



Nonexistence of solutions for singular nonlinear ordinary inequalities

Xiaohong Li 

College of Mathematics and Information Science, Shandong Institute of Business and Technology,
Yantai, 264005, Shandong, China

Received 27 March 2015, appeared 19 October 2015

Communicated by Patrizia Pucci

Abstract. In this paper we prove nonexistence theorems of nonnegative nontrivial solutions for a singular nonlinear ordinary inequality in bounded domains with singular points on the boundary. The proofs are based on the test function method developed by Mitidieri and Pohozaev. We also give the examples demonstrating that the conditions obtained are sharp in the case of the problem under consideration.

Keywords: nonlinear differential inequalities, nonexistence theorem, test function method.

2010 Mathematics Subject Classification: 35J60, 35J75.

1 Introduction

In this paper, we shall consider nonexistence of nontrivial weak solutions of the singular nonlinear differential inequality


$$\begin{cases} (|u'|^{p-2}u')' \geq a(x)u^q & \text{for } x \in (0, x_0], \\ u(x) \geq 0 & \text{for } x \in (0, x_0], \\ u'(x_0) < 0, \end{cases} \quad (1.1)$$

where $x_0 > 0$, $p > 1$, $q > p - 1$, and the function $a \in C((0, x_0])$ satisfies the estimate

$$a(x) \geq cx^{-\alpha} \quad (1.2)$$

for some constants $\alpha \in \mathbb{R}$ and $c > 0$.

There have been many results on the nonexistence of nonnegative nontrivial solutions for nonlinear differential inequalities (systems), see [1–32] and references therein. Tools based on different forms of the maximum principle like the moving planes method or moving spheres method, nonlinear capacity estimates and Pohozaev type identities, energy methods and

 Email: xh0535@sina.com

Harnack inequality type argument, have been proved to be very successful for solving interesting problems related to applications and to the general theory of partial differential equations.

Mitidieri and Pohozaev (see [22]) have developed a new effective approach to these problems on the basis of a special choice of test functions. By integration technique which uses suitable test functions, they have established a priori estimates of solutions and obtained the nonexistence results. This approach not only provides simple, accurate, and more general results but also is essentially different from the comparison method. Moreover, it can be applied to a wide class of nonlinear differential inequalities (see [5–7, 16–23, 25–29]) and systems (see [12, 14, 15, 24]). In particular, it was shown in [22] that the inequalities

$$\pm \Delta_p u \geq |x|^{-\alpha} u^q \quad \text{in } \mathbb{R}^N \quad (1.3)$$

have no weak positive solutions, and then, G. Caristi [3] perfected the results, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. In fact this method was applied to more general operators, including the generalized mean curvature operator (see [2, 16], [22–32]) and a wide class of anisotropic quasilinear operator (see [5, 6]). Later, by using refined techniques, Filippucci, Pucci and Rigoli (see [8–11]) proved very significant existence and nonexistence results for the coercive case.

In the present paper, by modifying the method developed by Mitidieri and Pohozaev in [22] and Galakhov in [16], we will show nonexistence theorems for the nonlinear differential inequality (1.1) with singular points on the boundary.

We understand solutions to problem (1.1) in the sense of distributions and define the class of admissible solutions to problem (1.1) as

$$X((0, x_0]) := \{u : (0, x_0] \rightarrow \mathbb{R}_+, \quad a(x)u^q, \quad |u'|^p \in L^1_{loc}((0, x_0])\}.$$

We prove the following theorems.

Theorem 1.1. *Suppose that the function $a \in C((0, x_0])$ is nonnegative and satisfies inequality (1.2), and $q > p - 1$. If $\alpha > p$, then the problem (1.1) has no nontrivial nonnegative solutions in $X((0, x_0])$.*

Theorem 1.2. *Under the assumptions of Theorem 1.1, the problem (1.1) with $\alpha = p$ has no nontrivial nonnegative solutions in $X((0, x_0]) \cap C((0, x_0])$.*

Remark 1.3. For $\alpha < p$ and $q > p - 1$, a solution of problem (1.1) with $a(x) = x^{-\alpha}$ can be written down explicitly as $u(x) = Cx^{\frac{\alpha-p}{q-p+1}}$ with an appropriate constant $C > 0$. Thus, the assumption $\alpha \geq p$ is essential to deal with nonexistence results.

2 Proofs of Theorems 1.1 and 1.2

In this section, we will prove the two theorems. In doing so we will follow the argument of Theorem 2.1 in [22] and Theorem 3.4 in [16].

To establish a priori estimates of the solutions, we need to define some test functions that will be widely used in the sequel. We consider the test function $\zeta \in C^1([0, x_0]; [0, 1])$ that satisfies

$$\zeta(x) = \begin{cases} 1, & \eta < x < x_0, \\ 0, & 0 < x < \eta/2, \end{cases} \quad (2.1)$$

and

$$|\xi'(x)| \leq c \eta^{-1}, \quad \forall x \in (0, x_0), \quad (2.2)$$

where $\eta \in (0, x_0)$ is a parameter and $c > 0$ is a constant. Set

$$\chi(x) = \xi^\lambda(x), \quad (2.3)$$

where $\lambda > 0$ is a parameter to be chosen later according to the nature of the problem.

To prove the main results of this section, we need the following lemma.

Lemma 2.1. *Assume that $a \in C((0, x_0])$ is a nonnegative function. Let χ be defined as (2.3) and $q > p - 1$. Then each nontrivial nonnegative solution to problem (1.1) in $X((0, x_0])$ satisfies a priori estimate*

$$\int_0^{x_0} a(x) u^{q+\gamma} \chi dx \leq C \int_{\eta/2}^{\eta} a(x)^{-\frac{p-1+\gamma}{q-p+1}} |\xi'|^{\frac{p(q+\gamma)}{q-p+1}} dx \quad (2.4)$$

for $\gamma > 0$, with a constant $C > 0$ independent of u .

Proof. Without loss of generality, we suppose $u > 0$. If u is allowed to vanish at some points, we consider $u_\delta = u + \delta$ with arbitrary $\delta > 0$ and then pass to the limit as $\delta \rightarrow 0^+$. Let $\gamma \in \mathbb{R}$ be a parameter to be chosen later. Multiplying (1.1) by $u^\gamma \chi$ and integrating by parts, we get

$$\int_0^{x_0} a(x) u^{q+\gamma} \chi dx + \gamma \int_0^{x_0} |u'|^p u^{\gamma-1} \chi dx \leq |u'|^{p-2} u' u^\gamma \chi|_0^{x_0} + \int_0^{x_0} |u'|^{p-1} u^\gamma |\chi'| dx. \quad (2.5)$$

Applying Young's inequality with exponents $l = \frac{p}{p-1}, l' = p > 1, \varepsilon > 0$ to the second integral on the right-hand side of (2.5), we obtain

$$\begin{aligned} & \int_0^{x_0} a(x) u^{q+\gamma} \chi dx + \gamma \int_0^{x_0} |u'|^p u^{\gamma-1} \chi dx \\ & \leq |u'|^{p-2} u' u^\gamma \chi|_0^{x_0} + \varepsilon \int_0^{x_0} |u'|^p u^{\gamma-1} \chi dx + \varepsilon^{1-p} \int_0^{x_0} u^{p+\gamma-1} \frac{|\chi'|^p}{\chi^{p-1}} dx. \end{aligned} \quad (2.6)$$

Taking $\varepsilon = \gamma/2$, we have

$$\int_0^{x_0} a(x) u^{q+\gamma} \chi dx + \frac{\gamma}{2} \int_0^{x_0} |u'|^p u^{\gamma-1} \chi dx \leq |u'|^{p-2} u' u^\gamma \chi|_0^{x_0} + \left(\frac{\gamma}{2}\right)^{1-p} \int_0^{x_0} u^{p+\gamma-1} \frac{|\chi'|^p}{\chi^{p-1}} dx. \quad (2.7)$$

By Hölder's inequality with exponents $m = \frac{q+\gamma}{p-1+\gamma} > 1, m' = \frac{q+\gamma}{q-p+1} > 1$ for every $\gamma > 0$ to the second integral on the right-hand side of (2.7) (since, by assumption, $q > p - 1$), we get

$$\begin{aligned} & \int_0^{x_0} a(x) u^{q+\gamma} \chi dx \\ & \leq |u'|^{p-2} u' u^\gamma \chi|_0^{x_0} + \left(\frac{\gamma}{2}\right)^{1-p} \left(\int_0^{x_0} a(x) u^{q+\gamma} \chi dx \right)^{\frac{1}{m}} \left(\int_0^{x_0} a(x)^{-\frac{m'}{m}} \frac{|\chi'|^{pm'}}{\chi^{pm'-1}} dx \right)^{\frac{1}{m'}}, \end{aligned} \quad (2.8)$$

i.e.,

$$\begin{aligned} & \int_0^{x_0} a(x) u^{q+\gamma} \chi dx \\ & \leq |u'(x_0)|^{p-2} u'(x_0) u^\gamma(x_0) \\ & \quad + \left(\frac{\gamma}{2}\right)^{1-p} \left(\int_0^{x_0} a(x) u^{q+\gamma} \chi dx \right)^{\frac{1}{m}} \left(\int_0^{x_0} a(x)^{-\frac{m'}{m}} \frac{|\chi'|^{pm'}}{\chi^{pm'-1}} dx \right)^{\frac{1}{m'}}. \end{aligned} \quad (2.9)$$

Since $u'(x_0) < 0$, we get

$$\int_0^{x_0} a(x)u^{q+\gamma}\chi dx \leq \left(\frac{\gamma}{2}\right)^{1-p} \left(\int_0^{x_0} a(x)u^{q+\gamma}\chi dx\right)^{\frac{1}{m}} \left(\int_0^{x_0} a(x)^{-\frac{m'}{m}} \frac{|\chi'|^{pm'}}{\chi^{pm'-1}} dx\right)^{\frac{1}{m'}}. \quad (2.10)$$

Consequently, the above inequality yields

$$\int_0^{x_0} a(x)u^{q+\gamma}\chi dx \leq \left(\frac{\gamma}{2}\right)^{1-p} \int_0^{x_0} a(x)^{-\frac{m'}{m}} \frac{|\chi'|^{pm'}}{\chi^{pm'-1}} dx. \quad (2.11)$$

Recalling the definition of the function χ in (2.2), we get

$$\frac{|\chi'|^{pm'}}{\chi^{pm'-1}} = \lambda^{pm'} \xi^{\lambda-pm'} |\xi'|^{pm'}, \quad (2.12)$$

which leads to

$$\int_0^{x_0} a(x)u^{q+\gamma}\chi dx \leq \left(\frac{\gamma}{2}\right)^{1-p} \lambda^{pm'} \int_0^{x_0} a(x)^{-\frac{m'}{m}} \xi^{\lambda-pm'} |\xi'|^{pm'} dx. \quad (2.13)$$

Since $\xi \in C^1([0, x_0]; [0, 1])$ satisfy (2.1), then

$$\begin{aligned} \int_0^{x_0} a(x)u^{q+\gamma}\chi dx &\leq \left(\frac{\gamma}{2}\right)^{1-p} \lambda^{pm'} \int_{\eta/2}^{\eta} a(x)^{-\frac{m'}{m}} \xi^{\lambda-pm'} |\xi'|^{pm'} dx \\ &\leq \left(\frac{\gamma}{2}\right)^{1-p} \lambda^{\frac{p(q+\gamma)}{q-p+1}} \int_{\eta/2}^{\eta} a(x)^{-\frac{p-1+\gamma}{q-p+1}} |\xi'|^{\frac{p(q+\gamma)}{q-p+1}} dx, \end{aligned} \quad (2.14)$$

by choosing λ large enough. Hence (2.4) holds with a constant $C = \left(\frac{\gamma}{2}\right)^{1-p} \lambda^{\frac{p(q+\gamma)}{q-p+1}}$. The lemma is proved. \square

Proof of Theorem 1.1. Now let $\xi \in C^1([0, x_0]; [0, 1])$ satisfy (2.1) and (2.2). Then (2.4) takes the form

$$\begin{aligned} \int_{\eta}^{x_0} a(x)u^{q+\gamma} dx &\leq \int_0^{x_0} a(x)u^{q+\gamma}\chi dx \\ &\leq C\lambda^{\frac{p(q+\gamma)}{q-p+1}} \int_{\eta/2}^{\eta} x^{\frac{\alpha(p-1+\gamma)}{q-p+1}} \eta^{-\frac{p(q+\gamma)}{q-p+1}} dx \\ &\leq C\lambda^{\frac{p(q+\gamma)}{q-p+1}} \eta^{-\frac{p(q+\gamma)}{q-p+1}} \int_{\eta/2}^{\eta} x^{\frac{\alpha(p-1+\gamma)}{q-p+1}} dx \\ &\leq C'\lambda^{\frac{p(q+\gamma)}{q-p+1}} \eta^{\sigma}, \end{aligned} \quad (2.15)$$

where

$$\sigma = \frac{q-p+1-pq+\alpha(p-1)+\gamma(\alpha-p)}{q-p+1}.$$

If we choose γ large enough, then the assumption $\alpha > p$ implies $\sigma > 0$. Hence

$$0 \leq \int_{\eta}^{x_0} a(x)u^{q+\gamma} dx \leq C'\lambda^{\frac{p(q+\gamma)}{q-p+1}} \eta^{\sigma}. \quad (2.16)$$

Letting $\eta \rightarrow 0$ in (2.16), we get

$$\int_0^{x_0} a(x)u^{q+\gamma} dx = 0. \quad (2.17)$$

Thus $u \equiv 0$. This completes the proof. \square

Proof of Theorem 1.2. If $\alpha = p$, one has $\sigma = 1 - p$ for every $\gamma > 0$ in (2.15). Now fix a number $b > 0$. We may choose the parameters γ and λ in (2.13) so that

$$pm' < \lambda < b^{\frac{q-p+1}{p}}. \quad (2.18)$$

For $u \in C((0, x_0])$, we can consider the set

$$M_{\eta,b} = \{x \in (\eta, x_0) : u(x) \geq b\}. \quad (2.19)$$

Due to (2.15) with $\alpha = p$, we get

$$cx_0^{-p}b^{q+\gamma}\mu(M_{\eta,b}) \leq \int_{M_{\eta,b}} cx^{-p}u^{q+\gamma}dx \leq \int_{M_{\eta,b}} a(x)u^{q+\gamma}dx \leq \int_{\eta}^{x_0} a(x)u^{q+\gamma}dx \quad (2.20)$$

and by (2.16)

$$cx_0^{-p}b^{q+\gamma}\mu(M_{\eta,b}) \leq C'\lambda^{\frac{p(q+\gamma)}{q-p+1}}\eta^{1-p}, \quad (2.21)$$

which leads to

$$\mu(M_{\eta,b}) \leq c^{-1}C'x_0^p\eta^{1-p} \left(\frac{\lambda^{\frac{p}{q-p+1}}}{b} \right)^{q+\gamma} \rightarrow 0 \quad (2.22)$$

for each b, η fixed and $\gamma \rightarrow \infty$, since the fraction in parentheses is less than 1 by (2.18). For each b and η , one obtains

$$\mu(M_{\eta,b}) = 0,$$

which means $u \equiv 0$. Thus we obtain the conclusion. \square

Acknowledgements

This work is supported by the National Natural Science Foundation of China (No. 11301301, No. 11571295 and No. 11401347) and the Foundation for Outstanding Middle-Aged and Young Scientists of Shandong Province (No. BS2013SF027). The author thanks the anonymous reviewer for his/her careful review and helpful suggestions for improvement.

References

- [1] H. BREZIS, X. CABRÉ, Some simple nonlinear PDE's without solutions, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)* **1**(1998), 223–262. [MR1638143](#)
- [2] M. F. BIDAUT-VÉRON, S. POHOZAEV, Nonexistence results and estimates for some nonlinear elliptic problems, *J. Anal. Math.* **84**(2001), 1–49. [MR1849197](#); [url](#)
- [3] G. CARISTI, On the absence of solutions of systems of quasilinear elliptic inequalities (in Russian), *Differ. Uravn.* **38**(2002), No. 3, 356–364, 430; translation in *Differ. Equ.* **38**(2002), 375–383. [MR2005073](#); [url](#)
- [4] G. CARISTI, E. MITIDIERI, Nonexistence of positive solutions of quasilinear equations, *Adv. Differential Equations* **2**(1997), 319–359. [MR1441847](#)
- [5] L. D'AMBROSIO, Liouville theorems for anisotropic quasilinear inequalities, *Nonlinear Anal.* **70**(2009), 2855–2869. [MR2509374](#); [url](#)

- [6] L. D'AMBROSIO, E. MITIDIERI, A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic inequalities, *Adv. Math.* **224**(2010), 967–1020. [MR2628800](#); [url](#)
- [7] R. FILIPPUCCI, Nonexistence of positive weak solutions of elliptic inequalities, *Nonlinear Anal.* **70**(2009), 2903–2916. [MR2509378](#); [url](#)
- [8] R. FILIPPUCCI, P. PUCCI, M. RIGOLI, Non-existence of entire solutions of degenerate elliptic inequalities with weights, *Arch. Ration. Mech. Anal.* **188**(2008), 155–179. [MR2379656](#); [url](#)
- [9] R. FILIPPUCCI, P. PUCCI, M. RIGOLI, On weak solutions of nonlinear weighted p -Laplacian elliptic inequalities, *Nonlinear Anal.* **70**(2009), 3008–3019. [MR2509387](#); [url](#)
- [10] R. FILIPPUCCI, P. PUCCI, M. RIGOLI, On entire solutions of degenerate elliptic differential inequalities with nonlinear gradient terms, *J. Math. Anal. Appl.* **356**(2009), 689–697. [MR2524301](#); [url](#)
- [11] R. FILIPPUCCI, P. PUCCI, M. RIGOLI, Nonlinear weighted p -Laplacian elliptic inequalities with gradient terms, *Commun. Contemp. Math.* **12**(2010), 501–535. [MR2661276](#); [url](#)
- [12] R. FILIPPUCCI, Nonexistence of nonnegative solutions of elliptic systems of divergence type, *J. Differential Equations* **250**(2011), 572–595. [MR2737854](#); [url](#)
- [13] E. GALAKHOV, Some nonexistence results for quasilinear elliptic problems, *J. Math. Anal. Appl.* **252**(2000), 256–277. [MR1797855](#)
- [14] E. GALAKHOV, Positive solutions of some semilinear differential inequalities and systems, *Differ. Equ.* **40**(2004), 711–722. [MR2162479](#)
- [15] E. GALAKHOV, Positive solutions of some quasilinear partial differential inequalities and systems, *Math. Nachr.* **279**(2006), 831–842. [MR2228657](#); [url](#)
- [16] E. GALAKHOV, Some nonexistence results for quasilinear PDE's, *Commun. Pure Appl. Anal.* **6**(2007), 141–161. [MR2276334](#)
- [17] E. GALAKHOV, On differential inequalities with point singularities on the boundary *Proc. Steklov Inst. Math.* **260**(2008), No. 1, 112–122. [MR2489507](#); [url](#)
- [18] G. G. LAPTEV, On the absence of solutions to a class of singular semilinear differential inequalities, *Proc. Steklov Inst. Math.* **232**(2001), 216–228. [MR1851451](#)
- [19] X. LI, F. LI, A priori estimates for nonlinear differential inequalities and applications, *J. Math. Anal. Appl.* **378**(2011), No. 2, 723–733. [MR2773280](#); [url](#)
- [20] X. LI, F. LI, Nonexistence of solutions for nonlinear differential inequalities with gradient nonlinearities, *Commun. Pure Appl. Anal.* **11**(2012), No. 3, 935–943. [MR2968602](#)
- [21] X. LI, F. LI, Nonexistence of solutions for singular quasilinear differential inequalities with a gradient nonlinearity, *Nonlinear Anal.* **75**(2012), No. 5, 2812–2822. [MR2878476](#); [url](#)
- [22] E. MITIDIERI, S. I. POHOZAEV, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, *Proc. Steklov Inst. Math.* **234**(2001), 1–362. [MR1879326](#)

- [23] E. MITIDIERI, S. I. POHOZAEV, The absence of global positive solutions to quasilinear elliptic inequalities, *Dokl. Akad. Nauk.* **57**(1998), 250–253. [MR1668404](#)
- [24] E. MITIDIERI, S. I. POHOZAEV, Nonexistence of positive solutions for systems of quasilinear elliptic equations and inequalities in \mathbb{R}^N , *Dokl. Akad. Nauk.* **59**(1999), 351–355. [MR1723262](#)
- [25] E. MITIDIERI, S. I. POHOZAEV, Absence of positive solutions for quasilinear elliptic problems in \mathbb{R}^N , *Proc. Steklov Inst. Math.* **227**(1999), 186–216. [MR1784347](#)
- [26] E. MITIDIERI, S. I. POHOZAEV, Nonexistence of weak solutions for some degenerate elliptic and parabolic problems on \mathbb{R}^n , *J. Evol. Equ.* **1**(2001) 189–220. [MR1846746](#); [url](#)
- [27] E. MITIDIERI, S. I. POHOZAEV, Towards a unified approach to nonexistence of solutions for a class of differential inequalities, *Milan J. Math.* **72**(2004), 129–162. [MR2099130](#); [url](#)
- [28] S. I. POHOZAEV, A general approach to the theory of the nonexistence of global solutions of nonlinear partial differential equations and inequalities, *Proc. Steklov Inst. Math.* **236**(2002), 273–284. [MR1931028](#)
- [29] S. I. POHOZAEV, A. TESEI, Nonexistence of local solutions to semilinear partial differential inequalities, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21**(2004), 487–502. [MR2069634](#); [url](#)
- [30] J. SERRIN, H. ZOU, Cauchy–Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities, *Acta Math.* **189**(2002), 79–142. [MR1946918](#); [url](#)
- [31] J. SERRIN, H. ZOU, Existence and nonexistence results for ground states of quasilinear elliptic equations, *Arch. Rational Mech. Anal.* **121**(1992), 101–130. [MR1188490](#); [url](#)
- [32] J. SERRIN, Entire solutions of quasilinear elliptic equations, *J. Math. Anal. Appl.* **352**(2009), 3–14. [MR2499881](#); [url](#)