}(\sigma)\{g(x_j(t)) - g(x_j(t-\sigma))\} d\sigma \\
& \qquad\qquad \left. + \sum_{j=1}^{n} m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma)\{g(y_j(t)) - g(y_j(t-\sigma))\} d\sigma \right) \\
=\ & -\sum_{k=1}^{n} v_k(\mu_{k1} + \tilde{l}_{kk} + l_{kk})S_k^* \left\{ g(x_k(t)) + g\left(\frac{1}{x_k(t)}\right) \right\} \\
& + \sum_{k=1}^{n} v_k \sum_{j=1}^{n} l_{kj} S_j^* \int_0^{h_2} g_{kj}(\sigma) \left\{ g(x_j(t)) - g\left(\frac{x_j(t-\sigma)}{x_k(t)}\right) + g\left(\frac{1}{x_k(t)}\right) \right\} d\sigma \\
& - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \left[\beta_{kj} S_k^* G(I_j^*) \left\{ g\left(\frac{1}{x_k(t)}\right) + \int_0^{h_1} f_{kj}(\tau) g\left(\frac{x_k(t)\tilde{y}_j(t-\tau)}{y_k(t)}\right) d\tau \right\} \right. \\
& \qquad\qquad \left. + m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma) g\left(\frac{y_j(t-\sigma)}{y_k(t)}\right) d\sigma \right] \\
& + \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \left(\beta_{kj} S_k^* G(I_j^*)\{g(\tilde{y}_j(t)) - g(y_k(t))\} + m_{kj} I_j^*\{g(y_j(t)) - g(y_k(t))\} \right) \\
\leq\ & -\sum_{k=1}^{n} v_k(\mu_{k1} + \tilde{l}_{kk} + l_{kk})S_k^* \left\{ g(x_k(t)) + g\left(\frac{1}{x_k(t)}\right) \right\} \\
& + \sum_{k=1}^{n} v_k \sum_{j=1}^{n} l_{kj} S_j^* \int_0^{h_2} g_{kj}(\sigma) \left\{ g(x_j(t)) - g\left(\frac{x_j(t-\sigma)}{x_k(t)}\right) + g\left(\frac{1}{x_k(t)}\right) \right\} d\sigma \\
& - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \left[\beta_{kj} S_k^* G(I_j^*) \left\{ g\left(\frac{1}{x_k(t)}\right) + \int_0^{h_1} f_{kj}(\tau) g\left(\frac{x_k(t)\tilde{y}_j(t-\tau)}{y_k(t)}\right) d\tau \right\} \right. \\
& \qquad\qquad \left. + m_{kj} I_j^* \int_0^{h_2} g_{kj}(\sigma) g\left(\frac{y_j(t-\sigma)}{y_k(t)}\right) d\sigma \right] \\
& + \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \left(\beta_{kj} S_k^* G(I_j^*)\{g(y_j(t)) - g(y_k(t))\} + m_{kj} I_j^*\{g(y_j(t)) - g(y_k(t))\} \right). \quad (5.4)
\end{aligned}
$$

Here, we use the relation $g(\tilde{y}_j(t)) \leq g(y_j(t))$, $j = 1, 2, \ldots, n$ (see Enatsu *et al.* [6, Lemma 3.2]), because $y_j(t) = I_j(t)/I_j^* \geq 1$ if and only if

$$y_j(t) - \tilde{y}_j(t) = \frac{I_j(t)}{I_j^*} - \frac{G(I_j(t))}{G(I_j^*)} = \frac{G(I_j(t))}{I_j^*}\left(\frac{I_j(t)}{G(I_j(t))} - \frac{I_j^*}{G(I_j^*)}\right) \geq 0,$$

which yields

$$
\begin{aligned}
\frac{dU(t)}{dt} \leq &- \sum_{k=1}^{n} v_k(\mu_{k1} + \tilde{l}_{kk} + l_{kk})S_k^*\left\{g(x_k(t)) + g\left(\frac{1}{x_k(t)}\right)\right\} \\
&+ \sum_{k=1}^{n} v_k \sum_{j=1}^{n} l_{kj}S_j^* \int_0^{h_2} g_{kj}(\sigma)\left\{g(x_j(t)) - g\left(\frac{x_j(t-\sigma)}{x_k(t)}\right) + g\left(\frac{1}{x_k(t)}\right)\right\}d\sigma \\
&- \sum_{k=1}^{n} v_k \sum_{j=1}^{n}\left[\beta_{kj}S_k^* G(I_j^*)\left\{g\left(\frac{1}{x_k(t)}\right) + \int_0^{h_1} f_{kj}(\tau)g\left(\frac{x_k(t)\tilde{y}_j(t-\tau)}{y_k(t)}\right)d\tau\right\}\right. \\
&\left.+ m_{kj}I_j^* \int_0^{h_2} g_{kj}(\sigma)g\left(\frac{y_j(t-\sigma)}{y_k(t)}\right)d\sigma\right] \\
&+ \sum_{k=1}^{n} v_k \sum_{j=1}^{n}\left(\beta_{kj}S_k^*\frac{G(I_j^*)}{I_j^*} + (1-\delta_{kj})m_{kj}\right)I_j^*\{g(y_j(t)) - g(y_k(t))\}.
\end{aligned}
\tag{5.5}
$$

On the other hand, by (4.1), we have that

$$
\begin{aligned}
&\sum_{k=1}^{n} v_k \sum_{j=1}^{n}\left(\beta_{kj}S_k^*\frac{G(I_j^*)}{I_j^*} + (1-\delta_{kj})m_{kj}\right)I_j^*\{g(y_j(t)) - g(y_k(t))\} \\
&= \sum_{k=1}^{n} v_k \sum_{j=1}^{n}\left(\beta_{kj}S_k^*\frac{G(I_j^*)}{I_j^*} + (1-\delta_{kj})m_{kj}\right)I_j^* g(y_j(t)) \\
&\quad - \sum_{k=1}^{n} v_k\left\{\sum_{j=1}^{n}\left(\beta_{kj}S_k^*\frac{G(I_j^*)}{I_j^*} + (1-\delta_{kj})m_{kj}\right)I_j^*\right\}g(y_k(t)) \\
&= \sum_{j=1}^{n} v_j \sum_{k=1}^{n}\left(\beta_{jk}S_j^*\frac{G(I_j^*)}{I_j^*} + (1-\delta_{jk})m_{jk}\right)I_k^* g(y_k(t)) - \sum_{k=1}^{n} v_k(\mu_{k2} + \gamma_k + \tilde{m}_{kk})I_k^* g(y_k(t)) \\
&= \sum_{k=1}^{n}\left\{\sum_{j=1}^{n} v_j\left(\beta_{jk}S_j^*\frac{G(I_j^*)}{I_j^*} + (1-\delta_{jk})m_{jk}\right) - v_k(\mu_{k2} + \gamma_k + \tilde{m}_{kk})\right\}I_k^* g(y_k(t)).
\end{aligned}
\tag{5.6}
$$

From (5.5) and (5.6), we therefore obtain that

$$
\begin{aligned}
\frac{dU(t)}{dt} \leq &- \sum_{k=1}^{n} v_k(\mu_{k1} + \tilde{l}_{kk} + l_{kk})S_k^*\left\{g(x_k(t)) + g\left(\frac{1}{x_k(t)}\right)\right\} \\
&+ \sum_{k=1}^{n} v_k \sum_{j=1}^{n} l_{kj}S_j^* \int_0^{h_2} g_{kj}(\sigma)\left\{g(x_j(t)) - g\left(\frac{x_j(t-\sigma)}{x_k(t)}\right) + g\left(\frac{1}{x_k(t)}\right)\right\}d\sigma \\
&- \sum_{k=1}^{n} v_k \sum_{j=1}^{n}\left[\beta_{kj}S_k^* G(I_j^*)\left\{g\left(\frac{1}{x_k(t)}\right) + \int_0^{h_1} f_{kj}(\tau)g\left(\frac{x_k(t)\tilde{y}_j(t-\tau)}{y_k(t)}\right)d\tau\right\}\right. \\
&\left.+ m_{kj}I_j^* \int_0^{h_2} g_{kj}(\sigma)g\left(\frac{y_j(t-\sigma)}{y_k(t)}\right)d\sigma\right] \\
&+ \sum_{k=1}^{n}\left\{\sum_{j=1}^{n} v_j\left(\beta_{jk}S_j^*\frac{G(I_j^*)}{I_j^*} + (1-\delta_{jk})m_{jk}\right) - v_k(\mu_{k2} + \gamma_k + \tilde{m}_{kk})\right\}I_k^* g(y_k(t))
\end{aligned}
$$

$$
\begin{aligned}
= & -\sum_{k=1}^{n}\left\{v_k(\mu_{k1}+\tilde{l}_{kk}+l_{kk})-\sum_{j=1}^{n}v_j l_{jk}\right\}S_k^* g(x_k(t)) \\
& -\sum_{k=1}^{n}v_k\left\{\left(\sum_{j=1}^{n}\beta_{kj}G(I_j^*)+(\mu_{k1}+\tilde{l}_{kk}+l_{kk})\right)S_k^*-\sum_{j=1}^{n}l_{kj}S_j^*\right\}g\left(\frac{1}{x_k(t)}\right) \\
& -\sum_{k=1}^{n}v_k\sum_{j=1}^{n}l_{kj}S_j^*\int_0^{h_2}g_{kj}(\sigma)g\left(\frac{x_j(t-\sigma)}{x_k}\right)d\sigma \\
& -\sum_{k=1}^{n}v_k\sum_{j=1}^{n}\left\{\beta_{kj}S_k^*G(I_j^*)\int_0^{h_1}f_{kj}(\tau)\left(\frac{x_k(t)\tilde{y}_j(t-\tau)}{y_k(t)}\right)d\tau\right. \\
& \left.+m_{kj}I_j^*\int_0^{h_2}g_{kj}(\sigma)g\left(\frac{y_j(t-\sigma)}{y_k(t)}\right)d\sigma\right\} \\
& +\sum_{k=1}^{n}\left\{\sum_{j=1}^{n}v_j\left(\beta_{jk}S_j^*\frac{G(I_j^*)}{I_j^*}+(1-\delta_{jk})m_{jk}\right)-v_k(\mu_{k2}+\gamma_k+\tilde{m}_{kk})\right\}I_k^* g(y_k(t)).
\end{aligned}
$$

Hence, by (4.1),

$$
\begin{aligned}
\frac{dU}{dt} \leq & -\sum_{k=1}^{n}\left\{v_k(\mu_{k1}+\tilde{l}_{kk}+l_{kk})-\sum_{j=1}^{n}v_j l_{jk}\right\}S_k^* g(x_k(t)) \\
& -\sum_{k=1}^{n}v_k b_k g\left(\frac{1}{x_k(t)}\right)-\sum_{k=1}^{n}v_k\sum_{j=1}^{n}l_{kj}S_j^*\int_0^{h_2}g_{kj}(\sigma)g\left(\frac{x_j(t-\sigma)}{x_k}\right)d\sigma \\
& -\sum_{k=1}^{n}v_k\sum_{j=1}^{n}\left\{\beta_{kj}S_k^*G(I_j^*)\int_0^{h_1}f_{kj}(\tau)\left(\frac{x_k(t)\tilde{y}_j(t-\tau)}{y_k(t)}\right)d\tau\right. \\
& \left.+m_{kj}I_j^*\int_0^{h_2}g_{kj}(\sigma)g\left(\frac{y_j(t-\sigma)}{y_k(t)}\right)d\sigma\right\} \\
& +\sum_{k=1}^{n}\left\{\sum_{j=1}^{n}v_j\left(\beta_{jk}S_j^*\frac{G(I_j^*)}{I_j^*}+(1-\delta_{jk})m_{jk}\right)-v_k(\mu_{k2}+\gamma_k+\tilde{m}_{kk})\right\}I_k^* g(y_k(t)), \quad (5.7)
\end{aligned}
$$

from which one can obtain the following lemma.

Lemma 5.1. *For $\tilde{R}_0 > 1$, if there exists a positive n-column vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ such that (1.17) holds, then $U'(t) \leq 0$.*

Proof of Theorem 1.1. Theorem 3.1 denotes the first part $\tilde{R}_0 \leq 1$ of Theorem 1.1. For the case $\tilde{R}_0 > 1$, by Lemma 4.1, there exists at least one endemic equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$ and by Lemmas 2.1 and 4.4, system (1.6) is permanent in $\mathbf{\Gamma}^0$. We now suppose that there exists a positive n-column vector $\mathbf{v}$ such that (1.17) holds. By Lemma 5.1, it hold (5.7) for (5.2) and $U'(t) = 0$, if and only if

$$
x_k(t) = 1, \text{ and } y_k(t) = y_j(t-\sigma), \text{ for any } 0 \leq \sigma \leq h_2, \ t > 0, \ j = 1, 2, \ldots, n, \ k = 1, 2, \ldots, n.
$$

Then, there exists a positive constant c such that

$$
\frac{I_k(t)}{I_k^*} = c, \quad \text{for any } t > 0, \ j = 1, 2, \ldots, n, \ k = 1, 2, \ldots, n,
$$

and

$$S_k(t) = S_k^* \quad \text{and} \quad I_k(t) = cI_k^*, \quad \text{for any } t > 0, \ k = 1, 2, \ldots, n.$$

From the first equation of system (1.6), we obtain that

$$0 = b_k - (\mu_{k1} + \tilde{l}_{kk} + l_{kk}) + c \sum_{j=1}^{n} \beta_{kj} S_k^* G(I_j^*) - l_{kj} S_j^*, \quad \text{for any } k = 1, 2, \ldots, n. \qquad (5.8)$$

Since the right-hand side of (5.8) is strictly decreasing in c, (5.8) holds if and only if $c = 1$. This implies that $(S_1(t), I_1(t), S_2(t), I_2(t), \ldots, S_n(t), I_n(t)) = \mathbf{E}^*$. Then, the only compact invariant subset where $U'(t) = 0$ is the singleton $\{\mathbf{E}^*\}$. Therefore, by Lemmas 2.1, 2.2 and 4.4, and a similar argument as in Section 3, we obtain that for $\tilde{R}_0 > 1$, $\mathbf{E}^*$ is globally asymptotically stable in $\mathbf{\Gamma}^0$. Hence, the proof is completed. □

Concerning condition (1.18), we have the following lemma (see Guo *et al.* [8, Lemma 2.1]).

Lemma 5.2. *The following system:*

$$\sum_{j=1}^{n} v_j \left\{ \beta_{jk} S_j^* \frac{G(I_j^*)}{I_j^*} + (1 - \delta_{jk}) m_{jk} \right\} = v_k (\mu_{k2} + \gamma_k + \tilde{m}_{kk}), \qquad k = 1, 2, \ldots, n \qquad (5.9)$$

has a positive solution $(v_1, v_2, \ldots, v_n)$ defined by

$$(v_1, v_2, \cdots, v_n) = (C_{11}, C_{22}, \ldots, C_{nn}), \qquad (5.10)$$

where

$$\tilde{\beta}_{kj} = \left(\beta_{kj} S_k^* \frac{G(I_j^*)}{I_j^*} + (1 - \delta_{kj}) m_{kj} \right) I_j^*, \qquad 1 \le k, j \le n,$$

and

$$\tilde{\mathbf{B}} = \begin{bmatrix} \sum_{j \neq 1} \tilde{\beta}_{1j} & -\tilde{\beta}_{21} & \cdots & -\tilde{\beta}_{n1} \\ -\tilde{\beta}_{12} & \sum_{j \neq 2} \tilde{\beta}_{2j} & \cdots & -\tilde{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{\beta}_{1n} & -\tilde{\beta}_{2n} & \cdots & \sum_{j \neq n} \tilde{\beta}_{nj} \end{bmatrix},$$

and C_{kk} denotes the cofactor of the k-th diagonal entry of $\tilde{\mathbf{B}}$, $1 \le k \le n$.

Proof. Consider a basis for the solution space of the linear system:

$$\tilde{\mathbf{B}} \mathbf{v} = 0, \qquad (5.11)$$

which can be written as (5.10) (see for example, Berman and Plemmons [3]). By the irreducibility of $\mathbf{B}$, one can see that $\left[\tilde{\beta}_{kj} \right]_{n \times n}$ is irreducible and $v_k = C_{kk} > 0$, $k = 1, 2, \ldots, n$. Then, by (5.11), we have that

$$\begin{bmatrix} \tilde{\beta}_{11} & \tilde{\beta}_{21} & \cdots & \tilde{\beta}_{n1} \\ \tilde{\beta}_{12} & \tilde{\beta}_{22} & \cdots & \tilde{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\beta}_{1n} & \tilde{\beta}_{2n} & \cdots & \tilde{\beta}_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \left(\sum_{j=1}^{n} \tilde{\beta}_{1j} \right) v_1 \\ \left(\sum_{j=1}^{n} \tilde{\beta}_{2j} \right) v_2 \\ \vdots \\ \left(\sum_{j=1}^{n} \tilde{\beta}_{nj} \right) v_n \end{bmatrix},$$

from which we have that

$$\sum_{j=1}^{n} v_j \tilde{\beta}_{jk} = v_k \sum_{j=1}^{n} \tilde{\beta}_{kj}, \quad k = 1, 2, \ldots, n,$$

which is equivalent to

$$\sum_{j=1}^{n} v_j \left\{ \beta_{jk} S_j^* \frac{G(I_j^*)}{I_j^*} + (1 - \delta_{jk}) m_{jk} \right\} I_k^* = v_k \sum_{j=1}^{n} \left\{ \beta_{kj} S_k^* \frac{G(I_k^*)}{I_k^*} + (1 - \delta_{kj}) m_{kj} \right\} I_j^*$$

$$= v_k (\mu_{k2} + \gamma_k + \tilde{m}_{kk}) I_k^*,$$

for $k = 1, 2, \ldots, n$. By $I_k^* > 0$, one can see that (5.9) has a positive solution $(v_1, v_2, \ldots, v_n)$ defined by (5.10). $\qquad \square$

Proof of Corollary 1.2. For system (1.6), it follows from Lemma 5.2 that there exists a positive n-column vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ such that (1.18) and (1.19) hold. By Theorem 1.1, we obtain the conclusion of the first part of this corollary. If there exists positive n-column vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ such that (1.20) and (1.21) hold, then by Theorem 1.1, we obtain the conclusion of the second part of this corollary. $\qquad \square$

By Corollary 1.2, we obtain the following corollaries.

Corollary 5.3. *If $\tilde{R}_0 \leq 1$ and*

$$\begin{cases} \mu_{k1} + \tilde{l}_{kk} \geq \mu_{k2} + \gamma_k + \tilde{m}_{kk}, \quad \text{and} \\ \beta_{jk} S_j^0 + (1 - \delta_{jk}) m_{jk} \geq (1 - \delta_{jk}) l_{jk}, \text{ for any } j, k = 1, 2, \ldots, n, \\ \text{or,} \\ \mu_{k1} + \tilde{l}_{kk} \leq \mu_{k2} + \gamma_k + \tilde{m}_{kk}, \quad \text{and} \\ \beta_{jk} S_j^0 + (1 - \delta_{jk}) m_{jk} \leq (1 - \delta_{jk}) l_{jk}, \text{ for any } j, k = 1, 2, \ldots, n, \end{cases} \tag{5.12}$$

then $\mathbf{E}^0$ is globally asymptotically stable in Γ.
If $\tilde{R}_0 > 1$ and

$$\begin{cases} \mu_{k1} + \tilde{l}_{kk} \geq \mu_{k2} + \gamma_k + \tilde{m}_{kk}, \quad \text{and} \\ \beta_{jk} S_j^* \dfrac{G(I_k^*)}{I_k^*} + (1 - \delta_{jk}) m_{jk} \geq (1 - \delta_{jk}) l_{jk}, \text{ for any } j, k = 1, 2, \ldots, n, \\ \text{or,} \\ \mu_{k1} + \tilde{l}_{kk} \leq \mu_{k2} + \gamma_k + \tilde{m}_{kk}, \quad \text{and} \\ \beta_{jk} S_j^* \dfrac{G(I_k^*)}{I_k^*} + (1 - \delta_{jk}) m_{jk} \leq (1 - \delta_{jk}) l_{jk}, \text{ for any } j, k = 1, 2, \ldots, n, \end{cases} \tag{5.13}$$

then $\mathbf{E}^*$ is globally asymptotically stable in Γ^0.

Corollary 5.4. *If*

$$\begin{cases} \mu_{k1} + \tilde{l}_{kk} \geq \mu_{k2} + \gamma_k + \tilde{m}_{kk}, \quad \text{and} \\ m_{jk} \geq l_{jk}, \text{ for any } j \neq k, \ k = 1, 2, \ldots, n, \end{cases}$$

then for $\tilde{R}_0 \leq 1$, $\mathbf{E}^0$ is globally asymptotically stable in $\mathbf{\Gamma}$, and for $\tilde{R}_0 > 1$, $\mathbf{E}^*$ is globally asymptotically stable in $\mathbf{\Gamma}^0$.

If

$$
\begin{cases}
\mu_{k1} + \tilde{l}_{kk} \leq \mu_{k2} + \gamma_k + \tilde{m}_{kk}, \quad and \\
\beta_{jk}S_j^0 + (1 - \delta_{jk})m_{jk} \leq (1 - \delta_{jk})l_{jk}, \text{ for any } j, k = 1, 2, \ldots, n,
\end{cases}
\tag{5.14}
$$

then for $\tilde{R}_0 \leq 1$, $\mathbf{E}^0$ is globally asymptotically stable in $\mathbf{\Gamma}$, and for $\tilde{R}_0 > 1$, $\mathbf{E}^*$ is globally asymptotically stable in $\mathbf{\Gamma}^0$, where $\bar{I}_k^*$, $k = 1, 2, \ldots, n$ are determined by the following equation.

$$
(\mu_{k2} + \gamma_k + \tilde{m}_{kk})\bar{I}_k^* = \sum_{j=1}^{n} \{\beta_{kj}S_k^0 + (1 - \delta_{kj})m_{kj}\}\bar{I}_j^*, \qquad k = 1, 2, \ldots, n.
$$

Note that $G(I_k^*) < \bar{I}_k^*$, $k = 1, 2, \ldots, n$, if $I_k^* > 0$, $k = 1, 2, \ldots, n$ exist (see (4.1)).

6 Discussion

In this paper, we have formulated the multi-group SIR epidemic model (1.4) with nonlinear incidence rates and distributed delays, which is related to the heroin model in which the heroin users under treatment can return to untreated users depending on their different characters and external influences. We have simplified the model to (1.6) and investigated the global asymptotic stability of its equilibria. We have defined the threshold parameter $\tilde{R}_0$ as in (1.13) and proved that if $\tilde{R}_0 < 1$ and there exists a positive n-column vector $\mathbf{u}$ such that (1.16) holds, then the disease-free equilibrium $\mathbf{E}^0$ is globally asymptotically stable, while if $\tilde{R}_0 > 1$ and there exists a positive n-column vector $\mathbf{v}$ such that (1.17) holds, then the endemic equilibrium $\mathbf{E}^*$ is globally asymptotically stable (see Theorem 1.1).

The condition (5.14) in Corollary 5.4, as well as the latter part of the conditions (5.12) and (5.13) in Corollary 5.3, implies that $\beta_{kk} = 0$ for all $k = 1, 2, \ldots, n$. It is worth noting that the condition $\beta_{kk} = 0$ for all $k = 1, 2, \ldots, n$ is valid, since the matrix $\mathbf{B}$ is assumed to be irreducible. For the case $\mu_{k1} + \tilde{l}_{kk} \leq \mu_{k2} + \gamma_k + \tilde{m}_{kk}$ and $\beta_{kk} > 0$ for some $k = 1, 2, \ldots, n$, it would be an interesting open problem to find the sufficient conditions under which each of equilibria $\mathbf{E}^0$ and $\mathbf{E}^*$ is globally stable.

Acknowledgements

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