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Controllability of fractional order integro-differential inclusions with infinite delay

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Abstract. This paper concerns for controllability of fractional order integro-differential inclusions with infinite delay in Banach spaces. A theorem about the existence of mild solutions to the controllability of fractional order integro-differential inclusions is obtained based on Dhage fixed point theorem. An example is given to illustrate the existence result.

Keywords: controllability, Caputo fractional derivative, integro-differential inclusions, fixed point, semigroup, infinite delay.

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1 Introduction

Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, economics and science. We can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [26, 27, 35, 37]. In recent years, there has been a significant development in fractional differential equations. One can see the monographs of Abbas *et al.* [1, 2], Kilbas *et al.* [31], Lakshmikantham *et al.* [32], Miller and Ross [38], Podlubny [40], Zhou [46], and the papers [3–5, 10, 17–20, 24, 34, 41, 42] and the references therein.

On the other hand, the most important qualitative behavior of a dynamical system is controllability. It is well known that the issue of controllability plays an important role in control theory and engineering [7, 8, 12, 15] because they have close connections to pole assignment, structural decomposition, quadratic optimal control and observer design etc. In

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recent years, the problem of controllability for various kinds of fractional differential and integro-differential equations have been discussed in [6, 16, 44].

El-Sayed and Ibrahim initiated the study of fractional differential inclusions in [25]. Recently several qualitative results for fractional differential inclusion were obtained in [11, 14, 39]. Recently, Benchohra *et al.* [9] studied the existence and controllability results for fractional order integro-differential inclusions with state-dependent delay in Fréchet spaces. Wang and Zhou [45] investigated the existence and controllability results for fractional semilinear differential inclusions.

Motivated by the papers cited above, in this paper, we consider the controllability results for fractional order integro-differential inclusions with infinite delay described by the form

$$\begin{aligned} &D_t^q x(t) \in Ax(t) + Bu(t) + \int_0^t a(t,s)F(s,x_s,x(s))\,ds, && t \in J = [0,T], \\ &x_0 = \phi \in \mathcal{B}, && t \in \ (-\infty,0], \end{aligned} \tag{1.1}$$

where D_t^q is the Caputo fractional derivative of order $0 < q < 1$, A generates a compact and uniformly bounded linear semigroup $S(\cdot)$ on X, $F\colon J \times \mathcal{B} \times X \longrightarrow \mathcal{P}(X)$ is a multivalued map ($\mathcal{P}(X)$ is the family of all nonempty subsets of X), $a\colon D \to \mathbb{R}$ ($D = \{(t,s) \in [0,T] \times [0,T] : t \geq s\}$), $\phi \in \mathcal{B}$ where $\mathcal{B}$ is called phase space to be defined in Section 2. B is a bounded linear operator from X into X, the control $u \in L^2(J;X)$, the Banach space of admissible controls. For any function x defined on $(-\infty, T]$ and any $t \in J$, we denote by x_t the element of $\mathcal{B}$ defined by

$$x_t(\theta) = x(t+\theta), \qquad \theta \in \ (-\infty, 0].$$

Here x_t represents the history of the state up to the present time t.

Our results are based on the Dhage fixed point theorem and the semigroup theory. To our knowledge, very few results are available for controllability for fractional integro-differential inclusions. So the present results complement this literature.

The paper is organized as follows. In Section 2 some preliminary results are introduced. The main result is presented in Section 3, and an example illustrating the abstract theory is presented in Section 4.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space.
$C = C(J,X)$ be the space of all X-valued continuous functions on J.
$L(X)$ be the Banach space of all linear and bounded operators on X.
$L^1(J,X)$ the space of X-valued Bochner integrable functions on J with the norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\|\,dt.$$

$L^\infty(J,\mathbb{R})$ is the Banach space of essentially bounded functions, normed by

$$\|y\|_{L^\infty} = \inf\{d > 0 : |y(t)| \leq d, \text{ a.e. } t \in J\}.$$

Denote by $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in P(X) : Y \text{ compact}\}$, $P_{cp,c}(X) = \{Y \in P(X) : Y \text{ compact, convex}\}$.

A multivalued map $G\colon X \to P(X)$ is convex (closed) valued if $G(X)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set U of X containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq U$.

G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$).

For more details on multivalued maps see the books of Deimling [22], Górniewicz [28] and Hu and Papageorgiou [30] .

Definition 2.1. The multivalued map $F\colon J \times \mathcal{B} \times X \longrightarrow \mathcal{P}(X)$ is said to be an Carathéodory if

(i) $t \longmapsto F(t,x,y)$ is measurable for each $(x,y) \in \mathcal{B} \times X$;

(ii) $(x,y) \longmapsto F(t,x,y)$ is upper semicontinuous for almost all $t \in J$.

We need some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.2. Let $\alpha > 0$ and $f\colon \mathbb{R}_+ \to X$ be in $L^1(\mathbb{R}_+, X)$. Then the Riemann–Liouville integral is given by:

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

For more details on the Riemann–Liouville fractional derivative, we refer the reader to [21].

Definition 2.3 ([40])**.** The Caputo derivative of order α for a function $f\colon [0,+\infty) \to \mathbb{R}$ can be written as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\, ds = I^{n-\alpha} f^{(n)}(t), \quad t>0,\ n-1 \le \alpha < n.$$

If $0 < \alpha \le 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}}\, ds.$$

Obviously, the Caputo derivative of a constant is equal to zero.

In this paper, we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced by Hale and Kato [29]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, and satisfies the following axioms:

(A1) If $x\colon (-\infty, T\,] \longrightarrow X$ is continuous on J and $x_0 \in \mathcal{B}$, then $x_t \in \mathcal{B}$ and x_t is continuous in $t \in J$ and

$$\|x(t)\| \le C\|x_t\|_{\mathcal{B}}, \tag{2.1}$$

where $C \ge 0$ is a constant.

(A2) There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$ such that
$$\|x_t\|_{\mathcal{B}} \leq C_1(t) \sup_{s\in[0,t]} \|x(s)\| + C_2(t)\|x_0\|_{\mathcal{B}}, \tag{2.2}$$
for $t \in [0, T]$ and x as in **(A1)**.

(A3) The space $\mathcal{B}$ is complete.

Remark 2.4. Condition (2.1) in **(A1)** is equivalent to $\|\phi(0)\| \leq C\|\phi\|_{\mathcal{B}}$, for all $\phi \in \mathcal{B}$.

Let $S_{F,x}$ be a set defined by
$$S_{F,x} = \{v \in L^1(J, X) : v(t) \in F(t, x_t, x(t)) \text{ a.e. } t \in J\}.$$

Lemma 2.5 ([33]). *Let X be a Banach space. Let $F\colon J \times \mathcal{B} \times X \longrightarrow P_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Ψ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$, then the operator*
$$\begin{aligned} \Psi \circ S_F \colon C(J, X) &\longrightarrow P_{cp,c}(C(J, X)), \\ x &\longmapsto (\Psi \circ S_F)(x) := \Psi(S_{F,x}) \end{aligned}$$
is a closed graph operator in $C(J, X) \times C(J, X)$.

Proposition 2.6 ([13, Proposition III.4]). *If Γ_1 and Γ_2 are compact valued measurable multifunctions, then the multifunction $t \to \Gamma_1(t) \cap \Gamma_2(t)$ is measurable. If (Γ_n) is a sequence of compact valued measurable multifunctions, then $t \to \cap\Gamma_n(t)$ is measurable, and if $\overline{\cup\Gamma_n(t)}$ is compact, then $t \to \overline{\cup\Gamma_n(t)}$ is measurable.*

Definition 2.7. A multivalued operator $N\colon X \to P_{cl}(X)$ is called

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that
$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad \text{for each } x, y \in X,$$

b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Theorem 2.8 (Dhage theorem [23]). *Let E be a Banach space, $A\colon E \to P_{cl,cv,bd}(E)$ and $B\colon E \to P_{cp,cv}(E)$, two multivalued operators satisfying:*

1. *A is a contraction, and*

2. *B is completely continuous.*

Then either

(i) the operator inclusion $u \in Au + Bu$ has a solution , or

(ii) the set $\mathcal{E} = \{u \in E, u \in \lambda A(u) + \lambda B(u), \quad 0 \leq \lambda \leq 1\}$ is unbounded.

Let Ω be a set defined by
$$\Omega = \left\{x\colon (-\infty, T] \to X \text{ such that } x|_{(-\infty,0]} \in \mathcal{B},\ x|_J \in C(J, X)\right\}.$$

3 Main results

In this section, we state and prove the controllability results for the system (1.1). Now we define the mild solution for our problem.

Definition 3.1. A function $x \in \Omega$ is said to be a mild solution of (1.1) if there exists $v(\cdot) \in L^1(J, X)$, such that $v(t) \in F(t, x_t, x(t))$ a.e. $t \in [0, T]$, and x satisfies

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ -Q(t)\phi(0) + \int_0^t R(t-s)Bu(s)\,ds \\ + \int_0^t \int_0^s R(t-s)a(s,\tau)v(\tau)\,d\tau\,ds, & t \in J, \end{cases} \tag{3.1}$$

where

$$Q(t) = \int_0^\infty \xi_q(\sigma)S(t^q\sigma)\,d\sigma, \qquad R(t) = q\int_0^\infty \sigma t^{q-1}\xi_q(\sigma)S(t^q\sigma)\,d\sigma$$

and for $\sigma \in (0, \infty)$,

$$\xi_q(\sigma) = \frac{1}{q}\sigma^{-1-\frac{1}{q}}\varpi_q(\sigma^{-\frac{1}{q}}) \geq 0,$$

$$\varpi_q(\sigma) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sigma^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q).$$

Here, ξ_q is a probability density function defined on $(0, \infty)$ [36], that is

$$\xi_q(\sigma) \geq 0, \quad \sigma \in (0, \infty) \quad \text{and} \quad \int_0^\infty \xi_q(\sigma)d\sigma = 1.$$

It is not difficult to verify that

$$\int_0^\infty \sigma\xi_q(\sigma)\,d\sigma = \frac{1}{\Gamma(1+q)}.$$

Remark 3.2. Note that $\{S(t)\}_{t\geq 0}$ is a uniformly bounded semigroup, i.e,

there exists a constant $M > 0$ such that $\|S(t)\| \leq M$ for all $t \in [0, T]$.

Remark 3.3. Note that

$$\|R(t)\| \leq C_{q,M}t^{q-1}, \quad t > 0, \tag{3.2}$$

where $C_{q,M} = \frac{qM}{\Gamma(1+q)}$.

Definition 3.4. The problem (1.1) is said to be controllable on the interval J if for every initial function $\phi \in \mathcal{B}$ and $x_1 \in X$ there exists a control $u \in L^2(J, X)$ such that the mild solution $x(\cdot)$ of (1.1) satisfies $x(T) = x_1$.

We impose the following assumptions:

(H1) The multifunction $F\colon J \times \mathcal{B} \times X \longrightarrow P_{cp,cv}(X)$ is Carathéodory.

(H2) There exists a function $\mu \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi \colon \mathbb{R}^+ \to (0, +\infty)$ such that

$$\begin{aligned} \|F(t,x,y)\| &= \sup\{\|v\| : v \in F(t,x,y)\} \\ &\le \mu(t)\psi\left(\|x\|_{\mathcal{B}} + \|y\|_X\right), \quad (t,x,y) \in J \times \mathcal{B} \times X, \end{aligned}$$

with

$$\omega_2 \int_0^T \mu(s)\, ds < \int_{v(0)}^{+\infty} \frac{du}{\psi(u)}, \tag{3.3}$$

where

$$\omega_2 = \beta_1 \left(M_1 M_2 a^2 C_{q,M}^2 \frac{T^{2q}}{q^2} + a C_{q,M} \frac{T^q}{q} \right),$$

$$v(0) = \omega_1 = \beta_2 + \beta_1 M_1 M_2 a C_{q,M} \frac{T^q}{q} \left[\|x_1\| + M \|\phi\|_{\mathcal{B}} \right],$$

and

$$\beta_1 = C_1^* + 1, \quad \beta_2 = C_2^* \|\phi\|_{\mathcal{B}}.$$

(H3) There exists a function $k \in L^1(J, \mathbb{R}_+)$ such that

$$H_d(F(t,x_1,y_1), F(t,x_2,y_2)) \le k(t) \left[\|x_1 - x_2\|_{\mathcal{B}} + \|y_1 - y_2\|_X \right].$$

(H4) For each $t \in J$, $a(t,s)$ is measurable on $[0,t]$ and $a(t) = \operatorname{ess\,sup}\{|a(t,s)|, 0 \le s \le t\}$ is bounded on J. The map $t \to a_t$ is continuous from J to $L^\infty(J, \mathbb{R})$, here, $a_t(s) = a(t,s)$.

(H5) The linear operator $W \colon L^2(J, X) \to X$ defined by

$$Wu = \int_0^T R(T-s) B u(s)\, ds.$$

has an inverse operator W^{-1}, which takes values in $L^2(J,X)/\ker W$ and there exist two positive constants M_1 and M_2 such that

$$\|B\|_{L(X)} \le M_1, \quad \|W^{-1}\|_{L(X)} \le M_2. \tag{3.4}$$

Theorem 3.5. *Assume that the hypotheses* $(H1)$–$(H5)$ *hold. Then the problem* (1.1) *is controllable on the interval* $(-\infty, T]$ *provided that*

$$M_1\ M_2\ a^2\ C_{q,M}^2 \frac{T^{2q}}{q^2} (C_1^* + 1) \|k\|_{L^1} < 1. \tag{3.5}$$

Proof. We transform the problem (1.1) into a fixed-point problem. Consider the multivalued operator $N \colon \Omega \longrightarrow \mathcal{P}(\Omega)$ defined by $N(h) = \{h \in \Omega\}$ with

$$h(t) = \begin{cases} \phi(t), & t \in \ (-\infty, 0]; \\ \\ -Q(t)\phi(0) + \int_0^t R(t-s) B u(s)\, ds & \\ + \int_0^t \int_0^s R(t-s) a(s,\tau) v(\tau)\, d\tau\, ds, & t \in J. \end{cases}$$

Using hypothesis $(H5)$ for an arbitrary function $x(\cdot)$ define the control

$$u(t) = W^{-1}\Big[x_1 + Q(t)\phi(0) - \int_0^T \int_0^s R(T-s) a(s,\tau) v(\tau)\, d\tau\, ds \Big](t). \tag{3.6}$$

Obviously, fixed points of the operator N are mild solutions of the problem (1.1). For $\phi \in \mathcal{B}$, we will define the function $y(\cdot)\colon (-\infty, T] \longrightarrow X$ by

$$y(t) = \begin{cases} \phi(t), & t \in \ (-\infty, 0]; \\ -Q(t)\phi(0), & t \in J. \end{cases}$$

Then $y_0 = \phi$. For each function $z \in C(J, X)$ with $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & t \in (-\infty, 0]; \\ z(t), & t \in J. \end{cases}$$

If $x(\cdot)$ verifies (3.1), we can decompose it as $x(t) = y(t) + \overline{z}(t)$, for $t \in J$, which implies $x_t = y_t + \overline{z}_t$, for every $t \in J$ and the function $z(t)$ satisfies

$$z(t) = \int_0^t R(t-s)Bu_{y+\overline{z}}(s)\,ds + \int_0^t \int_0^s R(t-s)a(s,\tau)v(\tau)\,d\tau\,ds,$$

where

$$v \in S_{F,y+\overline{z}} = \Big\{ v \in L^1(J,X) : v(t) \in F(t, y_t + \overline{z}_t, y(t) + \overline{z}(t)) \text{ for a.e. } t \in J \Big\}.$$

Let

$$Z_0 = \{z \in \Omega : z_0 = 0\}.$$

For any $z \in Z_0$, we have

$$\|z\|_{Z_0} = \sup_{t \in J} \|z(t)\| + \|z_0\|_{\mathcal{B}} = \sup_{t \in J} \|z(t)\|.$$

Thus $(Z_0, \|\cdot\|_{Z_0})$ is a Banach space. We define the operator $P\colon Z_0 \longrightarrow \mathcal{P}(Z_0)$ by $P(z) = \{h \in Z_0\}$ with

$$h(t) = \int_0^t R(t-s)Bu_{y+\overline{z}}(s)\,ds + \int_0^t \int_0^s R(t-s)a(s,\tau)v(\tau)\,d\tau\,ds, \quad v(s) \in S_{F,y+\overline{z}},\ t \in J.$$

Obviously the operator N having a fixed point is equivalent to P having one, so it turns to prove that P has a fixed point. Let $r > 0$ and consider the set

$$B_r = \{z \in Z_0 : \|z\|_{Z_0} \le r\}.$$

We need the following lemma.

Lemma 3.6. *Set*

$$C_i^* = \sup_{t \in J} C_i(t) \quad (i = 1, 2). \tag{3.7}$$

Then for any $z \in B_r$ we have

$$\|y_t + \overline{z}_t\|_{\mathcal{B}} \le C_2^* \|\phi\|_{\mathcal{B}} + C_1^* r,$$

and

$$\|u(s)\| \le M_2 \left[\|x_1\| + M\|\phi\|_{\mathcal{B}} + aC_{q,M} \int_0^T \int_0^\tau (t-\tau)^{q-1} \|v(\iota)\|\,d\iota\,d\tau \right]. \tag{3.8}$$

Proof. Using (2.2), (3.4), (3.6) and (3.7), we obtain

$$\begin{aligned}\|y_t+\overline{z}_t\|_{\mathcal{B}} &\le \|y_t\|_{\mathcal{B}}+\|\overline{z}_t\|_{\mathcal{B}}\\ &\le C_1(t)\sup_{0\le\tau\le t}\|y(\tau)\|+C_2(t)\|y_0\|_{\mathcal{B}}+C_1(t)\sup_{0\le\tau\le t}\|z(\tau)\|+C_2(t)\|z_0\|_{\mathcal{B}}\\ &\le C_2(t)\|\phi\|_{\mathcal{B}}+C_1(t)\sup_{0\le\tau\le t}\|z(\tau)\|\\ &\le C_2^*\|\phi\|_{\mathcal{B}}+C_1^*r.\end{aligned}$$

Also, we get

$$\begin{aligned}\|u(s)\| &\le \|W^{-1}\|\left[\|x_1\|+\|Q(t)\phi(0)\|+\int_0^T\int_0^\tau\|R(t-\tau)\|\|a(\tau,\iota)\|\|v(\iota)\|\,d\iota\,ds\right]\\ &\le M_2\Big[\|x_1\|+M\|\phi\|_{\mathcal{B}}+aC_{q,M}\int_0^T\int_0^\tau(t-\tau)^{q-1}\|v(\iota)\|\,d\iota\,d\tau\Big].\end{aligned}$$

The lemma is proved.

Now, we define the following multivalued operators $P_1, P_2 : Z_0 \longrightarrow \mathcal{P}(Z_0)$ as

$$P_1(z)=\left\{h\in Z_0 : h(t)=\int_0^t R(t-s)Bu_{y+\overline{z}}(s)\,ds,\ t\in J\right\}$$

and

$$P_2(z)=\left\{h\in Z_0 : h(t)=\int_0^t\int_0^s R(t-s)a(s,\tau)v(\tau)\,d\tau\,ds,\ v(s)\in S_{F,y+\overline{z}},\ t\in J\right\}.$$

It is clear that $P=P_1+P_2$. The problem of finding solutions of (1.1) is reduced to finding solutions of the operator inclusion $z\in P_1(z)+P_2(z)$. We shall show that the operators P_1 and P_2 satisfy all conditions of the Theorem 2.8. The proof will be given in several steps.

Step 1: P_1 is a contraction.

Let $z, z^*\in Z_0$ and $h\in P_1(z)$. Then, there exists $v(t)\in F(t,y_t+\overline{z}_t,\ y(t)+\overline{z}(t))$ such that

$$h(t)=\int_0^t R(t-s)Bu_{y+\overline{z}}(s)\,ds,\quad t\in J.$$

From $(H3)$, it follows that

$$\begin{aligned}&H_d\left(F(t,y_t+\overline{z}_t,y(t)+\overline{z}(t)),F(t,y_t+\overline{z}_t^*,y(t)+\overline{z}^*(t))\right)\\ &\qquad\le k(t)\left[\|\overline{z}_t-\overline{z}_t^*\|_{\mathcal{B}}+\|\overline{z}(t)-\overline{z}^*(t)\|_X\right].\end{aligned}$$

Hence there is $\omega\in F(t,y_t+\overline{z}_t^*,y(t)+\overline{z}^*(t))$ such that

$$|v(t)-\omega|\le k(t)\left[\|\overline{z}_t-\overline{z}_t^*\|_{\mathcal{B}}+\|\overline{z}(t)-\overline{z}^*(t)\|_X\right].$$

Consider $U\colon J\to\mathcal{P}(E)$ given by

$$u(t)=\{\omega\in E : |v(t)-\omega|\le k(t)\left[\|\overline{z}_t-\overline{z}_t^*\|_{\mathcal{B}}+\|\overline{z}(t)-\overline{z}^*(t)\|_X\right].$$

Since the multivalued operator $V(t)=U(t)\cap F(t,y_t+\overline{z}_t^*,y(t)+\overline{z}^*(t))$ is measurable (see Proposition 2.6), there exists a function $v_*(t)$, which is a measurable selection for V. So, $v_*(t)\in F(t,y_t+\overline{z}_t^*,y(t)+\overline{z}^*(t))$, and using **(A2)**, for each $t\in J$, we obtain

$$\begin{aligned}\|v(t)-v^*(t)\| &\le k(t)\left[\|\overline{z}_t-\overline{z}_t^*\|_{\mathcal{B}}+\|\overline{z}(t)-\overline{z}^*(t)\|_X\right]\\ &\le k(t)[C_1^*\|\overline{z}(t)-\overline{z}^*(t)\|+\|\overline{z}(t)-\overline{z}^*(t)\|]\\ &\le k(t)(C_1^*+1)\|\overline{z}(t)-\overline{z}^*(t)\|\\ &\le k(t)(C_1^*+1)\|z(t)-z^*(t)\|.\end{aligned}$$

Let us define for each $t \in J$

$$h^*(t) = \int_0^t R(t-s)Bu_{y+\bar{z}^*}(s)ds.$$

Then we have

$$\begin{aligned}
\|h(t) - h^*(t)\| &\leq \int_0^t \|R(t-s)\| \|Bu_{y+\bar{z}}(s) - Bu_{y+\bar{z}^*}(s)\| \, ds \\
&\leq M_1 \; a \; C_{q,M} \int_0^t (t-s)^{q-1} \|u_{y+\bar{z}}(s) - u_{y+\bar{z}^*}(s)\| \, ds \\
&\leq M_1 \; a \; C_{q,M} \int_0^t (t-s)^{q-1} \Big\| W^{-1} \Big[x_1 + Q(t)\phi(0) \\
&\quad - \int_0^T \int_0^\tau R(t-\tau)a(\tau,\iota)v(\iota)\| \, d\iota \, d\tau \Big] \\
&\quad - W^{-1} \Big[x_1 + Q(t)\phi(0) - \int_0^T \int_0^\tau R(t-\tau)a(\tau,\iota)v^*(\iota)\| \, d\iota \, d\tau \Big] \Big\| \, ds \\
&\leq M_1 \; M_2 \; a^2 \; C_{q,M}^2 \int_0^t (t-s)^{q-1} \int_0^T \int_0^\tau (t-\tau)^{q-1} \|v(\iota) - v^*(\iota)\| \, d\iota \, d\tau \, ds \\
&\leq M_1 \; M_2 \; a^2 \; C_{q,M}^2 \int_0^t (t-s)^{q-1} \\
&\quad \times \int_0^T \int_0^\tau (t-\tau)^{q-1} k(\iota)(C_1^*+1) \|z(\iota) - z^*(\iota)\| \, d\iota \, d\tau \, ds \\
&\leq M_1 \; M_2 \; a^2 \; C_{q,M}^2 \frac{T^{2q}}{q^2} (C_1^*+1) \|k\|_{L^1} \|z - z^*\|.
\end{aligned}$$

By an analogous relation, obtained by interchanging the roles of z and z^*, it follows that

$$H_d(P_1(z), P_1(z^*)) \leq M_1 \; M_2 \; a^2 \; C_{q,M}^2 \frac{T^{2q}}{q^2} (C_1^*+1) \|k\|_{L^1} \|z - z^*\|.$$

By (3.5), the mapping P_1 is a contraction.

Step 2: P_2 has compact, convex values, and it is completely continuous. This will be given in several claims.

Claim 1: P_2 is convex for each $z \in Z_0$.

Indeed, if h_1 and h_2 belong to P_2, then there exist $v_1, v_2 \in S_{F,y+\bar{z}}$ such that, for $t \in J$, we have

$$h_i(t) = \int_0^t \int_0^s R(t-s)a(s,\tau)v_i(\tau) \, d\tau \, ds, \quad i = 1, 2.$$

Let $d \in [0,1]$. Then for each $t \in J$, we have

$$dh_1(t) + (1-d)h_2(t) = \int_0^t \int_0^s R(t-s)a(s,\tau) \left[dv_1(\tau) + (1-d)v_2(\tau) \right] \, d\tau \, ds.$$

Since $S_{F,y+\bar{z}}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in P_2.$$

Claim 2: P_2 maps bounded sets into bounded sets in Z_0.

Indeed, it is enough to show that for any $r > 0$, there exists a positive constant ℓ such that for each $z \in B_r = \{z \in Z_0 : \|z\|_{Z_0} \leq r\}$, we have $\|P_2(z)\|_{Z_0} \leq \ell$. Then for each $h \in P_2(z)$, there exists $v \in S_{F,y+\bar{z}}$ such that

$$h(t) = \int_0^t \int_0^s R(t-s)a(s,\tau)v(\tau) \, d\tau \, ds.$$

Using (H2) and Lemma 3.6 we have for each $t \in J$,

$$\begin{aligned}
\|h(t)\| &\leq \int_0^t \int_0^s \|R(t-s)a(s,\tau)v(\tau)\|\, d\tau\, ds \\
&\leq a\, C_{q,M} \int_0^t \int_0^s (t-s)^{q-1} \left[\mu(\tau)\psi\left(\|y_\tau + \overline{z}_\tau\| + \|y(\tau)+\overline{z}(\tau)\|\right)\right] d\tau\, ds \\
&\leq a\, C_{q,M} \int_0^t \int_0^s (t-s)^{q-1} \left[\mu(\tau)\psi\left(C_2^*\|\phi\|_{\mathcal{B}} + C_1^* r + r\right)\right] d\tau\, ds \\
&\leq a\, C_{q,M} \int_0^t \int_0^s (t-s)^{q-1} \left[\mu(\tau)\psi\left(C_2^*\|\phi\|_{\mathcal{B}} + (C_1^*+1) r\right)\right] d\tau\, ds \\
&\leq \frac{T^q a\, C_{q,M}}{q} \psi\left(C_2^*\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right) \int_0^T \mu(\tau)\, d\tau \\
&\leq \ell.
\end{aligned}$$

Hence $P_2(B_r)$ is bounded.
Claim 3: P_2 maps bounded sets into equicontinuous sets of Z_0.
Let $h \in P_2(z)$ for $z \in Z_0$ and let $\tau_1, \tau_2 \in [0,T]$, with $\tau_1 < \tau_2$, we have

$$\begin{aligned}
\|h(\tau_2) - h(\tau_1)\| &\leq \left\| \int_0^{\tau_2} \int_0^s [R(\tau_1 - s) - R(\tau_2 - s)] a(s,\tau) v(\tau)\, d\tau\, ds \right\| \\
&\quad + \int_{\tau_2}^{\tau_1} \int_0^s \|R(\tau_1 - s)\| \|a(s,\tau)\| \|v(\tau)\|\, d\tau\, ds \\
&\leq I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \left\| \int_0^{\tau_2} \int_0^s [R(\tau_1 - s) - R(\tau_2 - s)] a(s,\tau) v(\tau)\, d\tau\, ds \right\| \\
I_2 &= \int_{\tau_2}^{\tau_1} \int_0^s \|R(\tau_1 - s)\| \|a(s,\tau)\| \|v(\tau)\|\, d\tau\, ds.
\end{aligned}$$

For I_1, using (3.2) and (H2), we have

$$\begin{aligned}
I_1 &\leq a \int_0^{\tau_2} \int_0^s \|R(\tau_1 - s) - R(\tau_2 - s)\| \|v(\tau)\|\, d\tau\, ds \\
&\leq a\psi\left(C_2^*\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right) \|\mu\|_{L^1} \int_0^{\tau_2} \|R(\tau_1 - s) - R(\tau_2 - s)\|\, ds \\
&\leq a\psi\left(C_2^*\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right) \|\mu\|_{L^1} \\
&\quad \times [q \int_0^{\tau_2} \int_0^\infty \sigma \|[(\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1}] \xi_q(\sigma) S((\tau_1 - s)^q \sigma)\|\, d\sigma\, ds \\
&\quad + q \int_0^{\tau_2} \int_0^\infty \sigma (\tau_2 - s)^{q-1} \xi_q(\sigma) \|S((\tau_1 - s)^q\sigma) - S((\tau_2 - s)^q \sigma)\|\, d\sigma\, ds] \\
&\leq a\psi\left(C_2^*\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right) \|\mu\|_{L^1} \times [C_{q,M} \int_0^{\tau_2} \left|(\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1}\right| ds \\
&\quad + q \int_0^{\tau_2} \int_0^\infty \sigma (\tau_2 - s)^{q-1} \xi_q(\sigma) \|S((\tau_1 - s)^q\sigma) - S((\tau_2 - s)^q \sigma)\|\, d\sigma\, ds].
\end{aligned}$$

Clearly, the first term on the right-hand side of the above inequality tends to zero as $\tau_2 \to \tau_1$. From the continuity of $S(t)$ in the uniform operator topology for $t > 0$, the second term on the right-hand side of the above inequality tends to zero as $\tau_2 \to \tau_1$.

In view of (3.2), we have

$$\begin{aligned} I_2 &\le a\psi\left(C_2^*\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right)\|\mu\|_{L^1}\int_{\tau_2}^{\tau_1}\|R(\tau_1-s)\|\,ds \\ &\le aC_{q,M}\psi\left(C_2^*\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right)\|\mu\|_{L^1}\int_{\tau_2}^{\tau_1}(\tau_1-s)^{q-1}\,ds. \end{aligned}$$

As $\tau_2 \to \tau_1$, I_2 tends to zero.

So $P_2(B_r)$ is equicontinuous.

Claim 4: $(P_2B_r)(t)$ is relatively compact for each $t \in J$, where

$$(P_2B_r)(t) = \{h(t) : h \in P_2(B_r)\}.$$

Let $0 < t \le T$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For arbitrary $\delta > 0$, we define

$$\begin{aligned} h_{\varepsilon,\delta}(t) &= q\int_0^{t-\varepsilon}(t-s)^{q-1}\int_\delta^\infty \sigma\xi_q(\sigma)S((t-s)^q\sigma)\int_0^s a(s,\tau)v(\tau)\,d\tau\,d\sigma\,ds \\ &= qS(\varepsilon^q\delta)\int_0^{t-\varepsilon}(t-s)^{q-1}\int_\delta^\infty \sigma\xi_q(\sigma)S((t-s)^q\sigma-\varepsilon^q\delta)\int_0^s a(s,\tau)v(\tau)\,d\tau\,d\sigma\,ds, \end{aligned}$$

where $v \in S_{F,y+\bar{z}}$. Since $S(t)$ is a compact operator, the set

$$H_{\varepsilon,\delta} = \{h_{\varepsilon,\delta}(t) : h \in P_2(B_r)\}$$

is relatively compact. Moreover,

$$\begin{aligned} &\|h(t) - h_{\varepsilon,\delta}(t)\| \\ &\quad\le q\int_0^{t-\varepsilon}(t-s)^{q-1}\int_0^\delta \sigma\xi_q(\sigma)\|S((t-s)^q\sigma)\|\int_0^s\|a(s,\tau)\|\|v(\tau)\|\,d\tau\,d\sigma\,ds \\ &\qquad+q\int_{t-\varepsilon}^t(t-s)^{q-1}\int_0^\infty \sigma\xi_q(\sigma)\|S((t-s)^q\sigma)\|\int_0^s\|a(s,\tau)\|\|v(\tau)\|\,d\tau\,d\sigma\,ds \\ &\quad\le T^qMa\psi\left(C_2^*\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right)\|\mu\|_{L^1}\int_0^\delta\sigma\xi_q(\sigma)\,d\sigma \\ &\qquad+\frac{\varepsilon^qMa}{\Gamma(1+q)}\psi\left(C_2^*\|\phi\|_{\mathcal{B}} + (C_1^*+1)r\right)\|\mu\|_{L^1}. \end{aligned}$$

Therefore, $(P_2B_r)(t)$ is relatively compact.

As a consequence of Claim 2 to 4 together with the Arzelà–Ascoli theorem we can conclude that P_2 is completely continuous.

Claim 5: P_2 has a closed graph.

Let $z_n \to z_*, h_n \in P_2(z_n)$, and $h_n \to h_*$. We shall show that $h_* \in P_2(z_*)$. $h_n \in P_2(z_n)$ means that there exists $v_n \in S_{F,y_n+\bar{z}_n}$ such that

$$h_n(t) = \int_0^t\int_0^s R(t-s)a(s,\tau)v_n(\tau)\,d\tau\,ds, \quad t \in J.$$

We have to prove that there exists $v_* \in S_{F,y_*+\bar{z}_*}$ such that

$$h_*(t) = \int_0^t\int_0^s R(t-s)a(s,\tau)v_*(\tau)\,d\tau\,ds, \quad t \in J.$$

Consider the linear and continuous operator $\Upsilon\colon L^1(J,X) \longrightarrow C(J,X)$ defined by

$$(\Upsilon v)(t) = \int_0^t \int_0^s R(t-s)a(s,\tau)v(s)\,d\tau\,ds.$$

From Lemma 2.5 it follows that $\Upsilon \circ S_F$ is a closed graph operator and from the definition of Υ one has

$$h_n(t) \in \Upsilon(S_{F,y_n+\overline{z}_n}).$$

As $z_n \to z_*$ and $h_n \to h_*$, there is a $v_* \in S_{F,y_*+\overline{z}_*}$ such that

$$h_*(t) = \int_0^t \int_0^s R(t-s)a(s,\tau)v(s)\,d\tau\,ds.$$

Hence the multivalued operator P_2 is upper semi-continuous.
Claim 6: A priori bounds.
Now it remains to show that the set

$$\mathcal{E} = \{z \in Z_0 : z \in \lambda P_1(z) + \lambda P_2(z), \text{ for some } 0 < \lambda < 1\}$$

is bounded.
Let $z \in \mathcal{E}$ be any element, then there exists $v \in S_{F,y+\overline{z}}$ such that

$$z(t) = \lambda \int_0^t R(t-s)Bu(s)\,ds + \lambda \int_0^t \int_0^s R(t-s)a(s,\tau)v(s)\,d\tau\,ds \quad \text{for some } 0 < \lambda < 1.$$

Thus, by (3.8), (H2) and Lemma 3.6, for each $t \in J$ we have

$$\begin{aligned}
\|z(t)\| &\le \int_0^t \|R(t-s)\|\|Bu(s)\|ds + \int_0^t \int_0^s \|R(t-s)\|\|a(s,\tau)\|\|v(s)\|\,d\tau\,ds \\
&\le M_1M_2aC_{q,M}\int_0^t (t-s)^{q-1}\Big[\|x_1\| + M\|\phi\|_{\mathcal{B}} \\
&\quad + aC_{q,M}\int_0^T\int_0^\tau (t-\tau)^{q-1}\|v(\iota)\|\,d\iota\,d\tau\Big]\,ds + aC_{q,M}\int_0^t (t-s)^{q-1}\int_0^s \|v(\tau)\|\,d\tau\,ds \\
&\le M_1M_2aC_{q,M}\frac{T^q}{q}\Big[\|x_1\| + M\|\phi\|_{\mathcal{B}}\Big] \\
&\quad + M_1M_2a^2C_{q,M}^2\int_0^t (t-s)^{q-1}\int_0^T\int_0^\tau (t-\tau)^{q-1}\|v(\iota)\|\,d\iota\,d\tau\,ds \\
&\quad + aC_{q,M}\int_0^t (t-s)^{q-1}\int_0^s \|v(\tau)\|\,d\tau\,ds \\
&\le M_1M_2aC_{q,M}\frac{T^q}{q}\Big[\|x_1\| + M\|\phi\|_{\mathcal{B}}\Big] + M_1M_2a^2C_{q,M}^2 \\
&\quad \times \int_0^t (t-s)^{q-1}\int_0^T\int_0^\tau (t-\tau)^{q-1}\left[\mu(\iota)\psi(\|y_\iota + \overline{z}_\iota\| + \|y(\iota)+\overline{z}(\iota)\|)\right]d\iota\,d\tau\,ds \\
&\quad + aC_{q,M}\int_0^t (t-s)^{q-1}\int_0^s \left[\mu(\tau)\psi(\|y_\tau + \overline{z}_\tau\| + \|y(\tau)+\overline{z}(\tau)\|)\right]d\tau\,ds \\
&\le M_1M_2aC_{q,M}\frac{T^q}{q}\Big[\|x_1\| + M\|\phi\|_{\mathcal{B}}\Big] + M_1M_2a^2C_{q,M}^2 \\
&\quad \times \int_0^t (t-s)^{q-1}\int_0^T\int_0^\tau (t-\tau)^{q-1}\left[\mu(\iota)\psi(C_2^*\|\phi\|_{\mathcal{B}} + (C_1^*+1)\|z(\iota)\|)\right]d\iota\,d\tau\,ds \\
&\quad + aC_{q,M}\int_0^t (t-s)^{q-1}\int_0^s \left[\mu(\tau)\psi(C_2^*\|\phi\|_{\mathcal{B}} + (C_1^*+1)\|z(\tau)\|)\right]d\tau\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq M_1 M_2 a C_{q,M} \frac{T^q}{q} \Big[\|x_1\| + M\|\phi\|_{\mathcal{B}} \Big] \\
&\quad + M_1 M_2 a^2 C_{q,M}^2 \frac{T^{2q}}{q^2} \int_0^t [\mu(s)\psi(C_2^*\|\phi\|_B + (C_1^*+1)\|z(s)\|)]\, ds \\
&\quad + a C_{q,M} \frac{T^q}{q} \int_0^t [\mu(s)\psi(C_2^*\|\phi\|_B + (C_1^*+1)\|z(s)\|)]\, ds \\
&\leq M_1 M_2 a C_{q,M} \frac{T^q}{q} \Big[\|x_1\| + M\|\phi\|_{\mathcal{B}} \Big] \\
&\quad + \left(M_1 M_2 a^2 C_{q,M}^2 \frac{T^{2q}}{q^2} + a C_{q,M} \frac{T^q}{q} \right) \int_0^t [\mu(s)\psi(\beta_2 + \beta_1\|z(s)\|)]\, ds
\end{aligned}
$$

Then

$$
\begin{aligned}
\beta_2 + \beta_1\|z(t)\| &\leq \beta_2 + \beta_1 M_1 M_2 a C_{q,M} \frac{T^q}{q} \Big[\|x_1\| + M\|\phi\|_{\mathcal{B}} \Big] \\
&\quad + \beta_1 \left(M_1 M_2 a^2 C_{q,M}^2 \frac{T^{2q}}{q^2} + a C_{q,M} \frac{T^q}{q} \right) \int_0^t [\mu(s)\psi(\beta_2 + \beta_1\|z(s)\|)]\, ds \\
&\leq \omega_1 + \omega_2 \int_0^t [\mu(s)\psi(\beta_2 + \beta_1\|z(s)\|)]\, ds.
\end{aligned}
$$

Let

$$m(t) := \sup\left\{ \beta_2 + \beta_1\|z(s)\| \ : \ 0 \leq s \leq t \right\}, \quad t \in J.$$

By the previous inequality, we have

$$m(t) \leq \omega_1 + \omega_2 \int_0^t [\mu(s)\psi(m(s))]\, ds.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$m(t) \leq v(t) \quad \text{for all } t \in J,$$

with

$$v(0) = \omega_1,$$

and

$$v'(t) = \omega_2 \mu(t)\psi(m(t)), \quad \text{a.e. } t \in J.$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq \omega_2 \mu(t)\psi(v(t)), \quad \text{a.e. } t \in J.$$

Integrating from 0 to t we get

$$\int_0^t \frac{v'(s)}{\psi(v(s))}\, ds \leq \omega_2 \int_0^t \mu(s)\, ds.$$

By a change of variable we get

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq \omega_2 \int_0^t \mu(s)\, ds.$$

Using the condition (3.3), this implies that for each $t \in J$, we have

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq \omega_2 \int_0^t \mu(s)\, ds \leq \omega_2 \int_0^T \mu(s) ds < \int_{v(0)}^{+\infty} \frac{du}{\psi(u)}.$$

Thus, for every $t \in J$, there exists a constant Λ such that $v(t) \leq \Lambda$ and hence $m(t) \leq \Lambda$. Since $\|z\|_{Z_0} \leq m(t)$, we have $\|z\|_{Z_0} \leq \Lambda$.

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 2.8 we deduce that $P_1 + P_2$ has a fixed point z defined on the interval $(-\infty, T]$ which is the solution of problem (1.1). This completes the proof.

4 An example

Consider the following integro-differential equation with fractional derivative of the form

$$\begin{aligned} \frac{\partial^q}{\partial t^q} v(t,\zeta) \in \bigg(& \frac{\partial^2}{\partial \zeta^2} v(t,\zeta) + \mu(t,\zeta) \\ & + \int_0^t (t-s)^2 \int_{-\infty}^0 G(t, v(t+\theta,\zeta))\eta(t,\theta,\zeta)\, d\theta\, ds \bigg), && t \in [0,1],\ \zeta \in [0,\pi]; \quad (4.1) \\ v(t,0) = v(t,\pi) = 0, & && t \in [0,1]; \\ v(\theta,\zeta) = \varphi(\theta,\zeta), & && \theta \in (-\infty,0],\ \zeta \in [0,\pi], \end{aligned}$$

where $0 < q < 1, \mu \colon [0,1] \times [0,\pi] \to [0,\pi]$, and $G \colon [0,1] \times \mathbb{R} \to P(\mathbb{R})$ is an u.s.c. multivalued map with compact convex values.

Set $X = L^2([0,\pi])$ and define A by

$$D(A) = \{u \in X : u'' \in X, u(0) = u(\pi) = 0\},$$
$$Au = u''.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $(S(t))_{t\geq 0}$ on X [43]. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n \in \mathbb{N}$, and the corresponding normalized eigenfunctions are given by

$$u_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx).$$

In addition, $\{u_n : n \in \mathbb{N}\}$ is an orthogonal basis for X,

$$S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}(u, u_n)u_n, \quad \text{for all } u \in X \text{ and every } t \geq 0.$$

From these expressions it follows that $(S(t))_{t\geq 0}$ is uniformly bounded compact semigroup.

For the phase space, we choose $\mathcal{B} = \mathcal{B}_\gamma$ defined by

$$\mathcal{B}_\gamma := \left\{ \phi \in C((-\infty,0], X) : \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } X \right\}$$

endowed with the norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \theta \leq 0\}.$$

Notice that the phase space $\mathcal{B}_\gamma$ satisfies axioms **(A1)–(A3)**.

For $t \in [0,1], \zeta \in [0,\pi]$ and $\varphi \in \mathcal{B}_\gamma$, we set

$$\begin{aligned} x(t)(\zeta) &= v(t,\zeta), \\ a(t,s) &= (t-s)^2, \\ F(t,\varphi,x(t))(\zeta) &= \int_{-\infty}^{0} G(t,\varphi(\theta)(\zeta))\eta(t,\theta,\zeta)\,d\theta, \\ Bu(t)(\zeta) &= \mu(t,\zeta). \end{aligned}$$

With the above choices, we see that the system (4.1) is the abstract formulation of (1.1). Assume that the operator $W\colon L^2([0,1],X) \to X$ defined by

$$Wu(\cdot) = \int_0^1 R(1-s)\mu(s,\cdot)\,ds,$$

has a bounded invertible operator W^{-1} in $L^2([0,1],X)/\ker W$.

Thus all the conditions of Theorem 3.5 are satisfied. Hence, system (4.1) is controllable on $(-\infty,T]$.

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References

[1] S. Abbas, M. Benchohra, G. M. N'Guérékata, *Topics in fractional differential equations*, Springer, New York, 2012. MR2962045

[2] S. Abbas, M. Benchohra, G. M. N'Guérékata, *Advanced fractional differential and integral equations*, Nova Science Publishers, New York, 2014.

[3] R. P. Agarwal, M. Belmekki, M. Benchohra, A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative, *Adv. Differ. Equat.* **2009**, Art. ID 981728, 47 pp. MR2505633; url

[4] R. P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Appl. Math.* **109**(2010), 973–1033. MR2596185; url

[5] K. Aissani, M. Benchohra, Semilinear fractional order integro-differential equations with infinite delay in Banach spaces, *Arch. Math.* **49**(2013), 105–117. MR3118867; url

[6] K. Aissani, M. Benchohra, Controllability of impulsive fractional differential equations with infinite delay, *Lib. Math. (N.S.)* **33**(2013), 47–64. MR3177494

[7] K. Balachandran, J. P. Dauer, Controllability of nonlinear systems in Banach spaces: a survey, *J. Optim. Theory Appl.* **115**(2002), 7–28. MR1937343

[8] K. Balachandran, J. Y. Park, Controllability of fractional integrodifferential systems in Banach spaces, *Nonlinear Anal. Hybrid Syst.* **3**(2009), 363–367. MR2561653; url

[9] M. Benchohra, K. Ezzinbi, S. Litimein, The existence and controllability results for fractional order integro-differential inclusions in Fréchet spaces, *Proc. A. Razmadze Math. Inst.* **162**(2013), 1–23.

[10] M. Benchohra, J. Henderson, S. Ntouyas, A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, *J. Math. Anal. Appl.* **338**(2008), 1340–1350. MR2386501; url

[11] M. Benchohra, J. Henderson, S. K. Ntouyas, A. Ouahab, Existence results for fractional functional differential inclusions with infinite delay and application to control theory, *Fract. Calc. Appl. Anal*. **11**(2008), 35–56. MR2379272

[12] M. Benchohra, A. Ouahab, Controllability results for functional semilinear differential inclusions in Fréchet spaces, *Nonlinear Anal.* **61**(2005), 405–423. MR2123084; url

[13] C. Castaing, M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Mathematics, Vol. 580, Springer-Verlag, Berlin-New York, 1977. MR0467310

[14] A. Cernea, On a fractional differential inclusion with boundary condition, *Studia Univ. Babes Bolyai Math.* **55**(2010), 105–113. MR2764254

[15] Y. K. Chang, J. J. Nieto, W. S. Li, Controllability of semilinear differential systems with nonlocal initial conditions in Banach spaces, *J. Optim. Theory Appl*. **142**(2009), 267–273. MR2525790; url

[16] Y. Q. Chen, H. S. Ahu, D. Xue, Robust controllability of interval fractional order linear time invariant systems, *Signal Proces.* **86**(2006), 2794–2802. url

[17] M. A. Darwish, J. Henderson, S. K. Ntouyas, Fractional order semilinear mixed type functional differential equations and inclusions, *Nonlinear Stud.* **16**(2009), 1–22. MR2527181

[18] M. A. Darwish, S. K. Ntouyas, Boundary value problems for fractional functional differential equations of mixed type, *Commun. Appl. Anal.* **13**(2009), 31–38. MR2514985

[19] M. A. Darwish, S. K. Ntouyas, Semilinear functional differential equations of fractional order with state-dependent delay, *Electron. J. Differential Equations* **2009**, No. 38., 1–10. MR2495843

[20] M. A. Darwish, S. K. Ntouyas, On initial and boundary value problems for fractional order mixed type functional differential inclusions, *Comput. Math. Appl.* **59**(2010), 1253–1265. MR2579487; url

[21] L. Debnath, D. Bhatta, *Integral transforms and their applications*, 2nd edition, CRC Press, 2007. MR2253985

[22] K. Deimling, *Multivalued differential equations*, Walter De Gruyter, Berlin–New York, 1992. MR1189795

[23] B. C. Dhage, Fixed-point theorems for discontinuous multivalued operators on ordered spaces with applications, *Comput. Math. Appl.* **51**(2006), 589–604. MR2207444; url

[24] M. El-Borai, Some probability densities and fundamental solutions of fractional evolution equations, *Chaos Solitons Fractals* **14**(2002), 433–440. MR1903295; url

[25] A. M. A. El-Sayed, A. G. Ibrahim, Multivalued fractional differential equations, *Appl. Math. Comput.* **68**(1995), 15–25. MR1320806; url

[26] L. Gaul, P. Klein, S. Kemple, Damping description involving fractional operators, *Mech. Syst. Signal Process.* **5**(1991), 81–88. url

[27] W. G. Glockle, T. F. Nonnenmacher, A fractional calculus approach of self-similar protein dynamics, *Biophys. J.* **68**(1995), 46–53.

[28] L. Górniewicz, *Topological fixed point theory of multivalued mappings*, Mathematics and its Applications, Vol. 495, Kluwer Academic Publishers, Dordrecht, 1999. MR1748378

[29] J. K. Hale, J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.* **21**(1978), 11–41. MR0492721

[30] Sh. Hu, N. Papageorgiou, *Handbook of multivalued analysis, Volume I: Theory*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1997. MR1485775

[31] A. A. Kilbas, Hari M. Srivastava, Juan J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science B.V., Amsterdam, 2006. MR2218073

[32] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of fractional dynamic systems*, Cambridge Scientific Publishers, 2009.

[33] A. Lasota, Z. Opial, An application of the Kakutani–Ky Fan theorem in the theory of ordinary differential equations, *Bull. Acad. Pol. Sci. Sér. Sci. Math. Astronom. Phys.* **13**(1965), 781–786. MR0196178

[34] F. Li, J. Zhang, Existence of mild solutions to fractional integrodifferential equations of neutral type with infinite delay, *Adv. Difference Equ.* **2011**, Art. ID 963463, 15 pp. MR2774252; url

[35] F. Mainardi, Fractional calculus: Some basic problems in continuum and statistical mechanics, in: *A. Carpinteri, F. Mainardi (eds.), Fractals and fractional calculus in continuum mechanics*, Springer-Verlag, Wien, 1997, 291–348. MR1611582; url

[36] F. Mainardi, P. Paradisi, R. Gorenflo, Probability distributions generated by fractional diffusion equations, in: *J. Kertész, I. Kondor (eds.), Econophysics: an emerging science*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[37] F. Metzler, W. Schick, H. G. Kilian, T. F. Nonnenmache, Relaxation in filled polymers: A fractional calculus approach, *J. Chem. Phys.* **103**(1995), 7180–7186. url

[38] K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley, New York, 1993. MR1219954

[39] A. Ouahab, Some results for fractional boundary value problem of differential inclusions, *Nonlinear Anal.* **69**(2008), 3877–3896. MR2463341; url

[40] I. Podlubny, *Fractional differential equations*, Academic press, New York, NY, USA, 1993.

[41] G. M. Mophou, O. Nakoulima, G. M. N'Guérékata, Existence results for some fractional differential equations with nonlocal conditions, *Nonlinear Stud.* **17**(2010), 15–22. MR2647799

[42] G. M. Mophou, G. M. N'Guérékata, Existence of mild solution for some fractional differential equations with nonlocal conditions, *Semigroup Forum* **79**(2009), 315–322. url

[43] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983. MR0710486

[44] A. B. Shamardan, M. R. A. Moubarak, Controllability and observability for fractional control systems, *J. Fract. Calc.* **15**(1999), 25–34. MR1690077

[45] J. Wang, Y. Zhou, Existence and controllability results for fractional semilinear differential inclusions, *Nonlinear Anal. Real World Appl.* **12**(2011), 3642–3653. MR2832998; url

[46] Y. Zhou, *Basic theory of fractional differential equations*, World Scientific, Singapore, 2014.