Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **28**, 1–15; http://www.math.u-szeged.hu/ejqtde/

Asymptotic behaviour of positive large solutions of quasilinear logistic problems

Ramzi Alsaedi[1], **Habib Mâagli**[1], **Vicențiu D. Rădulescu**[✉ 1, 2] and **Noureddine Zeddini**[1]

[1]Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[2]Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

Received 13 February 2015, appeared 26 May 2015
Communicated by Jeff R. L. Webb

Abstract. We are interested in the asymptotic analysis of singular solutions with blow-up boundary for a class of quasilinear logistic equations with indefinite potential. Under natural assumptions, we study the competition between the growth of the variable weight and the behaviour of the nonlinear term, in order to establish the blow-up rate of the positive solution. The proofs combine the Karamata regular variation theory with a related comparison principle. The abstract result is illustrated with an application to the logistic problem with convection.

Keywords: asymptotic behaviour, positive solution, boundary blow-up, maximum principle.

2010 Mathematics Subject Classification: 35B44, 35B50, 35J62.

1 Introduction

Let Ω be a C^2-bounded domain of $\mathbb{R}^n$, $(n \geq 2)$. Let Δ_p denote the p-Laplace operator $(p > 1)$, that is,

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

We first recall some notations used in this paper. For some $\alpha \in (0,1)$, we denote by $C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ the Banach space of locally Hölder continuous functions, that is, real-valued functions defined on Ω which are uniformly Hölder continuous with exponent α on any compact subset of Ω. The local Hölder space $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ consists of functions whose first order derivatives are locally Hölder continuous with exponent α in Ω. Similarly, for $p > 1$, we denote by $W^{1,p}_{\mathrm{loc}}(\Omega)$ the Banach space of locally L^p-integrable functions with locally L^p-integrable weak derivatives, that is,

$$W^{1,p}_{\mathrm{loc}}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable; } u|_K \in W^{1,p}(K) \text{ for all compact set } K \subset \Omega \right\}.$$

✉ Corresponding author. Email: vicentiu.radulescu@imar.ro; vicentiu.radulescu@math.cnrs.fr

In this paper, we study the existence and the boundary behaviour of solutions for the following quasilinear elliptic problem

$$\begin{cases} \Delta_p u = a(x)\, f(u), \quad x \in \Omega, \\ u > 0 \quad \text{in } \Omega \\ \lim_{\delta(x)\to 0} u(x) = \infty, \end{cases} \tag{1.1}$$

where $p > 1$, $f\colon (0,\infty) \to (0,\infty)$ is a C^1 function, a is a positive function, locally Hölder continuous in Ω and satisfies some conditions related to Karamata regular variation theory and $\delta(x)$ denotes the Euclidean distance from x to the boundary $\partial\Omega$.

By a weak solution of (1.1), we mean a positive function $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C^{1,\beta}_{\mathrm{loc}}(\Omega)$ for some $0 < \beta < 1$ which satisfies in the distributional sense

$$-\int_\Omega |\nabla u|^{p-2}\,\nabla u \nabla \varphi\, dx = \int_\Omega a(x) f(u)\varphi\, dx, \quad \text{for any test function } \varphi \in C_c^\infty(\Omega).$$

A solution of (1.1) is called a large solution (or boundary blow-up or explosive solution).

Problems such as (1.1) arise in the study of the subsonic motion of a gas [35], the electric potential in some bodies [23], and Riemann geometry [5].

When $p = 2$, problem (1.1) becomes

$$\Delta u = a(x)\, f(u), \qquad x \in \Omega, \quad u > 0 \quad \text{in } \Omega, \quad \lim_{\delta(x)\to 0} u(x) = +\infty. \tag{1.2}$$

The subject of large solutions to (1.2) has received much attention starting with the pioneering works of Bieberbach in 1916 with $a(x) = 1$, $f(u) = e^u$, $n = 2$ and with $a(x) = 1$, $f(u) = e^u$ and $n = 3$ in Rademacher's work in 1943 (see [3] and [36]). In 1957, Keller [20] and Osserman [34] gave a necessary and sufficient condition for the existence of a solution to (1.2) when $a(x) = 1$ and Ω is bounded, namely $\int^\infty 1/\sqrt{F(s)}\, ds < \infty$, where $F'(s) = f(s)$ is an increasing nonlinearity. Later, many authors have considered questions such as existence, uniqueness and boundary behaviour of the solution and its normal derivative in different domains and for bounded positive weights $a(x)$.

Problem (1.2) arises from many branches of mathematics and applied mathematics, and has been discussed by many authors in many contexts. For $p = 2$, $f = 0$ and $u \in C^2$ one obtains the classical Laplace equation which was extensively studied in the literature (see, for example, [1,37,39–41]).

In a significant development, Cîrstea and Rădulescu [7] use Karamata's regular variation theory to study the blow-up rate and uniqueness, near the boundary to problem (1.2), in the case where $a(x)$ decays to zero on $\partial\Omega$ at a fixed rate along the entire boundary $\partial\Omega$ and f' varies regularly at infinity.

More recently, some results of existence and nonexistence of solutions to problem (1.2) are established when the weight $a(x)$ is unbounded near the boundary $\partial\Omega$ (see [2,6–8,12–16,23, 27,37,44–46] and the references therein).

In the general case (not necessarily $p = 2$), the problem (1.1) seems to have been first considered in [9] when $a(x) = 1$. The question of existence, uniqueness and boundary behaviour of solutions were dealt there. Since then, there have been some other papers which included similar results for different types of nonlinearities; we mention for instance [11,14,15,30–32]. We also point out the important contributions of Guo and Webb [17,18] in the understanding of the structure of boundary blow-up solutions for quasi-linear elliptic problems.

When the weight $a(x)$ is bounded, problem (1.1) has been considered by several authors. But when the weight $a(x)$ is not necessarily bounded very little is known about the global behaviour of the solution except for the case $p = 2$, see for example [11,30,32].

Our aim in this paper is to establish existence and asymptotic behaviour of solutions of (1.1) with more general nonlinearities $f(u)$ and weights $a(x)$. In particular, we give global estimates of solutions (1.1) in the case where $a(x)$ may be unbounded and satisfies some hypotheses related to the Karamata class of regularly varying functions at zero.

In order to use the method of sub- and super-solutions for (1.1), we begin by giving an auxiliary result in the case $f(u) = u^\alpha$, $\alpha > p-1$. More precisely, we prove the existence and asymptotic behaviour of a positive solution for the following problem

$$\begin{cases} \Delta_p u = b(x)u^\alpha, \quad x \in \Omega, \quad \alpha > p-1 > 0 \\ u > 0 \quad \text{in } \Omega \\ \lim_{\delta(x)\to 0} u(x) = \infty, \end{cases} \tag{1.3}$$

where $b(x)$ satisfies the following hypothesis

($\mathbf{B_1}$) $b\colon \Omega \to (0,\infty)$ belongs to $C^{0,\gamma}_{\text{loc}}(\Omega)$, $0 < \gamma < 1$ and there exists $C > 1$ such that for each $x \in \Omega$,

$$\frac{1}{C}\,\delta(x)^{-\lambda}\,L(\delta(x)) \le b(x) \le C\,\delta(x)^{-\lambda}\,L(\delta(x))\,,$$

where $\lambda \le p$, L is defined on $(0,\eta]$ for some $\eta > \operatorname{diam}(\Omega)$ with $\int_0^\eta \frac{L(t)^{1/(p-1)}}{t^{\frac{\lambda-1}{p-1}}}\,dt < \infty$ and L belongs to the set of Karamata functions $\mathcal{K}$ defined on $(0,\eta]$ by

$$L(t) := c\,\exp\left(\int_t^\eta \frac{z(s)}{s}\,ds\right)$$

with $c > 0$ and $z \in C([0,\eta])$ such that $z(0) = 0$.

Under this hypothesis, we state our first main result.

Theorem 1.1. *Let $p > 1$, $\alpha > \max\{p-1, 1\}$ and assume that b satisfies (B_1). Then problem (1.3) has a positive weak solution $u \in C^{1,\beta}_{\text{loc}}(\Omega)$, for some $0 < \beta < 1$, satisfying for each $x \in \Omega$,*

$$\frac{1}{C}\,\delta(x)^{\frac{p-\lambda}{p-1-\alpha}}\,\theta_{L,\lambda,p,\alpha}(\delta(x)) \le u(x) \le C\,\delta(x)^{\frac{p-\lambda}{p-1-\alpha}}\,\theta_{L,\lambda,p,\alpha}(\delta(x))\,, \tag{1.4}$$

where $C > 1$ is a constant and $\theta_{L,\lambda,p,\alpha}$ is the function defined on $(0,\eta]$ by

$$\theta_{L,\lambda,p,\alpha}(t) := \begin{cases} L(t)^{1/(p-1-\alpha)}, & \text{if } \lambda < p, \\ \left(\displaystyle\int_0^t \frac{L(s)^{1/(p-1)}}{s}\,ds\right)^{\frac{p-1}{p-1-\alpha}}, & \text{if } \lambda = p. \end{cases} \tag{1.5}$$

In order to establish our main result for problem (1.1), we assume that the functions f and a satisfy the following conditions:

($\mathbf{H_1}$) The function a is positive, belongs to $C^{0,\gamma}_{\text{loc}}(\Omega)$, $0 < \gamma < 1$ and there exist two γ-Hölder continuous functions a_1 and a_2 such that for each $x \in \Omega$,

$$a_1(\delta(x)) \le a(x) \le a_2(\delta(x))\,,$$

where $a_i(t) = t^{-\lambda_i} L_i(t)$, with $\lambda_i \le p$ and $L_i \in \mathcal{K}$ defined on $(0,\eta]$, $(\eta > \operatorname{diam}(\Omega))$ such that $\int_0^\eta \frac{(L_i(t))^{\frac{1}{p-1}}}{t^{\frac{\lambda_i-1}{p-1}}}\,dt < \infty$.

($\mathbf{H_2}$) The function $f\colon (0,\infty) \to (0,\infty)$ is of class C^1 and there exist constants k_1, k_2, α_1, α_2 with $0 < k_1 \leq k_2$, $0 < p-1 < \alpha_1 \leq \alpha_2$ such that

$$f(t) \leq k_2\, t^{\alpha_2} \quad \text{for } t > 0 \quad \text{and} \quad f(t) \geq k_1\, t^{\alpha_1} \quad \text{for } t \geq 1.$$

We now give an example of weight $a(x)$ that satisfies hypothesis ($\mathbf{H_1}$). Consider the simplest case corresponding to $\Omega = B(0,1) \subset \mathbb{R}^n$ and assume that $p > 1$, $\lambda < p$, and $\mu > p-1$. Then the function

$$a(x) = (1-|x|)^{-\lambda}\, \log\left(\frac{2}{1-|x|}\right) + (1-|x|)^{-p}\, \left[\log\left(\frac{2}{1-|x|}\right)\right]^{-\mu}, \qquad x \in \Omega$$

satisfies hypothesis ($\mathbf{H_1}$) with

$$a_1(x) = (1-|x|)^{-\lambda}\, \log\left(\frac{2}{1-|x|}\right) \quad \text{and} \quad a_2(x) = c\,(1-|x|)^{-p}\left[\log\left(\frac{2}{1-|x|}\right)\right]^{-\mu}$$

for some positive constant $c > 0$. We refer to [7] for more examples of functions belonging to the Karamata class.

We now state the second main result in this paper.

Theorem 1.2. *Under hypotheses* (H_1)–(H_2), *problem* (1.1) *has a weak solution* $u \in C^{1,\beta}(\Omega)$, *for some* $0 < \beta < 1$, *satisfying*

$$\frac{1}{C}\,\delta(x)^{\frac{p-\lambda_2}{p-1-\alpha_2}}\,\theta_{L_2,\lambda_2,p,\alpha_2}(\delta(x)) \leq u(x) \leq C\,\delta(x)^{\frac{p-\lambda_1}{p-1-\alpha_1}}\,\theta_{L_1,\lambda_1,p,\alpha_1}(\delta(x)), \tag{1.6}$$

where

$$\theta_{L_i,\lambda_i,p,\alpha_i}(t) := \begin{cases} (L_i(t))^{\frac{1}{p-1-\alpha_i}}, & \text{if } \lambda_i < p, \\ \left(\int_0^t \frac{(L_i(s))^{\frac{1}{p-1}}}{s}\, ds\right)^{\frac{p-1}{p-1-\alpha_i}}, & \text{if } \lambda_i = p \end{cases} \tag{1.7}$$

for $i \in \{1,2\}$ *and* $C > 1$.

Throughout this paper, we need the following notations.

For two nonnegative functions f and g defined on a set S, the notation $f(x) \approx g(x)$, $x \in S$, means that there exists $c > 0$ such that

$$\frac{1}{c}\, f(x) \leq g(x) \leq c\, f(x), \quad \text{for all } x \in S.$$

We denote by φ_1 the positive normalized (i.e., $\max_{x\in\Omega} \varphi_1(x) = 1$) eigenfunction corresponding to the first positive eigenvalue λ_1 of the p-Laplace operator $(-\Delta_p)$ in $W_0^{1,p}(\Omega)$. By definition, φ_1 is the unique normalized function satisfying the following eigenvalue problem

$$\begin{cases} -\Delta_p \varphi_1 = \lambda_1\, \varphi_1^{p-1}, & x \in \Omega, \\ \varphi_1 \geq 0 \quad \text{in } \Omega \quad \text{and} & \lim_{\delta(x)\to 0} \varphi_1(x) = 0. \end{cases}$$

We recall that, from Moser iterations [33] and [24, Theorem 1], $\varphi_1 \in C^{1,\beta}(\overline{\Omega})$, for some $0 < \beta < 1$, and from strong maximum principle for quasilinear operators (see [42, Theorem 10]), φ_1 satisfies

$$\varphi_1(x) \approx \delta(x) \quad \text{in } \Omega \tag{1.8}$$

and

$$\varphi_1^p(x) + |\nabla \varphi_1(x)|^p \approx 1 \quad \text{in } \Omega. \tag{1.9}$$

Our paper is organized as follows. In Section 2, we collect some useful properties of Karamata functions. Section 3 deals with the proof of our main results. The last section is reserved to some applications.

2 The Karamata class $\mathcal{K}$

To make the paper self-contained, we begin this section by recapitulating some properties of Karamata regular variation theory established by Karamata in 1930. This theory has been applied to study the asymptotic behaviour of solutions to differential equations. We refer to [7, 8, 26, 37, 43, 46] for more details.

Lemma 2.1. *The following hold.*

(i) Let $L \in \mathcal{K}$ and $\varepsilon > 0$, then we have

$$\lim_{t \to 0^+} t^{\varepsilon} L(t) = 0.$$

(ii) Let $L_1, L_2 \in \mathcal{K}$ and $p \in \mathbb{R}$. Then we have $L_1 + L_2 \in \mathcal{K}$, $L_1 L_2 \in \mathcal{K}$ and $L_1^p \in \mathcal{K}$.

Example 2.2. Let m be a positive integer. Let $c > 0$, $(\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^m$ and d be a sufficiently large positive real number such that the function

$$L(t) = c \prod_{k=1}^{m} \left(\log_k \left(\frac{d}{t} \right) \right)^{\mu_k}$$

is defined and positive on $(0, \eta]$, for some $\eta > 1$, where $\log_k x = \log \circ \log \circ \cdots \circ \log x$ (k times). Then $L \in \mathcal{K}$.

Lemma 2.3. *A function L is in $\mathcal{K}$ if and only if L is a positive function in $C^1((0, \eta])$ satisfying*

$$\lim_{t \to 0^+} \frac{t\, L'(t)}{L(t)} = 0. \tag{2.1}$$

Proof. Let $L \in \mathcal{K}$. Since $L(t) := c \exp\left(\int_t^{\eta} \frac{z(s)}{s}\, ds \right)$, then for $t \in (0, \eta]$, we have

$$\frac{t\, L'(t)}{L(t)} = -z(t)\,.$$

So, using the fact that $z(0) = 0$, we deduce (2.1).

Conversely, let L be a positive function in $C^2((0, \eta])$ satisfying (2.1). For $t \in (0, \eta]$, put

$$z(t) = -\frac{t\, L'(t)}{L(t)}, \tag{2.2}$$

then $z \in C((0, \eta])$ and $\lim_{t \to 0^+} z(t) = 0$. Moreover, we have

$$L(t) = L(\eta) \exp\left(\int_t^{\eta} \frac{z(s)}{s}\, ds \right).$$

This proves that $L \in \mathcal{K}$. □

Applying Karamata's theorem (see [29,38]), we get the following.

Lemma 2.4. *Let $\mu \in \mathbb{R}$ and L be a function in $\mathcal{K}$ defined on $(0,\eta]$. We have*

(i) If $\mu < -1$, then $\int_0^\eta s^\mu L(s)\,ds$ diverges and $\int_t^\eta s^\mu L(s)\,ds \sim_{t\to 0^+} -\frac{t^{1+\mu}L(t)}{\mu+1}$.

(ii) If $\mu > -1$, then $\int_0^\eta s^\mu L(s)\,ds$ converges and $\int_0^t s^\mu L(s)\,ds \sim_{t\to 0^+} \frac{t^{1+\mu}L(t)}{\mu+1}$.

Lemma 2.5 ([28])**.** *Let $L \in \mathcal{K}$ be defined on $(0,\eta]$. Then we have*

$$\lim_{t\to 0^+} \frac{L(t)}{\int_t^\eta \frac{L(s)}{s}\,ds} = 0. \tag{2.3}$$

If further $\int_0^\eta \frac{L(s)}{s}\,ds$ converges, then we have

$$\lim_{t\to 0^+} \frac{L(t)}{\int_0^t \frac{L(s)}{s}\,ds} = 0. \tag{2.4}$$

Remark 2.6. Let $L \in \mathcal{K}$ be defined on $(0,\eta]$, then using (2.1) and (2.3), we deduce that

$$t \to \int_t^\eta \frac{L(s)}{s}\,ds \in \mathcal{K}.$$

If further $\int_0^\eta \frac{L(s)}{s}\,ds$ converges, we have by (2.4) that

$$t \to \int_0^t \frac{L(s)}{s}\,ds \in \mathcal{K}.$$

Lemma 2.7. *Let $L \in \mathcal{K}$, $0 < \varepsilon < \eta$ and φ_1 be the first eigenfunction of $(-\Delta_p)$ in Ω. Then we have*

$$L(\varepsilon\,\varphi_1(x)) \approx L(\delta(x)), \qquad x \in \Omega. \tag{2.5}$$

Proof. Let $L \in \mathcal{K}$. Then there exist $c > 0$ and $z \in C([0,\eta])$ such that $z(0) = 0$ and for $t \in (0,\eta]$, $\eta > \operatorname{diam}(\Omega)$,

$$L(t) := c\exp\left(\int_t^\eta \frac{z(s)}{s}\,ds\right).$$

Let $M = \max_{s\in[0,\eta]} |z(s)|$. By using (1.8), there exists $c_1 > 0$ such that

$$\frac{1}{c_1}\,\delta(x) \le \varepsilon\,\varphi_1(x) \le c_1\,\delta(x).$$

Using this fact we deduce that

$$\left|\int_{\delta(x)}^{\varepsilon\,\varphi_1(x)} \frac{z(s)}{s}\,ds\right| \le M\,\log c_1.$$

Hence,

$$c_1^{-M}\,L(\delta(x)) \le L(\varepsilon\,\varphi_1(x)) \le c_1^M\,L(\delta(x)), \qquad x \in \Omega.$$

This ends the proof. □

We point out that the constants in asymptotic relation (2.5) depend on ε (the first one goes to zero but the second one goes to infinity as $\varepsilon \to 0^+$).

3 Proof of main results

First, we recall some classical results about the sub- and super-solution method.

Definition 3.1. A function $v \in W^{1,p}_{\text{loc}}(\Omega) \cap C^{1,\beta}_{\text{loc}}(\Omega)$, $0 < \beta < 1$, is called a weak sub-solution of (1.1) if $v = \infty$ on $\partial\Omega$ and

$$-\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi \, dx \geq \int_\Omega a(x) f(v) \varphi \, dx \qquad \forall \varphi \in C_c^\infty(\Omega) \text{ with } \varphi \geq 0.$$

If the above inequality is reversed, v is called a weak super-solution of (1.1).

We point out that this definition agrees with the sub- and super-solutions used in the proofs of Theorem 1.1 and Theorem 1.2, since the corresponding relations in those proofs are viewed in the weak sense.

The following property is an adaptation of Lemma 2.1 in [13]. In the statement of the next result, we can assume without loss of generality that the Hölder exponent is the same for all functions $\underline{u}$, $\overline{u}$, and u. Indeed, if the corresponding exponents are β_1, β_2 and β_3, it is enough to consider $\beta = \min\{\beta_1, \beta_2, \beta_3\}$.

Lemma 3.2. *Let $a(x)$ be a locally γ-Hölder continuous function in Ω, $0 < \gamma < 1$ and f be continuously differentiable on $[0, \infty)$. Assume that there exist a weak sub-solution $\underline{u}$ and a weak super-solution $\overline{u}$ to the problem (1.1) such that $\underline{u} \leq \overline{u}$. Then there exists at least one weak solution $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C^{1,\beta}_{\text{loc}}(\Omega)$, for some $0 < \beta < 1$, such that $\underline{u} \leq u \leq \overline{u}$.*

Proof. For $n \in \mathbb{N}$, we set

$$\Omega_n := \left\{ x \in \Omega : \delta(x) < \frac{1}{n} \right\}.$$

Consider the boundary value problem

$$\begin{cases} \Delta_p u = a(x) f(u), & x \in \Omega_n, \\ u|_{\partial\Omega_n} = \underline{u}. \end{cases} \tag{3.1}$$

Since $\underline{u}$ is a sub-solution and $\overline{u}$ is a super-solution, this problem has at least one positive weak solution u_n such that $\underline{u} \leq u_n \leq \overline{u}$, see Rădulescu [37]. This in particular gives bounds on any compact set $K \subset \Omega$ for the sequence u_n which in turn leads to bounds in $C^{1,\gamma}_{\text{loc}}(\Omega)$. Since the embedding of $C^{1,\gamma}(\overline{\Omega'})$ into $C^1(\overline{\Omega'})$ is compact for every $\overline{\Omega'} \subset \Omega$, then for every $k \in \mathbb{N}$, we can select a subsequence of u_n which converges in $C^1(\overline{\Omega_k})$. A diagonal procedure gives a subsequence (denoted again by u_n) which converges to a function $u \in C^1_{\text{loc}}(\Omega)$. Passing to the limit in (3.1) we see that u is a weak solution of the equation in (1.1), verifying $\underline{u} \leq u \leq \overline{u}$. In particular, we deduce that $u = \infty$ on $\partial\Omega$. This proves the lemma. □

Next, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let φ_1 be the positive normalized eigenfunction associated to the first eigenvalue λ_1 of $-\Delta_p$ in $W^{1,p}_0(\Omega)$ and let $0 < \varepsilon < \eta$. In order to construct a sub-solution $\underline{u}$ and a super-solution $\overline{u}$ of (1.1), we define the function

$$v(x) = \left(\int_0^{\varepsilon \varphi_1(x)} t^{\frac{1-\lambda}{p-1}} \, (L(t))^{\frac{1}{p-1}} \, dt \right)^{\frac{p-1}{p-1-\alpha}} \quad \text{if } x \in \Omega$$

and we will prove that $\Delta_p v(x) \approx b(x)\, v(x)^\alpha$.
A straightforward computation shows that

$$\nabla v = \varepsilon^{\frac{p-\lambda}{p-1}} \,(v(x))^{\frac{\alpha}{p-1}}\,(\varphi_1(x))^{\frac{1-\lambda}{p-1}}\,\,(L(\varepsilon\varphi_1(x)))^{\frac{1}{p-1}}\,\,\nabla\varphi_1$$

and

$$\Delta_p v = \varepsilon^{p-\lambda}\,\varphi_1^{-\lambda}(x) L(\varepsilon\varphi_1(x))(v(x))^\alpha \left[\lambda_1\varphi_1^p + |\nabla\varphi_1|^p\left(\alpha\frac{(\varepsilon\varphi_1(x))^{\frac{p-\lambda}{p-1}}\,(L(\varepsilon\varphi_1(x)))^{\frac{1}{p-1}}}{(v(x))^{\frac{p-1-\alpha}{p-1}}} + (\lambda-1) - \varepsilon\varphi_1(x)\frac{L'(\varepsilon\varphi_1(x))}{L(\varepsilon\varphi_1(x))}\right)\right].$$

Since $L \in \mathcal{K}$ and $\int_0^\eta \frac{(L(t))^{\frac{1}{p-1}}}{t^{\frac{\lambda-1}{p-1}}}\,dt < \infty$, then we deduce from Lemmas 2.3, 2.4 (ii) and 2.5 that

$$\lim_{\varepsilon\to 0}\sup_{x\in\Omega} \frac{(\varepsilon\varphi_1(x))^{\frac{p-\lambda}{p-1}}\,(L(\varepsilon\varphi_1(x)))^{\frac{1}{p-1}}}{\displaystyle\int_0^{\varepsilon\varphi_1(x)} t^{\frac{1-\lambda}{p-1}}\,(L(t))^{\frac{1}{p-1}}\,dt} = \frac{p-\lambda}{p-1}$$

and

$$\lim_{\varepsilon\to 0}\sup_{x\in\Omega} \frac{(\varepsilon\varphi_1(x))L'(\varepsilon\varphi_1(x))}{L(\varepsilon\varphi_1(x))} = 0.$$

Hence, there exists $\varepsilon > 0$ such that for each $x \in \Omega$,

$$-\frac{(p-\lambda)(\alpha-p+1)+(p-1)^2}{4(p-1)} \le -\varepsilon\varphi_1(x)\frac{L'(\varepsilon\varphi_1(x))}{L(\varepsilon\varphi_1(x))} \le \frac{(p-\lambda)(\alpha-p+1)+(p-1)^2}{4(p-1)}$$

and

$$\begin{aligned}\frac{(p-\lambda)(3\alpha+p-1)-(p-1)^2}{4(p-1)} &\le \alpha\frac{(\varepsilon\varphi_1(x))^{\frac{p-\lambda}{p-1}}\,(L(\varepsilon\varphi_1(x)))^{\frac{1}{p-1}}}{\displaystyle\int_0^{\varepsilon\varphi_1(x)} t^{\frac{1-\lambda}{p-1}}\,(L(t))^{\frac{1}{p-1}}\,dt} \\ &\le \frac{(p-\lambda)(5\alpha-p+1)+(p-1)^2}{4(p-1)}.\end{aligned}$$

This gives

$$\begin{aligned}\frac{(p-\lambda)(\alpha-p+1)+(p-1)^2}{2(p-1)} &\le \alpha\frac{(\varepsilon\varphi_1(x))^{\frac{p-\lambda}{p-1}}\,(L(\varepsilon\varphi_1(x)))^{\frac{1}{p-1}}}{\displaystyle\int_0^{\varepsilon\varphi_1(x)} t^{\frac{1-\lambda}{p-1}}\,(L(t))^{\frac{1}{p-1}}\,dt} - \varepsilon\varphi_1(x)\frac{L'(\varepsilon\varphi_1(x))}{L(\varepsilon\varphi_1(x))} + (\lambda-1) \\ &\le \frac{(p-\lambda)(3\alpha-p+1)+3(p-1)^2}{2(p-1)}.\end{aligned}$$

Therefore using these inequalities and (1.9) we obtain $\Delta_p v(x) \approx (\varphi_1(x))^{-\lambda}\, L(\varepsilon\varphi_1(x))(v(x))^\alpha$.
Now, using (1.8), Lemma 2.7 and hypothesis ($\mathbf{B_1}$) we obtain

$$\begin{aligned}\Delta_p v(x) &\approx (\varphi_1(x))^{-\lambda}\, L(\varepsilon\varphi_1(x))\,(v(x))^\alpha \\ &\approx \delta(x)^{-\lambda}\, L(\delta(x))(v(x))^\alpha \\ &\approx b(x)(v(x))^\alpha.\end{aligned}$$

This proves that for every $\lambda \leq p$, there exists $M > 0$ such that for every $x \in \Omega$, we have

$$\frac{1}{M}\, b(x) v^{\alpha}(x) \leq \Delta_p v(x) \leq M b(x) v^{\alpha}(x). \tag{3.2}$$

By putting $c = M^{\frac{1}{\alpha - p+1}}$, it follows from (3.2) that $\underline{u} = \frac{1}{c}\, v$ and $\overline{u} = c\, v$ are respectively sub-solution and super-solution of problem (1.3). Thus, we conclude by Lemma (3.2) that problem (1.3) has a positive solution u such that $\underline{u} \leq u \leq \overline{u}$. Applying Lemma 2.7, Remark 2.6, and Lemma 2.4, we deduce that

$$u(x) \approx \delta(x)^{\frac{p-\lambda}{p-1-\alpha}}\, \theta_{L,\lambda,p,\alpha}(\delta(x)).$$ □

The following proposition plays a key role in the proof of Theorem 1.2.

Proposition 3.3. *Let a_1, a_2 be the functions defined in hypothesis (H_1) and let α_1, α_2 be such that $0 < p-1 < \alpha_1 \leq \alpha_2$. Let u_i be the solution, given in Theorem 1.1, of the following problem*

$$\begin{cases} \Delta_p u_i = a_i(\delta(x))\, u^{\alpha_i}, & x \in \Omega, \\ u_i > 0 & \text{in } \Omega \\ \lim_{\delta(x)\to 0} u_i(x) = \infty. \end{cases} \tag{3.3}$$

Then there exists a constant $c_0 > 0$ such that

$$u_2 \leq c_0\, u_1 \quad \text{in } \Omega. \tag{3.4}$$

Proof. By Theorem 1.1, problem (3.3) has a solution u_i and there exist two constants $c_1 > 0$, $c_2 > 0$ such that for each $x \in \Omega$, we have,

$$\frac{1}{c_i}\, \psi_{L_i,\lambda_i,p,\alpha_i}(\delta(x)) \leq u_i(x) \leq c_i\, \psi_{L_i,\lambda_i,p,\alpha_i}(\delta(x))\,, \tag{3.5}$$

where for $i \in \{1,2\}$, $\psi_{L_i,\lambda_i,p,\alpha_i}$ is the function defined on $(0,\eta]$, by

$$\psi_{L_i,\lambda_i,p,\alpha_i}(t) = t^{\frac{p-\lambda_i}{p-1-\alpha_i}}\ \theta_{L_i,\lambda_i,p,\alpha_i}(t) \tag{3.6}$$

and $\theta_{L_i,\lambda_i,p,\alpha_i}$ is given by (1.7). To prove Proposition (3.3), it is enough to show that $\frac{\psi_{L_2,\lambda_2,p,\alpha_2}}{\psi_{L_1,\lambda_1,p,\alpha_1}}$ is bounded in $(0,\eta]$. Now, using Lemma 2.1 (i) and hypothesis (H_1), we deduce that $\lambda_1 \leq \lambda_2 \leq p$. On the other hand, since $p-1 < \alpha_1 \leq \alpha_2$, then we deduce that

$$0 \leq \frac{p-\lambda_2}{\alpha_2 - (p-1)} \leq \frac{p-\lambda_1}{\alpha_1 - (p-1)}.$$

Put $\sigma = \frac{(\alpha_2-\alpha_1)(p-\lambda_1)+(\lambda_2-\lambda_1)(\alpha_1-(p-1))}{(\alpha_1-(p-1))(\alpha_2-(p-1))}$. Then $\sigma \geq 0$ and for each $t \in (0,\eta]$ we have

$$\frac{\psi_{L_2,\lambda_2,p,\alpha_2}(t)}{\psi_{L_1,\lambda_1,p,\alpha_1}(t)} = t^{\sigma}\, \frac{\theta_{L_2,\lambda_2,p,\alpha_2}(t)}{\theta_{L_1,\lambda_1,p,\alpha_1}(t)}.$$

Now, using Lemma 2.1 and the definition of $\theta_{L_i,\lambda_i,p,\alpha_i}$, we deduce that

$$\frac{\theta_{L_2,\lambda_2,p,\alpha_2}}{\theta_{L_1,\lambda_1,p,\alpha_1}} \in \mathcal{K}.$$

So, we distinguish the following two cases.
Case 1. $\sigma > 0$. In this case, we conclude by Lemma 2.1 that

$$\lim_{t\to 0} \frac{\psi_{L_2,\lambda_2,p,\alpha_2}(t)}{\psi_{L_1,\lambda_1,p,\alpha_1}(t)} = 0.$$

Hence $\frac{\psi_{L_2,\lambda_2,p,\alpha_2}}{\psi_{L_1,\lambda_1,p,\alpha_1}}$ is bounded in $(0,\eta]$.
Case 2. $\sigma = 0$. This is equivalent to $\lambda_1 = \lambda_2 = p$ or $\lambda_1 = \lambda_2 < p$ and $\alpha_1 = \alpha_2$. In this case, we have $L_1 \leq L_2$ in $(0,\eta]$. So we will discuss two subcases:
• If $\lambda_1 = \lambda_2 = p$, then for each $t \in (0,\eta]$ we have

$$\begin{aligned}\frac{\psi_{L_2,\lambda_2,p,\alpha_2}(t)}{\psi_{L_1,\lambda_1,p,\alpha_1}(t)} &= \left(\int_0^t \frac{(L_1(s))^{\frac{1}{p-1}}}{s}\,ds\right)^{\frac{p-1}{\alpha_1-(p-1)}} \left(\int_0^t \frac{(L_2(s))^{\frac{1}{p-1}}}{s}\,ds\right)^{\frac{1-p}{\alpha_2-(p-1)}} \\ &\leq \left(\int_0^t \frac{(L_2(s))^{\frac{1}{p-1}}}{s}\,ds\right)^{\frac{p-1}{\alpha_1-(p-1)}} \left(\int_0^t \frac{(L_2(s))^{\frac{1}{p-1}}}{s}\,ds\right)^{\frac{1-p}{\alpha_2-(p-1)}} \\ &\leq \left(\int_0^t \frac{(L_2(s))^{\frac{1}{p-1}}}{s}\,ds\right)^{\frac{(p-1)(\alpha_2-\alpha_1)}{(\alpha_1-(p-1))(\alpha_2-(p-1))}}.\end{aligned}$$

Since $p-1 < \alpha_1 \leq \alpha_2$ and $0 < \int_0^\eta \frac{(L_2(s))^{\frac{1}{p-1}}}{s}\,ds < \infty$, then we deduce that $\frac{\psi_{L_2,\lambda_2,p,\alpha_2}}{\psi_{L_1,\lambda_1,p,\alpha_1}}$ is bounded in $(0,\eta]$.
• If $\lambda_1 = \lambda_2 < p$ and $\alpha_1 = \alpha_2$, then for each $t \in (0,\eta]$ we have

$$\frac{\psi_{L_2,\lambda_2,p,\alpha_2}(t)}{\psi_{L_1,\lambda_1,p,\alpha_1}(t)} = \frac{\theta_{L_2,\lambda_2,p,\alpha_2}(t)}{\theta_{L_1,\lambda_1,p,\alpha_1}(t)} = \frac{(L_2(t))^{\frac{1}{p-1-\alpha_2}}}{(L_1(t))^{\frac{1}{p-1-\alpha_1}}} = \left(\frac{L_1(t)}{L_2(t)}\right)^{\frac{1}{\alpha_1-(p-1)}} \leq 1.$$

This completes the proof of Proposition 3.3. □

Proof of Theorem 1.2. Let u_i be a solution of the problem (3.3) and let c_0 be a positive constant such that $u_2 \leq c_0\, u_1$. Since $\lim_{\delta(x)\to 0} u_1(x) = \infty$, then $\inf_{x\in\Omega} u_1(x) > 0$. Let $\mu_1,\ \mu_2$ be two positive constants chosen so that

$$\mu_1 \geq \max\left(\frac{1}{k_1^{\frac{1}{\alpha_1-(p-1)}}}, \frac{1}{\inf\limits_{x\in\Omega} u_1(x)}\right) \quad \text{and} \quad \mu_2 \leq \min\left(\frac{\mu_1}{c_0}, \frac{1}{k_2^{\frac{1}{\alpha_2-(p-1)}}}\right),$$

where k_1, k_2 are given in hypothesis (H_2). Put

$$\overline{u} = \mu_1\, u_1 \quad \text{and} \quad \underline{u} = \mu_2\, u_2. \tag{3.7}$$

Then using hypotheses (H_1) and (H_2), we obtain

$$\begin{cases} \Delta_p \overline{u} = \frac{1}{\mu_1^{\alpha_1-(p-1)}}\, a_1(\delta(x))\, \overline{u}^{\alpha_1} \leq a(x) f(\overline{u}), \quad x \in \Omega, \\ \overline{u} > 0 \quad \text{in } \Omega; \quad \lim_{\delta(x)\to 0} \overline{u}(x) = \infty \end{cases}$$

and

$$\begin{cases} \Delta_p\, \underline{u} = \dfrac{1}{k_2\, \mu_2^{\alpha_2-(p-1)}}\, a_2(\delta(x))\, k_2\, \underline{u}^{\alpha_2} \geq a(x) f(\underline{u}), \quad x \in \Omega, \\ \underline{u} > 0 \quad \text{in } \Omega; \quad \lim_{\delta(x)\to 0} \underline{u}(x) = \infty. \end{cases}$$

So $\underline{u}$ and $\overline{u}$ are respectively a sub-solution and a super-solution of problem (1.1). Moreover, for each $x \in \Omega$ we have

$$\underline{u}(x) = \mu_2\, u_2(x) \le \mu_2\, c_0\, u_1(x) \le \mu_1\, u_1(x) = \overline{u}.$$

Since $a \in C^{0,\gamma}_{\text{loc}}(\Omega)$ and $f \in C^1([0,\infty))$, we deduce from Lemma 3.2 that (1.1) has a weak solution $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C^{1,\beta}_{\text{loc}}(\Omega)$, for some $0 < \beta < 1$, satisfying

$$\underline{u} \le u \le \overline{u}.$$

This together with (3.5) and (3.7) implies that u satisfies (1.6). □

4 Application to the singular logistic problem with convection

Let a be a function satisfying (H_1) and let f be a function satisfying (H_2) and $\beta \in \mathbb{R}$ with $\beta < 1$. In this paragraph, we are interested in the following problem

$$\begin{cases} \Delta_p u - \dfrac{\beta(p-1)}{u}\,|\nabla u|^p = a(x)\,f(u), \quad x \in \Omega, \\ u > 0 \quad \text{in } \Omega \\ \lim\limits_{\delta(x)\to 0} u(x) = \infty. \end{cases} \tag{4.1}$$

By putting $v = u^{1-\beta}$, we obtain by a simple calculus that v satisfies

$$\begin{cases} \Delta_p v = (1-\beta)^{p-1} a(x) v^{-\frac{\beta(p-1)}{1-\beta}}\, f(v^{\frac{1}{1-\beta}}), \quad x \in \Omega, \\ v > 0 \quad \text{in } \Omega \\ \lim\limits_{\delta(x)\to 0} v(x) = \infty. \end{cases} \tag{4.2}$$

Let g be the function defined on $(0,\infty)$ by $g(v) = (1-\beta)v^{-\frac{\beta(p-1)}{1-\beta}}\, f\big(v^{\frac{1}{1-\beta}}\big)$ and put $\alpha_1^* = \frac{\alpha_1 - \beta(p-1)}{1-\beta}$ and $\alpha_2^* = \frac{\alpha_2 - \beta(p-1)}{1-\beta}$. Clearly $0 < p-1 < \alpha_1^* \le \alpha_2^*$ and the function g satisfies

$$(1-\beta)k_1\, r^{\alpha_1^*} \le g(r) \quad \text{for } r \ge 1 \quad \text{and} \quad g(r) \le (1-\beta)k_2\, r^{\alpha_2^*} \quad \text{for } r > 0.$$

Therefore, it follows from Theorem 1.2 that problem (4.2) has a positive weak solution $v \in W^{1,p}_{\text{loc}}(\Omega) \cap C^{1,\nu}_{\text{loc}}(\Omega)$, for some $0 < \nu < 1$, such that

$$\frac{1}{C}\,(\delta(x))^{\frac{(p-\lambda_2)(1-\beta)}{p-1-\alpha_2}}\ \theta_{L_2,\lambda_2,p,\alpha_2^*}(\delta(x)) \le v(x) \le C\ (\delta(x))^{\frac{(p-\lambda_1)(1-\beta)}{p-1-\alpha_1}}\ \theta_{L_1,\lambda_1,p,\alpha_1^*}(\delta(x))$$

for some constant $C > 1$. Consequently, we deduce that problem (4.1) has a solution $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C^{1,\nu}_{\text{loc}}(\Omega)$ satisfying

$$\frac{1}{C}\,(\delta(x))^{\frac{p-\lambda_2}{p-1-\alpha_2}}\ \theta_{L_2,\lambda_2,p,\alpha_2}(\delta(x)) \le u(x) \le C\ (\delta(x))^{\frac{p-\lambda_1}{p-1-\alpha_1}}\ \theta_{L_1,\lambda_1,p,\alpha_1}(\delta(x))$$

for some constant $C > 1$.

Authors' contribution

All authors contributed equally and significantly in the elaboration of this article and their names are written in alphabetical order. All authors read and approved the final document.

Acknowledgements

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under Grant No. 39-130-35-HiCi. The authors acknowledge with thanks DSR technical and financial support. The authors are indebted to the anonymous referee for useful comments.

References

[1] R. Alsaedi, N. Constantinescu, V. Rădulescu, Nonlinearities in elliptic curve authentication, *Entropy* **16**(2014), 5144–5158. MR3268005; url

[2] C. Bandle, M. Marcus, Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behaviour, *J. Anal. Math.* **58**(1992), 9–24. MR1226934

[3] L. Bieberbach, $\Delta u = e^u$ und die automorphen Funktionen, *Math. Ann.* **77**(1916), 173–212. MR1511854

[4] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular variation*, Encyclopedia of Mathematics and its Applications, Vol. 27, Cambridge University Press, Cambridge, 1987. MR0898871

[5] K. S. Cheng, W. M. Ni, On the structure of the conformal scalar curvature equation on $\mathbb{R}^n$, *Indiana Univ. Math. J.* **41**(1992), 261–278. MR1160913

[6] M. Chuaqui, C. Cortázar, M. Elgueta, J. Garcia-Melian, Uniqueness and boundary behavior of large solutions to elliptic problems with singular weights, *Comm. Pure Appl. Anal.* **3**(2004), 653–662. MR2106305; url

[7] F. Cîrstea, V. Rădulescu, Uniqueness of the blow-up boundary solution of logistic equations with absorption, *C. R. Acad. Sci. Paris, Ser. I* **335**(2002), 447–452. MR1937111

[8] F. Cîrstea, V. Rădulescu, Nonlinear problems with boundary blow-up: a Karamata regular variation theory approach, *Asympt. Anal.* **46**(2006), 275–298. MR2215886

[9] G. Diaz, R. Letelier, Explosive solutions of quasilinear elliptic equations: existence and uniqueness, *Nonlinear Anal.* **20**(1993), 97–125. MR1200384

[10] Y. Du, Asymptotic behavior and uniqueness results for boundary blow-up solutions, *Differential Integral Equations* **17**(2004), 819–834. MR2074688

[11] Y. Du, Z. Guo, Boundary blow-up solutions and their applications in quasilinear elliptic equations, *J. Anal. Math.* **89**(2003), 277–302. MR1981921

[12] J. García-Melián, R. Letelier-Albornoz, J. Sabina de Lis, Uniqueness and asymptotic behaviour for solutions of semilinear problems with boundary blow-up, *Proc. Amer. Math. Soc.* **129**(2001), 3593–3602. MR1860492; url

[13] J. García-Melián, A remark on the existence of large solutions via sub and supersolutions, *Electron. J. Differential Equations* **2003**, No. 110, 1–4. MR2011583

[14] J. García-Melián, Large solutions for equations involving the p-Laplacian and singular weights, *Z. Angew. Math. Phys.* **60**(2009), 594–607. MR2520601

[15] J. García-Melián, Quasilinear equations with boundary blow-up and exponential reaction, *Adv. Nonl. Stud.* **9**(2009), 149–160. MR2473154

[16] M. Ghergu, V. D. Rădulescu, *Nonlinear PDEs: Mathematical models in biology, chemistry and population genetics*, Springer Monographs in Mathematics, Springer, Heidelberg, 2012. MR2865669

[17] Z. Guo, J. R. L. Webb, Structure of boundary blow-up solutions of quasilinear elliptic problems. I. Large and small solutions, *Proc. Roy. Soc. Edinburgh Sect. A* **135**(2005), 615–642. MR2153439; url

[18] Z. Guo, J. R. L. Webb, Structure of boundary blow-up solutions of quasilinear elliptic problems. II. Small and intermediate solutions, *J. Differential Equations* **211**(2005), 187–217. MR2121114; url

[19] J. Karamata, Sur un mode de croissance régulière de fonctions. Théorèmes fondamentaux (in French), *Bull. Soc. Math. France* **61**(1933), 55–62. MR1504998

[20] J. B. Keller, On solutions of $\Delta u = f(u)$, *Comm. Pure Appl. Math.* **10**(1957), 503–510. MR0091407

[21] A. V. Lair, A necessary and sufficient condition for the existence of large solutions to semilinear elliptic equations, *J. Math. Anal. Appl.* **240**(1999), 205–218. MR2585080; url

[22] A. V. Lair, A. W. Wood, Large solutions of sublinear elliptic equations, *Nonlinear Anal.* **39**(2000), 745–753. MR1733126

[23] A. C. Lazer, P. J. McKenna, On a problem of Bieberbach and Rademacher, *Nonlinear Anal.* **21**(1993), 327–335. MR1237124

[24] G. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.* **12**(1988), 1203–1219. MR0969499

[25] Q. Lu, Z. Yang, E. H. Twizell, Existence of entire explosive positive solutions of quasilinear elliptic equations, *Appl. Math. Comput.* **148**(2004), 359–372. MR2015378; url

[26] H. Mâagli, Asymptotic behavior of positive solutions of a semilinear Dirichlet problem, *Nonlinear Anal.* **74**(2011), 2941–2947. MR2785389; url

[27] H. Mâagli, S. Ben Othman, R. Chemmam, Asymptotic behavior of positive large solutions of semilinear Dirichlet problems, *Electron. J. Qual. Theory Differ. Equ.* **2013**, No. 57 , 1–13. MR3117850

[28] H. Mâagli, N. Mhadhebi, N. Zeddini, Existence and exact asymptotic behavior of positive solutions for a fractional boundary value problem, *Abstr. Appl. Anal.* **2013**, Art. ID 420514, 6 pp. MR3034992; url

[29] V. Maric, *Regular variation and differential equations*, Lecture Notes in Mathematics, Vol. 1726, Springer-Verlag, Berlin, 2000. MR1753584

[30] J. Matero, Quasilinear elliptic equations with boundary blow-up, *J. Anal. Math.* **69**(1996), 229–247. MR1428101

[31] P. J. Mckenna, W. Reichel, W. Walter, Symmetry and multiplicity for nonlinear elliptic differential equations with boundary blow-up, *Nonlin. Anal.* **28**(1997), 1213–1225. MR1422811

[32] A. Mohammed, Existence and asymptotic behavior of blow-up solutions to weighted quasilinear equations, *J. Math. Anal. Appl.* **298**(2004), 621–637. MR2086979; url

[33] J. Moser, A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equation, *Comm. Pure. Appl. Math.* **13**(1960), 457–468. MR0170091

[34] R. Osserman, On the inequality $\Delta u \geq f(u)$, *Pacific J. Math.* **7**(1957), 1641–1647. MR0098239

[35] S. L. Pohozaev, The Dirichlet problem for the equation $\Delta u = u^2$, *Soviet Math. Dokl.* **1**(1960), 1143–1146. MR0124610

[36] H. Rademacher, Einige besondere Probleme partiellen Differentialgleichungen, in: *Die Differential- und Integralgleichungen der Mechanik und Physik. I: Mathematischer Teil* (in German), 2nd edition, (P. Frank und R. von Mises, eds.), Rosenberg, New York, 1943, 838–845. MR0150433

[37] V. Rădulescu, Singular phenomena in nonlinear elliptic problems: from boundary blow-up solutions to equations with singular nonlinearities, in: *Handbook of differential equations: Stationary partial differential equations*, Vol. 4, 2007, 483–591. MR2569336

[38] R. Seneta, *Regularly varying functions*, Lectures Notes in Math., Vol. 508, Springer-Verlag, Berlin, 1976. MR0453936

[39] S. Stević, On harmonic Hardy and Bergman spaces, *J. Math. Soc. Japan* **54**(2002), 983–996. MR1921096; url

[40] S. Stević, On harmonic function spaces, *J. Math. Soc. Japan* **57**(2005), 781–802. MR2139734; url

[41] S. Stević, Area type inequalities and integral means of harmonic functions on the unit ball, *J. Math. Soc. Japan* **59**(2007), 583–601. MR2326178; url

[42] J. L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **1**(1984), 1992–2002. MR0768629

[43] N. Zeddini, R. Alsaedi, H. Mâagli, Exact boundary behavior of the unique positive solution to some singular elliptic problems, *Nonlin. Anal.* **89**(2013), 146–156. MR3073320; url

[44] Z. Zhang, A remark on the existence of explosive solutions for a class of semilinear elliptic equations, *Nonlin. Anal.* **41**(2000), 143–148. MR1759143

[45] Z. Zhang, Existence and asymptotic behavior of explosive solutions for nonlinear elliptic problems with convection terms, *Chinese Annals Math. Ser. A* **23**(2002), 395–406. MR1929494

[46] Z. Zhang, Y. Ma, L. Mi, X. Li, Blow-up rates of large solutions for elliptic equations, *J. Differential Equations* **249**(2010), 180–199. MR2644132; url