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Loitering at the hilltop on exterior domains

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Abstract. In this paper we prove the existence of an infinite number of radial solutions of $\Delta u + f(u) = 0$ on the exterior of the ball of radius $R > 0$ centered at the origin and f is odd with $f < 0$ on $(0, \beta)$, $f > 0$ on (β, δ), and $f \equiv 0$ for $u > \delta$. The primitive $F(u) = \int_0^u f(t)\,dt$ has a "hilltop" at $u = \delta$ which allows one to use the shooting method to prove the existence of solutions.

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1 Introduction

In this paper we study radial solutions of:

$$\Delta u + f(u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

$$u \to 0 \quad \text{as } |x| \to \infty, \tag{1.3}$$

where $x \in \Omega = \mathbb{R}^N \backslash B_R(0)$ is the complement of the ball of radius $R > 0$ centered at the origin. We assume there exist β, γ, δ with $0 < \beta < \gamma < \delta$ such that f is odd, locally Lipschitz with $f(0) = f(\beta) = f(\delta) = 0$, and $F(u) = \int_0^u f(s)\,ds$ where:

$$f < 0 \text{ on } (0, \beta), \quad f > 0 \text{ on } (\beta, \delta), \quad f \equiv 0 \text{ on } (\delta, \infty), \quad F(\gamma) = 0, \quad \text{and} \quad F(\delta) > 0. \tag{1.4}$$

In addition we assume:

$$f'(\beta) > 0 \quad \text{if } N > 2. \tag{1.5}$$

In an earlier paper [6] we studied (1.1), (1.3) when $\Omega = \mathbb{R}^N$ and we proved existence of an infinite number of solutions – one with exactly n zeros for each nonnegative integer n such that $u \to 0$ as $|x| \to \infty$. Interest in the topic for this paper comes from some recent papers [5, 8, 10] about solutions of differential equations on exterior domains.

When f grows superlinearly at infinity i.e. $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, and $\Omega = \mathbb{R}^N$ then the problem (1.1), (1.3) has been extensively studied [1–3, 7, 11]. However, the type of nonlinearity addressed in this paper has not.

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Since we are interested in radial solutions of (1.1)–(1.3) we assume that $u(x) = u(|x|) = u(r)$ where $r = |x| = \sqrt{x_1^2 + \cdots + x_N^2}$ so that u solves:

$$u''(r) + \frac{N-1}{r}u'(r) + f(u(r)) = 0 \quad \text{on } (R, \infty) \text{ where } R > 0, \tag{1.6}$$

$$u(R) = 0, \qquad u'(R) = a > 0. \tag{1.7}$$

We will show that there are infinitely many solutions of (1.6)–(1.7) on $[R, \infty)$ such that:

$$\lim_{r\to\infty} u(r) = 0.$$

Main theorem. *There exists a positive number d^* and positive numbers a_i so that:*

$$0 < a_0 < a_1 < a_2 < \cdots < d^*$$

and $u(r, a_i)$ satisfies (1.6)–(1.7), $u(r, a_i)$ has exactly i zeros on (R, ∞), and $\lim_{r\to\infty} u(r, a_i) = 0$.

We will first show that there exists a $d^* > 0$ so that the corresponding solution, $u(r, d^*)$, of (1.6)–(1.7) satisfies: $u(r, d^*) > 0$ on (R, ∞) and $\lim_{r\to\infty} u(r, d^*) = \delta$. Once d^* is determined we will then find the a_i.

An important step in proving this result is showing that solutions can be obtained with more and more zeros by choosing a appropriately. Intuitively it can be of help to interpret (1.6) as an equation of motion for a point $u(r)$ moving in a double-well potential $F(u)$ subject to a damping force $-\frac{N-1}{r}u'$. This potential however becomes flat at $u = \pm\delta$. According to (1.7) the system has initial position zero and initial velocity $a > 0$. We will see that if $a > 0$ is sufficiently small then the solution will "fall" into the well at $u = \beta$ and – due to damping – it will be unable to leave the well whereas if $a > 0$ is sufficiently large the solution will reach the top of the hill at $u = \delta$ and will continue to move to the right indefinitely. For an appropriate value of a – which we denote d^* – the solution will reach the top of the hill at $u = \delta$ as $r \to \infty$. For values of a slightly less than d^* the solutions will not make it to the top of the hill at $u = \delta$ and they will nearly stop moving. Thus the solution "loiters" near the hilltop on a sufficiently long interval and will usually "fall" into the positive well at $u = \beta$ or the negative well at $u = -\beta$ after passing the origin several times. The closer a is to d^* with $a < d^*$ the more times the solution passes the origin. Given $n \geq 0$ for the right value of a – which we denote as a_n – the solution will pass the origin n times and come to rest at the local maximum of the function $F(u)$ at the origin as $r \to \infty$.

In contrast to a double-well potential that goes off to infinity as $|u| \to \infty$ – for example $F(u) = u^2(u^2 - 4)$ – the solutions behave quite differently. Here as a increases the number of zeros of u increases as $a \to \infty$. Thus the number of times that u reaches the local maximum of $F(u)$ at the origin increases as the parameter a increases. See for example [7,9].

2 Preliminaries

Since $R > 0$ existence of solutions of (1.6)–(1.7) on $[R, R + \epsilon)$ for some $\epsilon > 0$ follows from the standard existence–uniqueness theorem [4] for ordinary differential equations. For existence on $[R, \infty)$ we consider:

$$E(r) = \frac{1}{2}u'^2 + F(u), \tag{2.1}$$

and using (1.6) we see that:

$$E'(r) = -\frac{N-1}{r}u'^2 \leq 0 \tag{2.2}$$

so E is nonincreasing. Therefore:

$$\frac{1}{2}u'^2 + F(u) = E(r) \leq E(R) = \frac{1}{2}a^2 \quad \text{for } r \geq R. \tag{2.3}$$

It follows from the definition of f in (1.4) that F is bounded from below and so there exists a real number, F_0, so that:

$$F(u) \geq F_0 \quad \text{for all } u. \tag{2.4}$$

Therefore (2.3)–(2.4) imply u' and hence (from (1.6)) u'' are uniformly bounded wherever they are defined. It follows from this then that u, u', and u'' are defined and continuous on $[R, \infty)$.

Lemma 2.1. *Let $u(r,a)$ be a solution of (1.6)–(1.7) with $a > 0$ and suppose $M_a \in (R, \infty)$ is a positive local maximum of $u(r,a)$. Then $|u(r,a)| < u(M_a, a)$ for $r > M_a$.*

Proof. If there were an $r_0 > M_a$ such that $|u(r_0, a)| = u(M_a, a)$ then integrating (2.2) on (M_a, r_0) and noting that $u'(M_a, a) = 0$ and F is even (since f is odd) we obtain:

$$\begin{aligned} F(u(M_a,a)) = F(u(r_0,a)) &\leq \frac{1}{2}u'^2(r_0,a) + F(u(r_0,a)) \\ &\quad + \int_{M_a}^{r_0} \frac{N-1}{r}u'^2\,dr = E(M_a) = F(u(M_a,a)). \end{aligned}$$

Thus:

$$\int_{M_a}^{r_0} \frac{N-1}{r}u'^2\,dr = 0$$

so that $u'(r,a) \equiv 0$ on (M_a, r_0) and hence by uniqueness of solutions of initial value problems it follows that $u(r,a)$ is constant on $[R, \infty)$. However, $u'(R,a) = a > 0$ and thus $u(r,a)$ is not constant. Therefore we obtain a contradiction and the lemma is proved. □

Lemma 2.2. *Let $u(r,a)$ be a solution of (1.6)–(1.7) with $a > 0$ on $(R, T_a]$ where $u(T_a, a) = \delta$ and $u'(r,a) > 0$ on $[R, T_a)$. Then $u'(r,a) > 0$ on $[R, \infty)$.*

Proof. Since $u'(r,a) > 0$ on $[R, T_a)$ then by continuity we have $u'(T_a, a) \geq 0$. If $u'(T_a, a) = 0$ then since $u(T_a, a) = \delta$ we have $f(u(T_a, a)) = 0$ and therefore by (1.6) we have $u''(T_a, a) = 0$ which would imply $u(r,a) \equiv \delta$ (by uniqueness of solutions of initial value problems) contradicting $u'(R,a) = a > 0$. Thus we see $u'(T_a, a) > 0$. Therefore $u(r,a) > \delta$ on $(T_a, T_a + \epsilon)$ for some $\epsilon > 0$ and so $f(u(r,a)) \equiv 0$ on this set. Then from (1.6) we have $u'' + \frac{N-1}{r}u' = 0$ and thus:

$$r^{n-1}u'(r,a) = T_a^{n-1}u'(T_a, a) > 0 \tag{2.5}$$

on $(T_a, T_a + \epsilon)$. It follows from this that $u(r,a)$ continues to be greater than δ so $f(u(r,a)) \equiv 0$ and therefore (1.6) reduces to $u'' + \frac{N-1}{r}u' = 0$ so that (2.5) continues to hold on $[R, \infty)$. This completes the proof. □

Lemma 2.3. *Let $u(r,a)$ be a solution of (1.6)–(1.7) with $a > 0$. Then there is an $r_a > R$ such that $u'(r,a) > 0$ on $[R, r_a]$ and $u(r_a, a) = \beta$. In addition, if $u(r,a)$ has a positive local maximum, M_a, with $\beta < u(M_a, a) < \delta$ then there exists $r_{a_2} > M_a$ such that $u'(r,a) < 0$ on $(M_a, r_{a_2}]$ and $u(r_{a_2}, a) = \beta$.*

Proof. Since $u'(R,a) = a > 0$ we see that $u(r,a)$ is increasing for values of r close to R. If $u(r,a)$ has a first critical point, $t_a > R$, with $u'(r,a) > 0$ on $[R, t_a)$ then we must have $u'(t_a,a) = 0$, $u''(t_a,a) \le 0$ and in fact $u''(t_a,a) < 0$ (by uniqueness of solutions of initial value problems). Therefore from (1.6) it follows that $f(u(t_a,a)) > 0$ so that $u(t_a,a) > \beta$. Thus the existence of r_a is established by the intermediate value theorem provided that $u(r,a)$ has a critical point. On the other hand, if $u(r,a)$ has no critical point then $u'(r,a) > 0$ for all $r \ge R$ so $\lim_{r\to\infty} u(r,a) = L$ where $0 < L \le \infty$. If $L = \infty$ then again we see by the intermediate value theorem that r_a exists. If $L < \infty$ then since E is nonincreasing by (2.2) and bounded below by (2.4), it follows that $\lim_{r\to\infty} E(r)$ exists which implies $\lim_{r\to\infty} u'(r,a)$ exists. This limit must be zero for if $u' \to A > 0$ as $r \to \infty$ then integrating this on (r_0, r) for large r_0 and r implies $u \to \infty$ as $r \to \infty$ but we know u is bounded by $L < \infty$. Thus it must be the case that $\lim_{r\to\infty} u'(r,a) = 0$. It follows then from (1.6) that $\lim_{r\to\infty} u''(r,a)$ exists and by an argument similar to the proof that $\lim_{r\to\infty} u'(r,a) = 0$ it follows that $\lim_{r\to\infty} u''(r,a) = 0$ so that by (1.6) we have $f(L) = 0$. Since $L > 0$ it follows from the definition of f that $L = \beta$ or $L = \delta$. If $L = \delta > \beta$ then again we see by the intermediate value theorem that r_a exists and so the only case we need to consider is if $u'(r,a) > 0$ and $L = \beta$. In this case we see that $f(u(r,a)) \le 0$ for all $r \ge R$ so that $u'' + \frac{N-1}{r}u' \ge 0$ by (1.6). Thus, $(r^{N-1}u'(r,a))' \ge 0$ and so $r^{N-1}u'(r,a) \ge R^{N-1}u'(R,a) = aR^{N-1} > 0$ for $r \ge R$ and hence if $1 \le N < 2$ then $u(r,a) = u(r,a) - u(R,a) \ge \frac{aR^{n-1}}{2-N}(r^{2-N} - R^{2-N}) \to \infty$ as $r \to \infty$ and if $N = 2$ then $u(r,a) = u(r,a) - u(R,a) \ge aR\ln(r/R) \to \infty$ as $r \to \infty$. These however contradict that $u(r,a) \le \beta$ and so it follows then in both of these situations that r_a exists and so we now only need to consider the case where $N > 2$ with $u'(r,a) > 0$ and $\lim_{r\to\infty} u(r,a) = \beta$. So suppose $u'(r,a) > 0$ and $u(r,a) - \beta < 0$ for $r \ge R$. Rewriting (1.6) we see:

$$u'' + \frac{N-1}{r}u' + \frac{f(u)}{u-\beta}(u-\beta) = 0.$$

Recalling (1.5) we see that:

$$\lim_{r\to\infty} \frac{f(u(r,a))}{u(r,a)-\beta} = \lim_{u\to\beta} \frac{f(u)}{u-\beta} = f'(\beta) > 0.$$

Thus $\frac{f(u(r,a))}{u(r,a)-\beta} \ge \frac{1}{2}f'(\beta)$ for $r > r_0$ where r_0 is sufficiently large. Next suppose v is a solution of:

$$v'' + \frac{N-1}{r}v' + \frac{1}{2}f'(\beta)(v-\beta) = 0$$

with $v(r_0) = u(r_0)$ and $v'(r_0) = u'(r_0)$.

Then it is straightforward to show that:

$$v(r) - \beta = r^{-\frac{N-2}{2}} J\left(\sqrt{\frac{1}{2}f'(\beta)}\, r\right)$$

where J is a solution of Bessel's equation of order $\frac{N-2}{2}$:

$$J'' + \frac{1}{r}J' + \left(1 - \frac{(\frac{N-2}{2})^2}{r^2}\right) J = 0.$$

It is well-known [4] that J has an infinite number of zeros on $(0,\infty)$ and so in particular there is an $r_1 > r_0$ where $v(r_1) - \beta = 0$. It then follows by the Sturm comparison theorem [4] that

$u(r,a) - \beta$ has a zero on (r_0, r_1) contradicting our assumption that $u(r,a) - \beta < 0$ for $r \geq R$. This therefore completes the proof of the first part of the lemma.

Suppose now that $u(r,a)$ has a maximum, M_a, so that $u'(M_a,a) = 0$ and $\beta < u(M_a,a) < \delta$. A similar argument using the Sturm comparison theorem shows that $u(r,a)$ again must equal β for some $r > M_a$. This completes the proof of the lemma. □

3 Proof of the Main theorem

Before proceeding to the proof of the main theorem, we will first show that there is a $d^* > 0$ such that $u'(r,d^*) > 0$ for $r \geq R$, $0 < u(r,a) < \delta$ for $r > R$, and $u(r,a) \to \delta$ as $r \to \infty$.

Let ϵ be chosen so that $0 < \epsilon < \delta - \gamma$. (Recall that $\beta < \gamma < \delta$ and $F(\gamma) = 0$).

Lemma 3.1. *Let $u(r,a)$ be a solution of (1.6)–(1.7) with $a > 0$. If $0 < a < \sqrt{2F(\delta-\epsilon)}$ then $u(r,a) < \delta - \epsilon$ on $[R,\infty)$.*

Proof. Since $E' \leq 0$ by (2.2) we see for $r \geq R$ that:

$$F(u(r,a)) \leq \frac{1}{2}u'^2(r,a) + F(u(r,a)) = E(r) \leq E(R) = \frac{1}{2}a^2 < F(\delta - \epsilon). \tag{3.1}$$

Now if there is an $r_0 > R$ such that $u(r_0,a) = \delta - \epsilon$ then substituting in (3.1) gives: $F(\delta-\epsilon) \leq \frac{1}{2}a^2 < F(\delta - \epsilon)$ which is impossible. □

Lemma 3.2. *Let $u(r,a)$ be a solution of (1.6)–(1.7) with $a > 0$. If $0 < \epsilon < \delta - \gamma$ and $0 < a < \sqrt{2F(\delta-\epsilon)}$ then there exists an $M_a > R$ such that $u(r,a)$ has a local maximum at M_a with $u(M_a,a) < \delta$ and $u'(r,a) > 0$ on $[R, M_a)$.*

Proof. From Lemma 3.1 we see that since $0 < \epsilon < \delta - \gamma$ and $0 < a < \sqrt{2F(\delta-\epsilon)}$ then $u(r,a) < \delta - \epsilon$ on $[R,\infty)$. Also $u(r,a)$ is increasing near $r = R$ since $u'(R,a) = a > 0$. We suppose now by the way of contradiction that $u'(r,a) > 0$ for all $r \geq R$. Then by Lemma 3.1 there is an $L > 0$ such that $\lim_{r\to\infty} u(r,a) = L \leq \delta - \epsilon$. Since E is bounded from below by (2.4), $E' \leq 0$ by (2.2), and $\lim_{r\to\infty} u(r,a) = L$, it follows that $\lim_{r\to\infty} u'(r,a)$ exists and in fact this must be zero (as in the proof of Lemma 2.3). From (1.6) it follows that $\lim_{r\to\infty} u''(r,a) = -f(L)$ and in fact this must also be zero (as in the proof that $\lim_{r\to\infty} u'(r,a) = 0$ from Lemma 2.3) and therefore $f(L) = 0$. Since $0 < L \leq \delta - \epsilon$ it then follows that $L = \beta$. However, from Lemma 2.3 we know that $u(r,a)$ must equal β for some $r_a > R$ and since we are assuming $u'(r,a) > 0$ for $r \geq R$ we see that $u(r,a)$ exceeds β for large r so that $L > \beta$ – a contradiction. Thus there is an $M_a > R$ with $u(M_a,a) < \delta - \epsilon$, $u'(r,a) > 0$ on $[R, M_a)$, $u'(M_a,a) = 0$, and $u''(M_a,a) \leq 0$. We have in fact that $u''(M_a,a) < 0$ (by uniqueness of solutions of initial value problems) and therefore M_a is a local maximum for $u(r,a)$. This completes the proof. □

Lemma 3.3. *Let $u(r,a)$ be a solution of (1.6)–(1.7). For sufficiently large $a > 0$ there exists $T_a > R$ such that $u(T_a,a) = \delta$, $u(r,a) < \delta$ on $[R, T_a)$, and $u'(r,a) > 0$ on $[R,\infty)$.*

Proof. Suppose $u(r,a) < \delta$ for all $r \geq R$ for all sufficiently large a. We first show that $|u(r,a)| < \delta$ for all $r \geq R$. If $u(r,a)$ is nondecreasing for all $r \geq R$ then of course we have $u(r,a) > 0 > -\delta$ and so $|u(r,a)| < \delta$ for all $r \geq R$. On the other hand if u is nondecreasing on $[R, M_a)$ such that $u(r,a)$ has a local maximum at M_a with $u(M_a,a) < \delta$ then by Lemma 2.1 we have $|u(r,a)| < u(M_a,a) < \delta$ for $r > M_a$. Thus in either case we see that:

$$|u(r,a)| < \delta \quad \text{for all } r \geq R. \tag{3.2}$$

Now we let $v_a(r) = \frac{u(r,a)}{a}$. Then v_a satisfies:

$$v_a'' + \frac{N-1}{r} v_a' + \frac{1}{a} f(a v_a) = 0, \tag{3.3}$$

$$v_a(R) = 0, \qquad v_a'(R) = 1. \tag{3.4}$$

It also follows from (2.2)–(2.3) that:

$$\left(\frac{1}{2} v_a'^2 + \frac{1}{a^2} F(a v_a) \right)' \leq 0 \quad \text{for } r \geq R,$$

and so integrating this on $[R, r)$ gives:

$$\frac{1}{2} v_a'^2 + \frac{1}{a^2} F(a v_a) \leq \frac{1}{2} \quad \text{for } r \geq R. \tag{3.5}$$

From (3.2) we know $|v_a| = |\frac{u(r,a)}{a}| < \frac{\delta}{a}$ and since F is bounded from below by (2.4) it follows from (3.5) that the $\{v_a'\}$ are uniformly bounded for large values of a. From (3.3) it also follows that the $\{v_a''\}$ are uniformly bounded for large values of a and so by the Arzelà–Ascoli theorem there is a subsequence of $\{v_a\}$ and $\{v_a'\}$ (still denoted $\{v_a\}$ and $\{v_a'\}$) such that $v_a \to v$ and $v_a' \to v'$ uniformly on compact subsets of $[R, \infty)$ as $a \to \infty$. But clearly $v \equiv 0$ (since $|v_a| = |\frac{u(r,a)}{a}| < \frac{\delta}{a}$ by (3.2) thus $|v_a| \to 0$ as $a \to \infty$) whereas $v'(R) = 1$ – a contradiction.

Therefore it must be the case that if a is sufficiently large then there exists $T_a > R$ such that $u(T_a, a) = \delta$ and $u(r, a) < \delta$ on $[R, T_a)$. In addition, it must be the case that $u'(r, a) > 0$ on $[R, T_a)$ for if not then there exists an $M_a < T_a$ such that $u'(M_a, a) = 0$ and $u(M_a, a) < \delta$. But from Lemma 2.1 it would follow that $|u(r, a)| < u(M_a, a) < \delta$ for $r > M_a$ contradicting that $u(T_a, a) = \delta$. Thus $u'(r, a) > 0$ on $[R, T_a)$. Now from Lemma 2.2 it follows that $u'(r, a) > 0$ on $[R, \infty)$. This completes the proof. □

Now let:

$$S = \Big\{ a > 0 \mid \exists M_a \text{ with } M_a > R \mid u'(r,a) > 0 \text{ on } [R, M_a),$$
$$u'(M_a, a) = 0,\ u''(M_a, a) < 0, \text{ and } u(M_a, a) < \delta \Big\}.$$

From Lemma 3.2 it follows that S is nonempty and from Lemma 3.3 it follows that S is bounded above. Next we set:

$$0 < d^* = \sup S.$$

Lemma 3.4. *Let $u(r, d^*)$ be the solution of (1.6)–(1.7) with $a = d^*$. Then:*

$$0 < u(r, d^*) < \delta \quad \textit{for all } r > R,$$

$$u'(r, d^*) > 0 \quad \textit{for all } r \geq R, \textit{ and:}$$

$$\lim_{r \to \infty} u(r, d^*) = \delta.$$

Proof. We first note that $d^* \notin S$ for if $d^* \in S$ then by continuity with respect to initial conditions that $d^* + \epsilon \in S$ for $\epsilon > 0$ sufficiently small contradicting the definition of d^*. Thus $d^* \notin S$. Therefore there exist $a \in S$ with $a < d^*$ and a arbitrarily close to d^*.

Next we show $u(r, d^*) < \delta$ for all $r \geq R$. First since $u(r, a) < \delta$ for all $a < d^*$ then by continuity with respect to initial conditions it follows that $u(r, d^*) \leq \delta$. Now suppose that there exists $T_{d^*} > R$ such that $u(T_{d^*}, d^*) = \delta$ with $u(r, d^*) < \delta$ for $R \le r < T_{d^*}$. Then by Lemma 2.2 we have $u'(r, d^*) > 0$ on $[R, \infty)$. So there exists $r_0 > T_{d^*}$ such that $u(r_0, d^*) > \delta + \epsilon$ for some $\epsilon > 0$. Then by continuity with respect to initial conditions it follows that $u(r_0, a) > \delta + \frac{1}{2}\epsilon$ for $a < d^*$ and a sufficiently close to d^*. But this contradicts that for $a < d^*$ we have $u(r, a) < \delta$ by Lemma 2.1. Thus there is no such T_{d^*} and so:

$$u(r, d^*) < \delta \quad \text{for all } r \ge R. \tag{3.6}$$

Now for $a < d^*$ and $a \in S$ there is an M_a where $u(r, a)$ has a local maximum. If $u(r, d^*)$ has a local maximum, M_{d^*}, then $u(M_{d^*}, d^*) < \delta$ by (3.6) and $u''(M_{d^*}, d^*) \le 0$. In fact, $u''(M_{d^*}, d^*) < 0$ (by uniqueness of solutions to initial value problems) and so by continuity with respect to initial conditions this implies that:

$$u(r, a) \text{ has a local maximum, } M_a, \text{ for } a \text{ slightly larger than } d^*. \tag{3.7}$$

But for $a > d^*$ we have $a \notin S$ so either $u'(r, a) > 0$ on $[R, \infty)$ or there exists N_a such that $u'(N_a, a) = 0$ and $u(N_a, a) \ge \delta$.

Clearly the first option does not hold because this contradicts (3.7) so therefore the second must be true. Then since $u(N_a, a) \ge \delta$ we have $f(u(N_a, a)) = 0$ and since $u'(N_a, a) = 0$ then $u''(N_a, a) = 0$ (from (1.6)) which implies $u(r, a)$ is constant (by uniqueness of solutions of initial value problems). But $a > d^* > 0$ and thus $u'(R, a) = a > 0$ so that $u(r, a)$ is not constant. This contradiction implies that the second option does not hold either so $u(r, d^*)$ has no local maximum and therefore $u'(r, d^*) > 0$ for all $r \ge R$. Thus $u(r, d^*)$ is increasing and bounded above by δ so $\lim_{r\to\infty} u(r, d^*) = L$ with $0 < L \le \delta$ and as in the proof of Lemma 2.3 we see $\lim_{r\to\infty} u'(r, a) = \lim_{r\to\infty} u''(r, a) = 0$ and so $f(L) = 0$. Thus $L = \beta$ or $L = \delta$. By Lemma 2.3 we know that u must equal β for some $r > R$ and since $u'(r, a) > 0$ for $r \ge R$ we see that $u(r, a)$ exceeds β for large r. Thus we see that $L = \delta$. This completes the proof. □

Lemma 3.5. *Let $u(r, a)$ be a solution on (1.6)–(1.7). For $0 < a < d^*$ and $a \in S$, $u(r, a)$ has a local maximum, M_a, on (R, ∞) such that:*

$$\lim_{a\to {d^*}^-} M_a = \infty,$$

and:

$$\lim_{a\to {d^*}^-} u(M_a, a) = \delta.$$

Proof. Since $a \in S$ then we know that M_a exists. If the $\{M_a\}$ were bounded independent of a then there is a subsequence (still labeled $\{M_a\}$) and a real number M such that $M_a \to M$. Also, by (2.3) and since F is bounded from below by (2.4) it follows that $\{u'(r, a)\}$ are uniformly bounded. It then follows from (1.6) that $\{u''(r, a)\}$ are uniformly bounded. Also $0 < u(r, a) < \delta$ on (R, ∞) and so by the Arzelà–Ascoli theorem there is a subsequence of $\{u(r, a)\}$ and $\{u'(r, a)\}$ (still labeled $\{u(r, a)\}$ and $\{u'(r, a)\}$) such that $u(r, a) \to u(r, d^*)$ and $u'(r, a) \to u'(r, d^*)$ uniformly on compact sets and so in particular $u'(M, d^*) = 0$. However, we know from Lemma 3.4 that $u'(r, d^*) > 0$ for $r \ge R$ and so we obtain a contradiction. Thus $\lim_{a\to {d^*}^-} M_a = \infty$. Next since $\lim_{r\to\infty} u(r, d^*) = \delta$ by Lemma 3.4 then given $\epsilon > 0$ there is $r_0 > R$ such that $u(r_0, d^*) > \delta - \frac{\epsilon}{2}$. Since $u(r, a) \to u(r, d^*)$ uniformly on compact subsets of $[R, \infty)$ as $a \to d^*$ it then follows that for a sufficiently close to d^* there is some p_a close to r_0 with $u(r_p, a) > \delta - \epsilon$. And since $u(r, a)$ has its maximum at M_a we have $u(M_a, a) \ge u(p_a, a) > \delta - \epsilon$. Thus $\lim_{a\to {d^*}^-} u(M_a, a) = \delta$. □

Lemma 3.6. *Let $u(r,a)$ be a solution of (1.6)–(1.7). For sufficiently small $a>0$ we have $u(r,a)>0$ for all $r>R$.*

Proof. We observe that from (2.2):

$$\{r^{2N-2}E(r)\}' = (2N-2)r^{2N-3}F(u) \le 0 \quad \text{when } 0 \le u \le \gamma. \tag{3.8}$$

We denote r_{a_1} as the smallest value of $r>R$ such that $u(r_{a_1},a)=\frac{1}{2}\beta$ and r_a as the smallest value of $r>R$ such that $u(r_a,a)=\beta$. We know that these numbers exist by Lemma 2.3 and it also follows from Lemma 2.3 that $u'(r,a)>0$ on $[R,r_a]$. By the definition of f and F we see that on the set $[\frac{1}{2}\beta,\beta]$ there exists $c_0>0$ such that $F(u)\le -c_0<0$. Therefore integrating (3.8) on $[R,r_a]$ and estimating we obtain:

$$\begin{aligned} r_a^{2N-2}E(r_a) &= R^{2N-2}E(R) + \int_R^{r_a}(2N-2)r^{2N-3}F(u)\,dr \\ &\le \frac{1}{2}R^{2N-2}a^2 + \int_{r_{a_1}}^{r_a}(2N-2)r^{2N-3}F(u)\,dr \le \frac{1}{2}R^{2N-2}a^2 - c_0[r_a^{2N-2}-r_{a_1}^{2N-2}] \\ &\le \frac{1}{2}R^{2N-2}a^2 - (2N-2)c_0[r_a - r_{a_1}]r_{a_1}^{2N-3}. \end{aligned} \tag{3.9}$$

Recalling (2.3) and rewriting we have:

$$\frac{|u'|}{\sqrt{a^2-2F(u)}} \le 1 \quad \text{on } [R,\infty). \tag{3.10}$$

Integrating (3.10) on $[R,r_{a_1}]$ where $u'(r,a)>0$ gives:

$$\int_0^{\frac{\beta}{2}} \frac{ds}{\sqrt{a^2-2F(s)}} = \int_R^{r_{a_1}} \frac{u'}{\sqrt{a^2-2F(u)}}\,dt \le r_{a_1}-R. \tag{3.11}$$

On $[0,\beta]$ we have $2F(s)\ge -c_1^2s^2$ for some $c_1>0$ and therefore:

$$\int_0^{\frac{\beta}{2}} \frac{ds}{\sqrt{a^2-2F(s)}} \ge \int_0^{\frac{\beta}{2}} \frac{ds}{\sqrt{a^2+c_1^2s^2}} = \frac{1}{c_1}\ln\left(\frac{c_1\beta}{2a} + \sqrt{1+\left(\frac{c_1\beta}{2a}\right)^2}\right) \to \infty \quad \text{as } a\to 0^+. \tag{3.12}$$

Therefore by (3.11) and (3.12) we have:

$$r_{a_1} \to \infty \quad \text{as } a \to 0^+. \tag{3.13}$$

In addition, integrating (3.10) on $[r_{a_1},r_a]$ gives for small a:

$$\int_{\frac{\beta}{2}}^{\beta} \frac{ds}{\sqrt{a^2+c_1^2s^2}} \le \int_{\frac{\beta}{2}}^{\beta} \frac{ds}{\sqrt{a^2-2F(s)}} = \int_{r_{a_1}}^{r_a} \frac{u'}{\sqrt{a^2-2F(u)}}\,dt \le r_a - r_{a_1}. \tag{3.14}$$

The left-hand side of (3.14) approaches $\int_{\frac{\beta}{2}}^{\beta} \frac{ds}{c_1 s} = \frac{\ln(2)}{c_1} \ge \frac{1}{2c_1}$ as $a\to 0^+$ therefore it follows from (3.9) and (3.13)–(3.14) that:

$$r_a^{2N-2}E(r_a) \le \frac{1}{2}R^{2N-2}a^2 - \frac{(N-1)c_0 r_{a_1}^{2N-3}}{c_1} \to -\infty$$

as $a\to 0^+$. Thus for sufficiently small a we see that E becomes negative on $[R,r_a]$ and since E is nonincreasing by (2.2), E remains negative for all $r\ge r_a$. It follows that $u(r,a)$ cannot be zero for any $r>r_a$ because at any such point z we would have $E(z)=\frac{1}{2}u'^2(z,a)\ge 0$. We also know $u(r,a)$ is increasing on $[R,r_a]$ by Lemma 2.3 and so $u(r,a)>0$ on $[R,r_a]$. Thus $u(r,a)$ stays positive for all $r>R$ for small $a>0$. This completes the proof. □

Lemma 3.7. *There exists d_1 with $0 < d_1 < d^*$ such that $u(r, d_1)$ has at least one zero on $[R, \infty)$. In addition, if $a < d^*$ and a is sufficiently close to d^* then $u(r, a)$ has a local minimum, m_a, and $u(m_a, a) \to -\delta$ as $a \to d^{*-}$.*

Proof. Suppose first that $a \in S$ and $u'(r, a) < 0$ on (M_a, r). Then integrating (2.2) on (M_a, r), using (2.3)–(2.4), and using the fact from Lemma 2.1 that $-\delta < u(r, a) < \delta$ on (M_a, r) gives:

$$
\begin{aligned}
E(M_a) - E(r) &= \int_{M_a}^{r} \frac{N-1}{t} u'^2(t, a)\, dt \le \frac{N-1}{M_a} \int_{M_a}^{r} |u'(t, a)||u'(t, a)|\, dt \\
&\le \frac{N-1}{M_a} \int_{M_a}^{r} \sqrt{a^2 - 2F(u(t, a))}[-u'(t, a)]\, dt \\
&\le \frac{N-1}{M_a} \int_{u(r,a)}^{u(M_a,a)} \sqrt{a^2 - 2F(s)}\, ds \le \frac{2(N-1)\delta\sqrt{a^2 - 2F_0}}{M_a}.
\end{aligned}
$$

Thus we see:

$$
E(M_a) - E(r) \le \frac{2(N-1)\delta\sqrt{a^2 - 2F_0}}{M_a}. \tag{3.15}
$$

We now have two possibilities. Either:

(i) $u'(r, a) < 0$ for all $r > M_a$ for a sufficiently close to d^*,

or:

(ii) there exists $m_a > M_a$ such that $u'(r, a) < 0$ on (M_a, m_a) and $u'(m_a, a) = 0$ for a sufficiently close to d^*.

If (i) holds then $u(r, a) \to L$ and as in the proof of Lemma 2.3 it follows that $u'(r, a) \to 0$ and $u''(r, a) \to 0$ as $r \to \infty$ where $f(L) = 0$. By Lemma 2.1 we also have $|u(r, a)| < u(M_a, a) < \delta$ for $r > M_a$ so that $L = 0$ or $L = \pm\beta$. In particular, $|L| \le \beta$. Also as $r \to \infty$ we see from (3.15):

$$
0 < F(u(M_a, a)) - F(L) = E(M_a) - E(\infty) \le \frac{2(N-1)\delta\sqrt{a^2 - 2F_0}}{M_a}. \tag{3.16}
$$

As $a \to d^{*-}$ the right-hand side of (3.16) goes to 0 by Lemma 3.5. Also by Lemma 3.5, $F(u(M_a, a)) \to F(\delta) > 0$ as $a \to d^{*-}$ and therefore it follows from (3.16) that $F(L) > 0$ for a sufficiently close to d^*. This however implies that $|L| \ge \gamma > \beta$ which contradicts that $|L| \le \beta$. Therefore we see that (i) does not hold for a sufficiently close to d^*. Thus it must be the case that (ii) holds for a sufficiently close to d^*. With $r = m_a$ then we have from (3.15):

$$
F(u(M_a, a)) - F(u(m_a, a)) = E(M_a) - E(m_a) \le \frac{2(N-1)\delta\sqrt{a^2 - 2F_0}}{M_a}. \tag{3.17}
$$

As above the right-hand side of (3.17) goes to 0 by Lemma 3.5 and $F(u(M_a, a)) \to F(\delta) > 0$ as $a \to d^{*-}$. Therefore it follows that $F(u(m_a, a)) \to F(\delta) > 0$ and hence $|u(m_a, a)| \to \delta$ for $a \to d^*$. Also since $u'(m_a, a) = 0$ and $u'(r, a) < 0$ on (M_a, m_a) we must have $u''(m_a) \ge 0$ so that $f(u(m_a, a)) \le 0$. This implies $u(m_a, a) \le -\beta < 0$ thus $u(r, a) \to -\delta$ and in particular we see that $u(r, a)$ must be zero somewhere on the interval (M_a, m_a) provided a is sufficiently close to d^*. So there exists a d_1 with $0 < d_1 < d^*$ such that $u(r, d_1)$ has at least one zero on (R, ∞). This completes the proof of the lemma. □

Now let:

$$W_0 = \{0 < a < d_1 \mid u(r,a) > 0 \text{ on } [R,\infty)\}.$$

By Lemma 3.6 we know that W_0 is nonempty, and clearly W_0 is bounded above by d_1. So we let:

$$a_0 = \sup W_0.$$

Then we have the following lemma.

Lemma 3.8. *$u(r,a_0) > 0$ on $[R,\infty)$ and $\lim_{r\to\infty} u(r,a_0) = 0$. In addition, there is an M_{a_0} such that $u'(r,a_0) > 0$ on $[R, M_{a_0})$ and $u'(r,a_0) < 0$ on (M_{a_0},∞).*

Proof. If $u(r,a_0)$ has a zero, z, then $u'(z,a_0) \neq 0$ (by uniqueness of solutions of initial value problems) and so $u(r,a)$ will have a zero for a slightly larger than a_0 which contradicts the definition of a_0. Thus $u(r,a_0) > 0$ on $[R,\infty)$.

Next suppose that $u(r,a_0)$ has a positive local minimum, m_{a_0}, so that $u'(m_{a_0},a_0) = 0$, $u''(m_{a_0},a_0) \geq 0$, (and in fact $u''(m_{a_0},a_0) > 0$ by uniqueness of solutions of initial value problems), so therefore $f(u(m_{a_0},a_0)) < 0$. Then $0 < u(m_{a_0},a_0) < \beta$ and $E(m_{a_0}) = F(u(m_{a_0},a_0)) < 0$. Thus for $a > a_0$ and a close to a_0 then $u(r,a)$ must also have a positive local minimum, m_a, and $E(m_a) < 0$. But since $a > a_0$ then $u(r,a)$ must have a zero, z_a, with $z_a > m_a$. Since E is nonincreasing this implies $0 \leq \frac{1}{2}u'^2(z_a,a) = E(z_a) \leq E(m_a) < 0$ which is a contradiction.

Thus it must be that $u'(r,a_0) < 0$ for $r > M_{a_0}$. Since $u(r,a_0) > 0$ it follows then that $u(r,a_0) \to \beta$ or $u(r,a_0) \to 0$ as $r \to \infty$ but from Lemma 2.3 we know that $u(r,a_0)$ will become less than β for sufficiently large r. Thus $u(r,a_0) \to 0$ as $r \to \infty$. This completes the proof of the lemma. □

Proof of the Main theorem. Now for $a_0 < a < d^*$ it follows that $u(r,a)$ has *at least one zero* on $[R,\infty)$. By Lemma 4 from [9], for $a > a_0$ and a close to a_0 then $u(r,a)$ has *at most one zero* on $[R,\infty)$. Hence for $a > a_0$ and a sufficiently close to a_0 then $u(r,a)$ has *exactly one zero* on $[R,\infty)$.

Next we can use a similar argument as in Lemma 3.7 to prove that there exists d_2 with $d_1 \leq d_2 < d^*$ such that $u(r,d_2)$ has at least two zeros on $[R,\infty)$.

To see this, using a nearly identical argument as in Lemma 3.7 it follows that:

$$E(m_a) - E(r) \leq \frac{2(N-1)\delta\sqrt{a^2 - 2F_0}}{m_a} \tag{3.18}$$

where m_a is the minimum obtained in Lemma 3.7. Then either:

(i) $u'(r,a) > 0$ for $r > m_a$ for a sufficiently close to d^*,

or:

(ii) there exists $M_{2,a} > m_a$ such that $u'(r,a) > 0$ on $(m_a, M_{2,a})$ and $u'(M_{2,a}) = 0$ for a sufficiently close to d^*.

If (i) holds then it follows as in the proof of Lemma 3.7 that $u(r,a) \to L$ where $L = 0$ or $L = \pm\beta$. And as $r \to \infty$ we see from (3.18):

$$F(u(m_a,a)) - F(L) = E(m_a) - E(\infty) \leq \frac{2(N-1)\delta\sqrt{a^2 - 2F_0}}{m_a}. \tag{3.19}$$

As $a \to d^{*-}$ the right-hand side of (3.19) goes to zero since $m_a > M_a$ and $M_a \to \infty$ by Lemma 3.5. Also by Lemma 3.7, $F(u(m_a,a)) \to F(\delta) > 0$ as $a \to d^{*-}$ and so $F(L) > 0$ for a

sufficiently close to d^* which implies $|L| \geq \gamma > \beta$ which contradicts $|L| \leq \beta$. Thus it must be the case that (ii) holds and as in the proof of Lemma 3.7 it follows that $u(r,a)$ must be zero on $(m_a, M_{2,a})$. So there exists a d_2 with $d_1 < d_2 < d^*$ such that $u(r,d_2)$ has at least two zeros on (R,∞).

Then we define:

$$W_1 = \{a_0 < a < d_2 \mid u(r,a) \text{ has exactly one zero on } [R,\infty)\}.$$

Clearly W_1 is nonempty since from Lemma 3.7 we have $d_1 \in W_1$. Also W_1 is bounded above by d_2. Thus we set:

$$a_1 = \sup W_1.$$

Then it can be shown in an argument similar to the one in Lemma 3.8 that $u(r,a_1)$ has one zero on (R,∞) and $u(r,a_1) \to 0$ as $r \to \infty$. Proceeding inductively we can show for $n \geq 1$ that there exists a_n with $a_{n-1} < a_n < d^*$ such that $u(r,a_n)$ has exactly n zeros on (R,∞) and $u(r,a_n) \to 0$ as $r \to \infty$. This completes the proof of the main theorem. □

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