# Exponential Stability for a Class of Singularly Perturbed Itô Differential Equations

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**Abstract**

The problem of exponential stability in mean square of the zero solution for a class of singularly perturbed system of Itô differential equations is investigated.

Estimates of the block components of the fundamental random matrix are provided.

**Keywords:** Itô differential equations, singular perturbations, exponential stability in mean square.

**AMS classification: 34D15, 34F05** [1]

## 1 Problem formulation

Consider the system of Itô differential equations:

$$\begin{aligned} dx_1(t) &= [A_{11}x_1(t) + A_{12}x_2(t)]dt + \sum_{k=1}^{N} A_{11}^k x_1(t)dw_k(t) \\ \varepsilon dx_2(t) &= [A_{21}x_1(t) + A_{22}x_2(t)]dt + \sum_{k=1}^{N} A_{21}^k x_1(t)dw_k(t) \end{aligned} \tag{1.1}$$

[1]This paper is in the final form and no version of it will be submitted for publication elsewhere.

where $x_i \in \mathbf{R}^{n_i}, i = 1, 2, A_{ij}, A_{ij}^k$ are constant matrices with apropriate dimensions, $\varepsilon > 0$ is a small parameter; $w(t) = (w_1(t) \quad w_2(t)...w_N(t))^*$ is a standard Wiener process on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

For each $\varepsilon > 0$ a solution of the system (1.1) is a random process $x(t, \varepsilon) = \begin{pmatrix} x_1(t, \varepsilon) \\ x_2(t, \varepsilon) \end{pmatrix}$ which is continuous with probability 1, verifies (1.1) and for all $t, x(t, \varepsilon)$ is adapted to family of $\sigma$-algebras generated by the process $w(t)$.

**Definition 1.1** We say that the zero solution of the system (1.1) is mean square exponentially stable if there exist $\beta \geq 1, \alpha > 0$ such that for any solution $x(t, \varepsilon)$ of the system (1.1) with $x(0, \varepsilon) = x_0$ independent of $w(t)$, $t \geq 0$, we have:

$$E|x(t, \varepsilon)|^2 \leq \beta e^{-\alpha t} E|x(0)|^2$$

for all $t \geq 0$ $E$ being the expectation.

**Remark**

a) The constants $\alpha, \beta$ from the above definition may be dependent on the small parameter $\varepsilon > 0$.

b) It can be seen [11] that in the definition of exponential stability we may consider only the solutions with initial conditions $x(0) \in \mathbf{R}^n (n = n_1 + n_2)$.

The goal of this paper is to provide conditions assuring exponential stability of the zero solution of the system (1.1). Such conditions are expressed in terms of exponential stability of the zero solution of some subsytems of lower dimension not depending upon small parameter $\varepsilon$.

If $A_{22}$ is an invertible matrix we may associate the following system of Itô differential equations of lower dimension:

$$dx_1(t) = A_r x_1(t)dt + \sum_{k=1}^{N} A_r^k x_1(t) dw_k(t) \tag{1.2}$$

where

$$\begin{aligned} A_r &= A_{11} - A_{12}A_{22}^{-1}A_{21} \\ A_r^k &= A_{11}^k - A_{12}A_{22}^{-1}A_{21}^k, k = 1, 2, ..., N. \end{aligned}$$

We shall investigate the relationship between the exponential stability of the zero solution of the "reduced system" (1.2) and the exponential stability of the zero solution of the system (1.1) for $\varepsilon > 0$ small enough.

We extend to the class of stochastic systems (1.1) the well known result of Klimusev-Krasovski [9] from the ordinary differential equations.

The same problem was investigated in [6] in the case when the fast equation is

$$\varepsilon dx_2(t) = [A_{21}x_1(t) + A_{22}x_2(t)]dt + \sqrt{\varepsilon}\sum_{k=1}^{N}[A_{21}^k x_1(t) + A_{22}^k x_2(t)]dw_k(t).$$

The convergence in mean square of the slow component $x_1(t,\varepsilon)$ to a solution of the reduced system is proved in [4] when the system (1.1) is nonlinear.

# 2 Some preliminary results

Recall several results which are used in the next section.

**A.** For each $\varepsilon > 0$ we denote $\Phi(t,t_0,\varepsilon), t \geq t_0$ the fundamental random matrix solution of the system (1.1) (see [5]), the columns of $\Phi(t,t_0,\varepsilon)$ are solutions of the system (1.1) and $\Phi(t_0,t_0,\varepsilon) = I_{n_1+n_2}$.

If $x(t) = x(t,t_0,x_0,\varepsilon)$ is the solution of the system (1.1) which verifies $x(t_0,t_0,x_0,\varepsilon) = x_0$ then we have $x(t,\varepsilon) = \Phi(t,t_0,\varepsilon)x_0, x_0 \in \mathbf{R}^{n_1+n_2}$.

**B.** Consider the linear system of Itô differential equations

$$dx(t) = A_0x(t)dt + \sum_{k=1}^{N} A_kx(t)dw_k(t) \tag{2.1}$$

we have

**Lemma 2.1** [7] *The following are equivalent:*

*(i) the zero solution of the system (2.1) is exponentially stable in mean square.*

*(ii) there exists a positive matrix $X$ which solves the linear equation of Liapunov type*

$$AX + XA^* + \sum_{k=1}^{N} A^kX(A^k)^* + I = 0. \tag{2.2}$$

*Moreover if (i) holds then the unique positive solution of the equation (2.2) is given by*

$$X = E\int_0^\infty \Phi(t,0)\Phi^*(t,0)dt \tag{2.3}$$

*where $\Phi(t,0)$ is the fundamental random matrix of the system (2.1).*

**C.** If $I \subset \mathbf{R}$ is an interval, we denote $L^2_w(I, \mathbf{R}^n)$ the subspace of the measurable processes $f : I \times \Omega \to \mathbf{R}^n$ which are adapted with respect to the family of the $\sigma$-algebras generated by the Wiener process $w(t)$ and

$$E \int_I |f(t)|^2 dt < \infty.$$

Resoning as in the prove of proposition 1 in [2] we obtain:

**Lemma 2.2** *Assume that the zero solution of the system (2.1) is exponentially stable in mean square. If $x(t)$ is a solution of the affine system*

$$dx(t) = [A_0 x(t) + f_0(t)]dt + \sum_{k=1}^{N} [A_k x(t) + f_k(t)]dw_k(t),$$

$t \geq 0, x(t_0) = x_0$ *then*

$$E|x(t)|^2 \leq \beta(e^{-\alpha(t-t_0)}|x_0|^2 + \sum_{k=0}^{N} \int_{t_0}^{t} e^{-\alpha(t-s)} E|f_k(s)|^2 ds),$$

*for all* $t \in [t_0, t_1]$, $f_k \in L^2_w([t_0, t_1), \mathbf{R}^n)$, $\alpha, \beta$ *are positive constants not depending upon* $f_k, x_0$.

# 3 The main result

**Theorem 3.1** *Assume*

*(i) the zero solution of the reduced system (1.2) is exponentially stable in mean square;*

*(ii) the eigenvalues of the matrix $A_{22}$ are located in the half plane $Re\lambda < 0$.*

*Then there exists $\varepsilon_0 > 0$ such that for arbitrary $\varepsilon \in (0, \varepsilon_0)$ the zero solution of the system (1.1) is exponentially stable in mean square. Moreover, if*

$$\begin{pmatrix} \Phi_{11}(t, t_0, \varepsilon) & \Phi_{12}(t, t_0, \varepsilon) \\ \Phi_{21}(t, t_0, \varepsilon) & \Phi_{22}(t, t_0, \varepsilon) \end{pmatrix}$$

*is a partition of the fundamental "random" matrix $\Phi(t, t_0, \varepsilon)$ of the system (1.1) we have the estimates*

$$
\begin{array}{lcl}
E|\Phi_{11}(t,t_0,\varepsilon)|^2 & \leq & \beta_1 e^{-\alpha_1(t-t_0)} \\
E|\Phi_{12}(t,t_0,\varepsilon)|^2 & \leq & \beta_1 \varepsilon e^{-\alpha_1(t-t_0)} \\
E|\Phi_{21}(t,t_0,\varepsilon)|^2 & \leq & \frac{\beta_2}{\varepsilon}(e^{-\alpha_1(t-t_0)} - e^{-\alpha_2 \frac{t-t_0}{\varepsilon}}) + \beta_3 e^{-\alpha_1(t-t_0)} \\
E|\Phi_{22}(t,t_0,\varepsilon)|^2 & \leq & \beta_3 e^{-\alpha_1(t-t_0)}
\end{array}
\tag{3.1}
$$

*for all* $t \geq t_0, \beta_k, \alpha_k$ *positive constants not depending upon* $t, t_0, \varepsilon$.

**Proof:** Consider the nonlinear equation

$$
A_{21} + A_{22}T_{21} - \varepsilon T_{21}(A_{11} + A_{12}T_{21}) = 0. \tag{3.2}
$$

By a standard implicit functions argument, we deduce that there exists $\varepsilon_1 > 0$ and an anlytic function $T_{21}(\varepsilon)$ defined for $|\varepsilon| < \varepsilon_1$ which is a solution for equation (3.2) with $T_{21}(0) = -A_{22}^{-1}A_{21}$. Moreover $T_{21}(\varepsilon) = -A_{22}^{-1}A_{21} + \varepsilon\hat{T}_{21}(\varepsilon)$ with $|\hat{T}_{21}(\varepsilon)| \leq \hat{c} < \infty, (\forall)|\varepsilon| < \varepsilon_1$.

Let $T_{12}(\varepsilon)$ be the unique solution of the equation

$$
\varepsilon(A_{11} + A_{12}T_{21}(\varepsilon))T_{12} - T_{12}(A_{22} - \varepsilon T_{21}(\varepsilon)A_{12}) + A_{12} = 0. \tag{3.3}
$$

We have $T_{12}(\varepsilon) = A_{12}A_{22}^{-1} + \varepsilon\hat{T}_{12}(\varepsilon)$ where $|\hat{T}_{12}(\varepsilon)| \leq \hat{c} < \infty$ for all $|\varepsilon < \varepsilon_1$.

If $\begin{pmatrix} x_1(t,\varepsilon) \\ x_2(t,\varepsilon) \end{pmatrix}$ is a solution of the system (1.1) we define

$$
\begin{pmatrix} \xi_1(t,\varepsilon) \\ \xi_2(t,\varepsilon) \end{pmatrix} = \begin{pmatrix} I_{n_1} & -\varepsilon T_{12}(\varepsilon) \\ 0 & I_{n_2} \end{pmatrix} \begin{pmatrix} I_{n_1} & 0 \\ -T_{21}(\varepsilon) & I_{n_2} \end{pmatrix} \begin{pmatrix} x_1(t,\varepsilon) \\ x_2(t,\varepsilon) \end{pmatrix}. \tag{3.4}
$$

Using the Itô formula ([5]) we deduce that $\begin{pmatrix} \xi_1(t,\varepsilon) \\ \xi_2(t,\varepsilon) \end{pmatrix}$ is a solution of the following system:

$$
\begin{array}{rcl}
d\xi_1(t) & = & A_1(\varepsilon)\xi_1(t)dt + \sum_{k=1}^{N}(\hat{A}_{11}^k(\varepsilon)\xi_1(t) + \varepsilon\hat{A}_{12}^k(\varepsilon)\xi_2(t))dw_k(t) \\
\varepsilon d\xi_2(t) & = & A_2(\varepsilon)\xi_2(t)dt + \sum_{k=1}^{N}(\hat{A}_{21}^k(\varepsilon)\xi_1(t) + \varepsilon\hat{A}_{22}^k(\varepsilon)\xi_2(t))dw_k(t)
\end{array}
\tag{3.5}
$$

where

$$
\begin{array}{rcl}
A_1(\varepsilon) & = & A_{11} + A_{12}T_{21}(\varepsilon) = A_r + O(\varepsilon) \\
A_2(\varepsilon) & = & A_{22} - \varepsilon T_{21}(\varepsilon)A_{12} \\
\hat{A}_{21}^k(\varepsilon) & = & A_{21}^k - \varepsilon T_{21}(\varepsilon)A_{11}^k \\
A_{11}^k(\varepsilon) & = & A_r^k + O(\varepsilon), \quad \hat{A}_{12}^k(\varepsilon) = \hat{A}_{11}^k(\varepsilon)T_{12}(\varepsilon) \\
\hat{A}_{22}^k(\varepsilon) & = & \hat{A}_{21}^k(\varepsilon)T_{12}(\varepsilon).
\end{array}
\tag{3.6}
$$

Since the zero solution of the reduced system (1.2) is exponentially stable in mean square it follows that the linear equation

$$A_r X + X A_r^* + \sum_{k=1}^{N} A_r^* X (A_r^k)^* + I_{n_1} = 0 \tag{3.7}$$

has a unique symmetric solution $X_r > 0$.

Also, since $A_{22}$ is stable, it follows that the Liapunov equation

$$A_{22} X_2 + X_2 A_{22}^* + I_{n_2} = 0$$

has a unique solution $X_2 > 0$ (see (2.3)).

By a standard argument based on implicit function theorem we deduce that the linear equations

$$\begin{aligned} A_1(\varepsilon) X_1 + X_1 A_1^*(\varepsilon) + \sum_{k=1}^{N} \hat{A}_{11}^k(\varepsilon) X_1 (A_{11}^k(\varepsilon))^* + I_{n_1} &= 0 \\ A_2(\varepsilon) X_2 + X_2 A_2^*(\varepsilon) + \varepsilon \sum_{k=1}^{N} \hat{A}_{22}^k(\varepsilon) X_2 (\hat{A}_{22}^k(\varepsilon))^* + I_{n_2} &= 0 \end{aligned}$$

have positive solutions $X_1(\varepsilon), X_2(\varepsilon)$ respectively defined for all $\varepsilon \in (0, \varepsilon_2)$. Moreover, there exist $\mu_1, \mu_2, \nu_1, \nu_2$ not depending upon $\varepsilon$ such that

$$0 < \mu_1 I_{n_1} \leq X_1(\varepsilon) \leq \mu_2 I_{n_1};$$

$$0 < \nu_1 I_{n_2} \leq X_2(\varepsilon) \leq \nu_2 I_{n_2}.$$

Applying Lemma 2.1 we conclude that the zero solution of the system

$$d\xi_1(t) = A_1(\varepsilon)\xi_1(t)dt + \sum_{k=1}^{N} \hat{A}_{11}^k(\varepsilon)\xi_1(t)dw_k(t) \tag{3.8}$$

and the one of the system

$$\varepsilon d\xi_2(t) = A_2(\varepsilon)\xi_2(t)dt + \varepsilon \sum_{k=1}^{N} \hat{A}_{22}^k(\varepsilon)\xi_2(t)dw_k(t) \tag{3.9}$$

respectively, are exponentially stable in mean square.

Applying Lemma 2.2 to the first equation in the system (3.5) we deduce that there exist $\tilde{\beta}_1 > 0, \tilde{\alpha}_1 > 0$ such that

$$E|\xi_1(t,\varepsilon)|^2 \leq \tilde{\beta}_1 (e^{-\tilde{\alpha}_1(t-t_0)} E|\xi_1(t_0)|^2 + \varepsilon^2 \int_{t_0}^{t} e^{-\tilde{\alpha}_1(t-\sigma)} E|\xi_2(s,\varepsilon)|^2 ds. \tag{3.10}$$

Also, applying Lemma 2.2 to the second equation in the system (3.5) we deduce that there exist $\tilde{\beta}_2 > 0, \tilde{\alpha}_2 > 0$ such that

$$E|\xi_2(t,\varepsilon)|^2 \leq \tilde{\beta}_2[e^{-\tilde{\alpha}_2\frac{(t-t_0)}{\varepsilon}}E|\xi_2(t_0,\varepsilon)|^2 + \frac{1}{\varepsilon^2}\int_{t_0}^{t} e^{-\tilde{\alpha}_2\frac{(t-\sigma)}{\varepsilon}}E|\xi_1(\sigma,\varepsilon)|^2 d\sigma] \quad (3.11)$$

for all $t \geq t_0$.

Substituting (3.11) in (3.10) and changing the order of integration we get

$$\begin{aligned} E|\xi_1(t,\varepsilon)|^2 &\leq \tilde{\beta}_3 e^{-\tilde{\alpha}_1(t-t_0)}[E|\xi_1(t_0,\varepsilon)|^2 + \varepsilon E|\xi_2(t_0,\varepsilon)|^2] \\ &\quad + \tilde{\beta}_4\int_{t_0}^{t}[\varepsilon e^{-\tilde{\alpha}_1(t-s)} + e^{-\tilde{\alpha}_2\frac{(t-s)}{\varepsilon}}]E|\xi_1(s,\varepsilon)|^2 ds. \end{aligned}$$

By a standard argument in singular perturbation theory (see [1, 12]) we deduce that there exists $0 < \varepsilon_0 \leq \varepsilon_2$ such that for arbitrary $\varepsilon \in (0,\varepsilon_0)$ we have

$$E|\xi_1(t,\varepsilon)|^2 \leq \tilde{\beta}_5 e^{-\alpha_1(t-t_0)}[E|\xi_1(t_0,\varepsilon)|^2 + \varepsilon E|\xi_2(t_0,\varepsilon)|^2] \quad (3.12)$$

for all $t \geq t_0$, where $\tilde{\beta}_5 > 0, \alpha_1 \in (0,\tilde{\alpha}_1)$.

Using (3.12) in (3.11) we obtain

$$\begin{aligned} E|\xi_2(t,\varepsilon)| &\leq \tilde{\beta}_2 e^{-\tilde{\alpha}_2\frac{(t-t_0)}{\varepsilon}}[E|\xi_2(t_0,\varepsilon)|^2] \quad (3.13) \\ &+ \frac{\tilde{\beta}_6}{\varepsilon}[e^{-\alpha_1(t-t_0)} - e^{-\tilde{\alpha}_2\frac{(t-t_0)}{\varepsilon}}][E|\xi_1(t_0,\varepsilon)|^2 + \varepsilon E|\xi_2(t_0,\varepsilon)|^2]. \end{aligned}$$

Reversing (3.4) and taking into account (3.12), (3.13) we get:

$$\begin{aligned} E|x_1(t,\varepsilon)|^2 &\leq \beta_1 e^{-\alpha_1(t-t_0)}[E|x_1(t_0,\varepsilon)|^2 + \varepsilon E|x_2(t_0,\varepsilon)|^2] \\ E|x_2(t,\varepsilon)|^2 &\leq \beta_2 e^{-\alpha_2\frac{(t-t_0)}{\varepsilon}}E|x_2(t_0,\varepsilon)| \quad (3.14) \\ &\quad + \frac{\tilde{\beta}_2}{\varepsilon}(e^{-\alpha_1(t-t_0)} - e^{-\alpha_2\frac{(t-t_0)}{\varepsilon}})(E|x_1(t_0,\varepsilon)|^2 + \varepsilon E|x_2(t_0,\varepsilon)|^2) \end{aligned}$$

which means the exponential stability of the zero solution of the system (1.1).

The estimations (3.1) follows from (3.14) replacing $\begin{pmatrix} x_1(t_0,\varepsilon) \\ x_2(t_0,\varepsilon) \end{pmatrix}$ by $\begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ I_{n_2} \end{pmatrix}$ respectively.

**Remark.** The estimates of the block components of the fundamental matrix solution $\Phi(t,t_0,\varepsilon)$ of the system (1.1) obtained in the above theorem differ essentially from the ones obtained in the deterministic framework (see [1]).

This is due to the intensity of the white noise from the second equation of the system (1.1).

Let us consider the system of differential equations:

$$dx_1(t) = [A_{11}x_1(t)+A_{12}x_2(t)]dt+\sum_{k=1}^{N}[A_{11}^k x_1(t)+\varepsilon A_{12}^k x_1(t)]dw_k(t) \quad (3.15)$$
$$\varepsilon dx_2(t) = [A_{21}x_1(t)+A_{22}x_2(t)]dt+\sqrt{\varepsilon}\sum_{k=1}^{N}[A_{21}^k x_1(t)+A_{22}^k x_2(t)]dw_k(t).$$

In this case the reduced system is :

$$dx_1(t) = [A_{11} - A_{12}A_{22}^{-1}A_{21}]x_1(t)dt + \sum_{k=1}^{N} A_{11}^k x_1(t)dw_k(t). \quad (3.16)$$

We associate to system (3.15) the following system:

$$dx_2(\tau) = A_{22}x_2(\tau)d\tau + \sum_{k=1}^{N} A_{22}^k x_2(\tau)dw_k(\tau) \quad (3.17)$$

which is called “the boundary layer subsystem”.

If in the system (3.15) we perform the coordinates transformation (3.4) we get:

$$d\xi_1(t) = A_1(\varepsilon)\xi_1(t)dt + \sum_{k=1}^{N}(\hat{A}_{11}^k(\varepsilon)\xi_1(t) + \varepsilon\hat{A}_{12}^k(\varepsilon\xi_2(t))dw_k(t) \quad (3.18)$$
$$\varepsilon d\xi_2(t) = A_2(\varepsilon)\xi_2(t)dt + \sqrt{\varepsilon}\sum_{k=1}^{N}(\hat{A}_{21}^k(\varepsilon)\xi_1(t) + \hat{A}_{22}^k(\varepsilon)\xi_2(t))dw_k(t).$$

With the same arguments as in the proof of Theorem 3.1 we obtain:

**Theorem 3.1'** *Assume*

*a) $A_{22}$ is a invertible matrix.*

*b) The zero solution of the reduced system (3.16) and the one of the boundary layer system (3.17), respectively, is exponentially stable in mean square.*

*Under these assumptions the block components of the fundamental random matrix solution of the system (3.15) verify:*

$$\begin{aligned} E|\Phi_{11}(t,t_0,\varepsilon)|^2 &\leq \beta_1 e^{-\alpha_1(t-t_0)} \\ E|\Phi_{12}(t,t_0,\varepsilon)|^2 &\leq \beta_1 \varepsilon e^{-\alpha_1(t-t_0)} \\ E|\Phi_{21}(t,t_0,\varepsilon)|^2 &\leq \beta_2 e^{-\alpha_1(t-t_0)} \\ E|\Phi_{22}(t,t_0,\varepsilon)|^2 &\leq \beta_2 (e^{-\alpha_2\frac{(t-t_0)}{\varepsilon}} + \varepsilon e^{-\alpha_1(t-t_0)}) \end{aligned} \quad (3.19)$$

*for all $t \geq t_0, \varepsilon > 0$ small enough; $\alpha_k, \beta_k$ positive constants not depending upon $t, t_0, \varepsilon$.*

The estimates obtained in Theorem 3.1' are similar to the ones obtained in deterministic framework [1]. In the case of system (3.15) we are able to prove the following converse result:

**Theorem 3.2** *Assume that $A_{22}$ is an invertible matrix. If the fundamental random matrix solution $\Phi(t, t_0, \varepsilon)$ of the system (3.15) verifies the estimates (3.19), then the zero solution of the reduced subsystem (3.16) and the one of the boundary layer subsystem (3,17) respectively is exponentially stable in mean square.*

**Proof.** Denote $X(\varepsilon) = E\int_0^\infty \Phi(t,0,\varepsilon)\mathcal{I}(\varepsilon)\Phi^*(t,0,\varepsilon)dt$ where $\mathcal{I}(\varepsilon) = \begin{pmatrix} I_{n_1} & 0 \\ 0 & \varepsilon^{-1}I_{n_2} \end{pmatrix}$. If $\begin{pmatrix} X_{11}(\varepsilon) & X_{12}(\varepsilon) \\ X_{12}^*(\varepsilon) & X_{22}(\varepsilon) \end{pmatrix}$ is the partition of $X(\varepsilon)$ it is easy to see that

$$\begin{aligned}
X_{11}(\varepsilon) &= E\int_0^\infty (\Phi_{11}(t,0,\varepsilon)\Phi_{11}^*(t,0,\varepsilon) + \varepsilon^{-1}\Phi_{12}(t,0,\varepsilon)\Phi_{12}^*(t,0,\varepsilon))dt \\
X_{12}(\varepsilon) &= E\int_0^\infty (\Phi_{11}(t,0,\varepsilon)\Phi_{21}^*(t,0,\varepsilon) + \varepsilon^{-1}\Phi_{12}(t,0,\varepsilon)\Phi_{22}^*(t,0,\varepsilon))dt \\
X_{22}(\varepsilon) &= E\int_0^\infty (\Phi_{21}(t,0,\varepsilon)\Phi_{21}^*(t,0,\varepsilon) + \varepsilon^{-1}\Phi_{22}(t,0,\varepsilon)\Phi_{22}^*(t,0,\varepsilon))dt.
\end{aligned}$$

Using the estimates (3.19) we see hat the integral from definition of $X(\varepsilon)$ is convegent. Moreover we get:

$$X_{ij}(\varepsilon) \leq \hat{c} < \infty \tag{3.20}$$

for $\varepsilon > 0$ small enough, $\hat{c}$ not depending upon $\varepsilon$.

Based on Itô formula [5] we conclude that $X(\varepsilon)$ verifies the equation

$$A(\varepsilon)X(\varepsilon) + X(\varepsilon)A^*(\varepsilon) + \sum_{k=1}^{N} A^k(\varepsilon)X(\varepsilon)(A^k(\varepsilon))^* + \mathcal{I}(\varepsilon) = 0 \tag{3.21}$$

where

$$\begin{aligned}
A(\varepsilon) &= \begin{pmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{pmatrix} \\
A^k(\varepsilon) &= \begin{pmatrix} A_{11}^k & \varepsilon A_{12}^k \\ \varepsilon^{-\frac{1}{2}}A_{21}^k & \varepsilon^{-\frac{1}{2}}A_{22}^k \end{pmatrix}.
\end{aligned}$$

If

$$T(\varepsilon) = \begin{pmatrix} I_{n_1} & -\varepsilon T_{12}(\varepsilon) \\ 0 & I_{n_2} \end{pmatrix}\begin{pmatrix} I_{n_1} & 0 \\ -T_{21}(\varepsilon) & I_{n_2} \end{pmatrix}$$

is the matrix used in (3.4) we denote

$$\begin{pmatrix} Y_{11}(\varepsilon) & Y_{12}(\varepsilon) \\ Y_{12}^*(\varepsilon) & Y_{22}(\varepsilon) \end{pmatrix} = T(\varepsilon)X(\varepsilon)T^*(\varepsilon).$$

Based on (3.20) we have

$$|Y_{ij}(\varepsilon)| \leq \tilde{c} < \infty \tag{3.22}$$

for $\varepsilon > 0$ small enough, $\tilde{c}$ not depending upon $\varepsilon$. Using (3.21) we obtain that $Y_{ij}(\varepsilon)$ solve the following system

$$\begin{aligned}
A_1(\varepsilon)Y_{11}(\varepsilon) + Y_{11}(\varepsilon)A_1^*(\varepsilon) + \sum_{i=1}^{N}[\hat{A}_{11}^i(\varepsilon)Y_{11}(\varepsilon)(\hat{A}_{11}^i(\varepsilon))^* \\
+\varepsilon\hat{A}_{12}^i(\varepsilon)Y_{12}^*(\varepsilon)(\hat{A}_{11}^i(\varepsilon))^* + \varepsilon\hat{A}_{11}^i(\varepsilon)Y_{12}(\varepsilon)(\hat{A}_{12}^i(\varepsilon))^* \\
+\varepsilon^2\hat{A}_{12}^i(\varepsilon)Y_{22}(\varepsilon)(\hat{A}_{12}^i(\varepsilon))^*] + M_{11}(\varepsilon) = 0 \\
\varepsilon A_1(\varepsilon)Y_{12}(\varepsilon) + Y_{12}(\varepsilon)A_2^*(\varepsilon) + \sum_{i=1}^{N}[\varepsilon^{\frac{1}{2}}\hat{A}_{11}^i(\varepsilon)Y_{11}(\varepsilon)(\hat{A}_{21}^i(\varepsilon))^* \\
+\varepsilon^{\frac{3}{2}}\hat{A}_{12}^i(\varepsilon)Y_{12}^*(\varepsilon)(\hat{A}_{21}^i(\varepsilon))^* + \varepsilon^{\frac{1}{2}}\hat{A}_{11}^i(\varepsilon)Y_{12}(\varepsilon)(\hat{A}_{22}^i(\varepsilon))^* \\
+\varepsilon^{\frac{3}{2}}\hat{A}_{12}^i(\varepsilon)Y_{22}(\varepsilon)(\hat{A}_{22}^i(\varepsilon))^* + M_{12}(\varepsilon) = 0 \\
A_2(\varepsilon)Y_{22}(\varepsilon) + Y_{22}(\varepsilon)A_2^*(\varepsilon) + \sum_{i=1}^{N}[\hat{A}_{21}^i(\varepsilon)Y_{11}(\varepsilon)(\hat{A}_{21}^i(\varepsilon))^* \\
+\hat{A}_{22}^i(\varepsilon)Y_{12}^*(\varepsilon)(\hat{A}_{21}^i(\varepsilon))^* + \hat{A}_{21}^i(\varepsilon)Y_{12}(\varepsilon)(\hat{A}_{22}^i(\varepsilon))^* \\
+\hat{A}_{22}^i(\varepsilon)Y_{22}(\varepsilon)(\hat{A}_{22}^i(\varepsilon))^*] + M_{22}(\varepsilon) = 0
\end{aligned} \tag{3.23}$$

where

$$\begin{aligned}
M_{11}(\varepsilon) &= [I_{n_1} + \varepsilon T_{12}(\varepsilon)T_{21}(\varepsilon)][I_{n_1} + \varepsilon T_{12}(\varepsilon)T_{21}(\varepsilon)]^* + \varepsilon T_{12}(\varepsilon)T_{12}^*(\varepsilon) \\
M_{12}(\varepsilon) &= -[I_{n_1} + \varepsilon T_{12}(\varepsilon)T_{21}(\varepsilon)]T_{21}^*(\varepsilon) - T_{12}(\varepsilon) \\
M_{22}(\varepsilon) &= I_{n_2} + \varepsilon T_{21}(\varepsilon)T_{21}^*(\varepsilon).
\end{aligned}$$

If $\varepsilon_k, k \geq 0$ is a sequence with $\lim_{k\to\infty}\varepsilon_k = 0$ then from (3.22) it follows that there exist a subsequence $\varepsilon_{k_l}$ such that $\lim_{l\to\infty} Y_{ij}(\varepsilon_{k_l})$ are well defined.

Set $Y_{ij}^0 = \lim_{l\to\infty} Y_{ij}(\varepsilon_{k_l})$.Replacing $\varepsilon$ by $\varepsilon_{k_l}$ in (3.23) and taking the limit for $l \to \infty$ we get

$$\begin{aligned}
A_r Y_{11}^0 + Y_{11}^0 A_r^* + \sum_{i=1}^{N} A_r^i Y_{11}^0 (A_r^i)^* + I_{n_1} = 0 \\
A_{22}Y_{22}^0 + Y_{22}^0 A_{22}^* + \sum_{i=1}^{N} A_{22}^i Y_{22}^0 (A_{22}^i)^* + I_{n_2} + \sum_{i=1}^{N} A_{21}^i Y_{11}^0 (A_{21}^i)^* = 0 \\
Y_{12}^0 = 0.
\end{aligned} \tag{3.24}$$

The conclusion follows using Lemma 2.1 and the first two equations of the system (3.24).

# 4 Application

In this section we shall present an application of the result of Theorem 3.1 designing of a stabilizing state feedback for the following class of controlled stochastic linear systems:

$$\begin{aligned} dx_1(t) &= [A_{11}x_1(t)+A_{12}x_2(t)+B_1u(t)]dt+\sum_{k=1}^{N} A_{11}^k x_1(t)dw_k(t) \\ \varepsilon dx_2(t) &= [A_{21}x_1(t)+A_{22}x_2(t)+B_2u(t)]dt+\sum_{k=1}^{N} A_{21}^k x_1(t)dw_k(t) \end{aligned} \quad (4.1)$$

$u \in \mathbf{R}^m$ is the control, $x_i \in \mathbf{R}^{n_i}, i = 1, 2$ are the states and $A_{ij}^k, B_j$ are real constant matrices of appropriate dimensions.

Our aim is to construct a control law of the form:

$$u(t) = F_1x_1(t) + F_2x_2(t) \quad (4.2)$$

with the property that the zero solution of the corresponding closed-loop system:

$$\begin{aligned} dx_1(t) &= [(A_{11}+B_1F_1)x_1(t)+(A_{12}+B_1F_2)x_2(t)]dt \\ &\quad +\sum_{k=1}^{N} A_{11}^k x_1(t)dw_k(t) \\ \varepsilon dx_2(t) &= [(A_{21}+B_2F_1)x_1(t)+(A_{22}+B_2F_2)x_2(t)]dt \\ &\quad +\sum_{k=1}^{N} A_{21}^k x_1(t)dw_k(t) \end{aligned} \quad (4.3)$$

be mean square exponentially stable. To avoind the ill conditioning due to the presence of the small parameter $\varepsilon$ in the system (4.1) the designing of the stabilizing state feedback (4.2) is made in two steps. In each step we shall design a stabilizing feedback gain for a controlled system of lower dimensions and independent of the small parameter $\varepsilon$.

To the system (4.1) with $A_{22}$ invertible we associate the following reduced subsystem:

$$dx_1(t) = [A_rx_1(t) + B_ru(t)]dt + \sum_{k=1}^{N} A_r^k x_1(t)dw_k(t) \quad (4.4)$$

where $A_r$ and $A_r^k$ are the same as in (1.2) and $B_r = B_1 = A_{12}A_{22}^{-1}B_2$.

The corresponding boundary layer subsystem of the system (4.1) is:

$$x_2'(\sigma) = A_{22}x_2(\sigma) + B_2u(\sigma) \quad (4.5)$$

$\sigma = \frac{t}{\varepsilon}$. If the pair $(A_{22}, B_2)$ is stabilizable (see [10]) we choose $F_2 \in \mathbf{R}^{m\times n_2}$ such that the matrix $A_{22} + B_2F_2$ have the eigenvalues located in the half plane $Re\lambda < 0$.

Now, using this choice of the feedback gain $F_2$ we consider the following system:

$$dx_1(t) \quad = \quad [A_r x_1(t) + B_r u(t)]dt + \sum_{k=1}^{N} A_r^k(F_2)x_1(t)dw_k(t) \tag{4.6}$$

where $A_r^k(F_2) = A_{11}^k - [A_{12} + B_1F_2][A_{22} + B_2F_2]^{-1}A_{21}^k$.

Assuming that the system (4.6) is stochastically stabilizable we choose the matrix $F_r \in \mathbf{R}^{m\times n_1}$ such that the zero solution of the closed-loop system

$$\frac{dx_1}{dt} = [A_r + B_rF_r]x_1(t)dt + \sum_{k=1}^{N} A_r^k(F_2)x_1(t)dw_k(t) \tag{4.7}$$

be mean square exponentially stable.

Set

$$F_1 = [I_m + F_2A_{22}^{-1}B_2]F_r + F_2A_{22}^{-1}A_{21}. \tag{4.8}$$

The matrix $F_1$ given by (4.8) and the matrix $F_2$ obtained before, can be used to provide a stabilizing control law (4.2).

To show that such a control stabilizes the system (4.1) for $\varepsilon > 0$ small enough, we shall use the result of the Theorem 3.1 to the closed-loop system (4.3). To this end we show that the assumption of the Theorem 3.1 are fulfild.

Indeed the matrix coefficients of the boundary layer subsystem is now $A_{22} + B_2F_2$ which is stable. On the other hand the reduced subsystem associated to the closed-loop system is

$$\begin{aligned} dx_1(t) \quad = \quad & [A_{11}+B_1F_1-(A_{12}+B_1F_2)(A_{22}+B_2F_2)^{-1}(A_{21}+B_2F_1)]x_1(t)dt \\ & + \sum_{k=1}^{N} A_r^k(F_2)x_1(t)dw_k(t). \end{aligned} \tag{4.9}$$

By algebraic manipulations, similar with the ones in deterministic framework, (see e.g. [3, 8]) we obtain that (4.9) is equivalent to (4.7) and hence the zero solution of the system (4.9) is exponentially stable in mean square.

Thus we obtain that the assumptions from Theorem 3.1 hold and hence the zero solution of the closed-loop system is mean square exponentially stable for arbitrary $\varepsilon > 0$ small enough.

**Remark:** In the case of the stochastic systems (4.1) the designing of the feedback gain $F_r$ cannot be performed independently of the stabilizing feedback gain $F_2$ as it happens in the deterministic framework [3, 8].

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