




Note on fractional difference Gronwall inequalities

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Abstract. This note is a reaction on a bunch of fractional inequalities that appeared in the last few years, and that are all based on what claims to be a fractional discrete Gronwall inequality. However, we show by a counterexample that this inequality is not correct. Stimulated by this, the main aim of this note is to propose new inequalities and illustrate the results on examples. Asymptotic properties of a solution of a linear equation are studied as well. Moreover, a brief discussion of other related results is given.

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1 Introduction

Recently, Deekshitulu and Mohan published a set of papers on fractional difference inequalities. All the main results in [12, 13, 15] are proved using a fractional discrete version of Gronwall inequality given in [11]. In this short note, we show that the proof of this inequality is not correct, the inequality does not hold, and hence the validity of the implied results is questionable. Stimulated by these results, the main aim of this paper is to prove new inequalities of Gronwall type for fractional difference inequalities with linear right-hand side and constant or variable coefficients. Asymptotic properties of a solution of a linear homogeneous fractional difference equation with constant coefficient are also studied. In our results, we use a convenient Green function. In Section 4, we illustrate our results on a linear example from [14] which is also corrected in this note.

Throughout the present paper, ∇ denotes the backward difference operator defined as $\nabla u(n) = u(n) - u(n - 1)$, and having the next properties.

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Lemma 1.1. Let $\nabla_k F(k, j) := F(k, j) - F(k - 1, j)$. The following holds true

$$\begin{aligned} \nabla \left[\sum_{j=a}^k F(k, j) \right] &= F(k - 1, k) + \sum_{j=a}^k \nabla_k F(k, j), \\ \sum_{j=1}^k A(k - j) \nabla f(j) &= \sum_{j=0}^{k-1} A(j) \nabla_k f(k - j), \\ \sum_{j=a}^b f(j) \nabla g(j) &= [f(j)g(j)]_{j=a-1}^b - \sum_{j=a-1}^{b-1} \nabla f(j+1)g(j). \end{aligned}$$

So we will consider fractional differences corresponding to backward difference operator. Of course, other related achievements are already done than the above-mentioned: we refer the reader to similar interesting results in [1, 8, 16]. We only note here that in [1], inequality of the form

$$\nabla_*^{\mu} u(k) \leq a(k)u(k), \quad k \in \mathbb{N}_0$$

was investigated (cf. equation (4.1)); in [8], the authors used Riemann–Liouville type fractional difference; and in [16], the author focused on fractional difference corresponding to forward difference operator. Hence our results are not covered in these papers.

We denote by $\mathbb{N}_a = \{a, a + 1, \dots\}$ the shifted set of positive integers, for simplicity $\mathbb{N} = \mathbb{N}_1$, and $-\mathbb{N}_a = \{\dots, a - 1, a\}$. We assume the property of empty sum and empty product, i.e., $\sum_{j=a}^b f(j) = 0$, $\prod_{j=a}^b f(j) = 1$ whenever $a, b \in \mathbb{Z}$ are such that $a > b$.

2 Caputo like fractional difference

In this section, we recall some definitions of ∇ -based fractional operators, and we show that fractional difference considered in [9] is of Caputo type.

Definition 2.1 (see [18]). Let $\alpha \in \mathbb{C}$, $p = \max\{0, p_0\}$, $p_0 \in \mathbb{Z}$ be such that $0 < \operatorname{Re}(\alpha + p_0) \leq 1$, and function f be defined on \mathbb{N}_{a-p} . We define the α -th fractional sum as

$$\Sigma^{\alpha} f(k) := \frac{\nabla^p}{\Gamma(p + \alpha)} \sum_{j=a}^k (k - \rho(j))^{\overline{p+\alpha-1}} f(j) \quad (2.1)$$

for $k \in \mathbb{N}_a$, where $\rho(j) = j - 1$ and

$$(\alpha)^{\bar{\beta}} = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}, & \alpha, \alpha + \beta \notin \{\dots, -1, 0\}, \\ 1, & \alpha = \beta = 0, \\ 0, & \alpha = 0, \beta \notin \{\dots, -1, 0\}, \\ \text{undefined,} & \text{otherwise} \end{cases}$$

for $\alpha, \beta \in \mathbb{C}$.

If $p \in \mathbb{N}$, $\alpha \notin -\mathbb{N}_0$, $k \in \mathbb{N}_a$ and f is defined on \mathbb{N}_{a-p} , formula (2.1) can be simplified to the case $p = 0$ (cf. equation (2.2) in [18]) using

$$\frac{\nabla^p}{\Gamma(p + \alpha)} \sum_{j=a}^k (k - \rho(j))^{\overline{p+\alpha-1}} f(j) = \frac{1}{\Gamma(\alpha)} \sum_{j=a}^k (k - \rho(j))^{\overline{\alpha-1}} f(j), \quad (2.2)$$

i.e.,

$$\Sigma^\alpha f(k) = \frac{1}{\Gamma(\alpha)} \sum_{j=a}^k (k - \rho(j))^{\overline{\alpha-1}} f(j) \quad (2.3)$$

for $k \in \mathbb{N}_a$, f defined on \mathbb{N}_{a-p} . We note that the latter equality is valid whenever $\alpha \in \mathbb{R} \setminus -\mathbb{N}_0$. Σ^α is sometimes denoted as $\nabla^{-\alpha}$.

Fractional sum (2.3) is a discrete analogue to Riemann–Liouville fractional integral (cf. [21]).

Analogically to forward difference based Riemann–Liouville [5] and Caputo [3, 4] like fractional difference, Deekshitulu and Mohan proposed in [9] the next definition of a ∇ -based fractional difference.

Definition 2.2. Let $\mu \in (0, 1)$ and f be defined on \mathbb{N}_0 . Then we define the μ -th fractional difference of a function f as

$$\nabla^\mu f(k) = \sum_{j=0}^{k-1} \binom{j-\mu}{j} \nabla_k f(k-j) \quad (2.4)$$

for $k \in \mathbb{N}$, where $\binom{b}{n}$ with $b \in \mathbb{R}$, $n \in \mathbb{Z}$ is a generalized binomial coefficient given by

$$\binom{b}{n} = \begin{cases} \frac{\Gamma(b+1)}{\Gamma(b-n+1)\Gamma(n+1)}, & n > 0, \\ 1, & n = 0, \\ 0, & n < 0. \end{cases}$$

Here we used the lower index k to denote the variable affected by operator ∇ .

From now on, we assume $\mu \in (0, 1)$. By [9, Remark 3.2], $\Sigma^{-\mu}u(n) = \nabla^\mu u(n)$ on \mathbb{N} if u is defined on \mathbb{N}_0 . However, this equivalence is obtained because of an incorrect application of operator ∇ . We provide a simple counterexample.

Example 2.3. Consider $u(k) = b^k$ for $k \in \mathbb{N}_0$ and $b > 0$. Then

$$[\nabla^\mu u(k)]_{k=1} = \left[\sum_{j=0}^{k-1} \binom{j-\mu}{j} \nabla_k u(k-j) \right]_{k=1} = \binom{-\mu}{0} [\nabla u(k)]_{k=1} = b - 1.$$

On the other side, $0 < 1 - \mu < 1$. Thus $p = 1$ in (2.1). Moreover, since u is defined on \mathbb{N}_0 , then $a = 1$. Therefore,

$$\begin{aligned} [\Sigma^{-\mu}u(k)]_{k=1} &= \left[\frac{\nabla}{\Gamma(1-\mu)} \sum_{j=1}^k (k - \rho(j))^{\overline{-\mu}} u(j) \right]_{k=1} \\ &= \left[\frac{1}{\Gamma(-\mu)} \sum_{j=1}^k (k - \rho(j))^{\overline{-\mu-1}} u(j) \right]_{k=1} \\ &= \left[\frac{1}{\Gamma(-\mu)} \sum_{j=1}^k \frac{\Gamma(k-j-\mu)}{\Gamma(k-j+1)} u(j) \right]_{k=1} = b. \end{aligned}$$

Here we applied the identity (2.2).

In reality, $\Sigma^{-\mu}$ is closely related to Riemann–Liouville like ∇ -based fractional difference discussed in [6], while the following lemma states that ∇^μ is of Caputo type.

Lemma 2.4. Let $\mu \in (0, 1)$, $\nu = 1 - \mu$ and function f be defined on \mathbb{N}_0 . Then $\nabla^\mu f(k) = \Sigma^\nu(\nabla f(k))$ on \mathbb{N} .

Proof. For $k \in \mathbb{N}$ we expand the left-hand side by (2.4), and apply Lemma 1.1 to get

$$\begin{aligned}\nabla^\mu f(k) &= \sum_{j=0}^{k-1} \binom{j-\mu}{j} \nabla_k f(k-j) = \frac{1}{\Gamma(\nu)} \sum_{j=0}^{k-1} \frac{\Gamma(j+\nu)}{\Gamma(j+1)} \nabla_k f(k-j) \\ &= \frac{1}{\Gamma(\nu)} \sum_{j=1}^k \frac{\Gamma(k-j+\nu)}{\Gamma(k-j+1)} \nabla f(j) = \frac{1}{\Gamma(\nu)} \sum_{j=1}^k (k-\rho(j))^{\overline{\nu-1}} \nabla f(j)\end{aligned}$$

which, by (2.1) with $a = 1$, $p = 0$, is exactly what has to be proved. \square

In the sense of the above lemma, we add the lower index $*$ as done in [3,4] to denote the Caputo nature of the difference, i.e., $\nabla_*^\mu := \nabla^\mu$ in the rest of the paper. Next, in [9, Remark 3.2] the properties of Σ^μ were translated to ∇_*^μ . The above discussion implies that this is also a mistake, and ∇_*^μ does not have to possess such properties. Nevertheless, we do not go into details, as we do not need the properties in the present paper.

3 Linear fractional difference equation

In this section, we derive a solution of a nonhomogeneous linear fractional difference equation in terms of a Green function. Particular case of a constant coefficient at linear term is investigated in details.

First, we provide a lemma transforming a fractional difference equation to a corresponding fractional sum equation and a direct corollary.

Lemma 3.1. Let $\mu \in (0, 1)$, u, f be real functions defined on \mathbb{N}_0 , and $\mathbb{N}_0 \times \mathbb{R}$, respectively. For any $n \in \mathbb{N}$, if

$$\nabla_*^\mu u(k+1) = f(k, u(k)), \quad \forall k = 0, 1, \dots, n-1, \quad (3.1)$$

then

$$u(k) = u(0) + \sum_{j=0}^{k-1} A_\mu(k-1, j) f(j, u(j)), \quad \forall k \in 0, 1, \dots, n \quad (3.2)$$

with $A_\mu(k, j) = \binom{k-j+\mu-1}{k-j}$ for $0 \leq j \leq k$.

Proof. We show that applying Σ^μ to equation (3.1) results in equation (3.2). Let $k \in \{0, 1, \dots, n-1\}$ be arbitrary and fixed. By Lemma 2.4, $\Sigma^\mu(\nabla_*^\mu u(k+1)) = \Sigma^\mu(\Sigma^\nu(\nabla u(k+1)))$ for $k \in \mathbb{N}_0$, where $\nu = 1 - \mu$. Property 2.(ii) in [18] says that if $\mu \in \mathbb{C}$ and $\nu \notin \mathbb{N}$, then $\Sigma^\mu \Sigma^\nu = \Sigma^{\mu+\nu}$. Hence

$$\begin{aligned}\Sigma^\mu(\nabla_*^\mu u(k+1)) &= \Sigma^1(\nabla u(k+1)) = \sum_{j=0}^k \nabla u(j+1) = u(k+1) - u(0) \\ &= \Sigma^\mu f(k, u(k)) = \frac{1}{\Gamma(\mu)} \sum_{j=0}^k (k-\rho(j))^{\overline{\mu-1}} f(j, u(j)) \\ &= \sum_{j=0}^k \binom{k-j+\mu-1}{k-j} f(j, u(j)) = \sum_{j=0}^k A_\mu(k, j) f(j, u(j)).\end{aligned}$$

This completes the proof. \square

Corollary 3.2. Let $\mu \in (0, 1)$, u, f be real functions defined on \mathbb{N}_0 , and $\mathbb{N}_0 \times \mathbb{R}$, respectively. For any $n \in \mathbb{N}$, if

$$\nabla_*^\mu u(k+1) \leq f(k, u(k)), \quad \forall k = 0, 1, \dots, n-1,$$

then

$$u(k) \leq u(0) + \sum_{j=0}^{k-1} A_\mu(k-1, j) f(j, u(j)), \quad \forall k = 0, 1, \dots, n.$$

Proof. Since $A_\mu(k, j) > 0$ for each $0 \leq j \leq k < n$, the statement immediately follows from definition of Σ^μ . \square

Note that the above corollary holds true with \geq instead of \leq .

Next, we derive a solution of a linear initial value problem. Let $h(k)$ denote the solution of the problem

$$\begin{aligned} \nabla_*^\mu h(k+1) &= a(k)h(k), \quad k \in \mathbb{N}_0 \\ h(0) &= 1, \end{aligned} \tag{3.3}$$

and define a Green function $\{g_j(k)\}_{k \in \mathbb{N}_0, j \in \mathbb{N}_0}$ as

$$\begin{aligned} \nabla_*^\mu g_j(k+1) &= a(k)g_j(k) + \delta_j(k), \quad k \in \mathbb{N}_0 \\ g_j(0) &= 0, \end{aligned} \tag{3.4}$$

where $\delta_j(k) = 0$ for $j \neq k$ and $\delta_j(j) = 1$. Then $v(k) = u_0 h(k)$, $k \in \mathbb{N}_0$ solves

$$\begin{aligned} \nabla_*^\mu v(k+1) &= a(k)v(k), \quad k \in \mathbb{N}_0 \\ v(0) &= u_0, \end{aligned} \tag{3.5}$$

and

$$u(k) = \sum_{j=0}^{k-1} g_j(k) b(j), \quad k \in \mathbb{N}_0 \tag{3.6}$$

solves the equation

$$\begin{aligned} \nabla_*^\mu u(k+1) &= a(k)u(k) + b(k), \quad k \in \mathbb{N}_0 \\ u(0) &= 0. \end{aligned} \tag{3.7}$$

Here we note that $g_j(k) = 0$ for $0 \leq k \leq j$ and $g_j(j+1) = 1$. So setting

$$\tilde{g}_j(k) := g_j(k+1), \quad k = -1, 0, \dots,$$

formula (3.6) becomes

$$u(k) = \sum_{j=0}^{k-1} \tilde{g}_j(k-1) b(j), \quad k \in \mathbb{N}_0, \tag{3.8}$$

while (3.4) gives

$$\begin{aligned} \nabla_*^\mu \tilde{g}_j(k+1) &= a(k+1)\tilde{g}_j(k), \quad k \geq j \\ \tilde{g}_j(j) &= 1, \quad \tilde{g}_j(k) = 0, \quad -1 \leq k < j \end{aligned} \tag{3.9}$$

for $j \in \mathbb{N}_0$. Using $\nabla_*^\mu \tilde{g}_j(j) = 1$, we can directly verify that (3.8) solves (3.4). Thus there are two different ways, (3.6) and (3.8), how to define a solution u of (3.7). The following lemma uses (3.6), and concludes the above arguments.

Lemma 3.3. *The initial value problem*

$$\begin{aligned}\nabla_*^\mu u(k+1) &= a(k)u(k) + b(k), \quad k \in \mathbb{N}_0 \\ u(0) &= u_0\end{aligned}$$

has a solution

$$u(k) = u_0 h(k) + \sum_{j=0}^{k-1} g_j(k) b(j), \quad k \in \mathbb{N}_0.$$

Proof. The considered problem is decomposed to a homogeneous equation with a nontrivial initial condition, of the form (3.5), and a nonhomogeneous equation with a zero initial condition, of the form (3.7). Consequently, the superposition principle is applied. \square

The rest of this section is devoted to the case of constant function $a(k)$.

Proposition 3.4. *Let $\mu \in (0, 1)$, $a, u_0 \in \mathbb{R}$ and u fulfill*

$$\begin{aligned}\nabla_*^\mu u(k+1) &= au(k), \quad k \in \mathbb{N}_0 \\ u(0) &= u_0.\end{aligned}\tag{3.10}$$

Then u has the form

$$u(k) = u_0 \left[1 + \sum_{j=1}^k \sum_{\substack{i_1, i_2, \dots, i_j \geq 0 \\ \sum_{l=1}^j i_l \leq k-j}} \prod_{l=1}^j \frac{a\Gamma(i_l + \mu)}{\Gamma(\mu)\Gamma(i_l + 1)} \right], \quad k \in \mathbb{N}_0.\tag{3.11}$$

Proof. First, we apply Lemma 3.1 to get a corresponding fractional sum equation

$$u(k) = u_0 + \sum_{j=0}^{k-1} A_\mu(k-1, j) au(j), \quad k \in \mathbb{N}_0.\tag{3.12}$$

Note that $A_\mu(k, j) = A_\mu(k + \alpha, j + \alpha) = \frac{\Gamma(k-j+\mu)}{\Gamma(\mu)\Gamma(k-j+1)}$ for any $\alpha \in [-j, \infty)$, $0 \leq j \leq k$. For simplicity, we denote $B_\mu(k-j) = aA_\mu(k, j)$ for $0 \leq j \leq k$, i.e.,

$$B_\mu(n) = \frac{a\Gamma(n + \mu)}{\Gamma(\mu)\Gamma(n + 1)}, \quad n \in \mathbb{N}_0.$$

Consequently,

$$u(k) = u_0 + \sum_{j=0}^{k-1} B_\mu(k-1-j)u(j), \quad k \in \mathbb{N}_0.\tag{3.13}$$

We claim that then

$$u(k) = u_0 \left[1 + \sum_{j=1}^k \sum_{\substack{i_1, i_2, \dots, i_j \geq 0 \\ \sum_{l=1}^j i_l \leq k-j}} \prod_{l=1}^j B_\mu(i_l) \right], \quad k \in \mathbb{N}_0\tag{3.14}$$

what is the statement of the proposition.

We prove the claim by induction with respect to k . If $k = 0$, then due to the empty sum property $u(0) = u_0$ in both (3.13) and (3.14). Now, let us assume that (3.14) holds for $0, 1, \dots, k$, and we show that it is true also for $k + 1$. Using (3.13) and the inductive hypothesis, we have

$$\begin{aligned}
 u(k+1) &= u_0 + \sum_{j=0}^k B_\mu(k-j)u(j) \\
 &= u_0 + \sum_{j=0}^k B_\mu(k-j)u_0 \left[1 + \sum_{q=1}^j \sum_{\substack{i_1, i_2, \dots, i_q \geq 0 \\ \sum_{l=1}^q i_l \leq j-q}} \prod_{l=1}^q B_\mu(i_l) \right] \\
 &= u_0 \left[1 + \underbrace{\sum_{j=0}^k B_\mu(k-j)}_{=:S_1} + \underbrace{\sum_{j=0}^k B_\mu(k-j) \sum_{q=1}^j \sum_{\substack{i_1, i_2, \dots, i_q \geq 0 \\ \sum_{l=1}^q i_l \leq j-q}} \prod_{l=1}^q B_\mu(i_l)}_{=:S_2} \right].
 \end{aligned} \tag{3.15}$$

Changing $k - j + 1 \rightarrow j$, we get

$$S_1 = \sum_{j=1}^{k+1} B_\mu(j-1) = \sum_{j=1}^1 \sum_{0 \leq i_j \leq k} \prod_{l=1}^1 B_\mu(i_j) = \sum_{j=1}^1 \sum_{\substack{i_1, i_2, \dots, i_j \geq 0 \\ \sum_{l=1}^j i_l \leq k+1-j}} \prod_{l=1}^j B_\mu(i_l).$$

Similarly in S_2 :

$$\begin{aligned}
 S_2 &= \sum_{j=1}^{k+1} B_\mu(j-1) \sum_{q=1}^{k+1-j} \sum_{\substack{i_1, i_2, \dots, i_q \geq 0 \\ \sum_{l=1}^q i_l \leq k+1-j-q}} \prod_{l=1}^q B_\mu(i_l) \\
 &= \sum_{j=1}^{k+1} \sum_{q=1}^{k+1-j} B_\mu(j-1) \sum_{\substack{i_1, i_2, \dots, i_q \geq 0 \\ \sum_{l=1}^q i_l + j - 1 \leq k - q}} \prod_{l=1}^q B_\mu(i_l).
 \end{aligned}$$

Now we switch the sums $\sum_{j=1}^{k+1} \sum_{q=1}^{k+1-j} = \sum_{q=1}^{k+1} \sum_{j=1}^{k+1-q}$. Note that the second sum is empty for $q = k + 1$. Hence

$$S_2 = \sum_{q=1}^k \sum_{j=1}^{k+1-q} B_\mu(j-1) \sum_{\substack{i_1, i_2, \dots, i_q \geq 0 \\ \sum_{l=1}^q i_l + j - 1 \leq k - q}} \prod_{l=1}^q B_\mu(i_l).$$

Note that $j - 1$ takes values $0, 1, \dots, k - q$. So we can denote $i_{q+1} = j - 1$ and merge the last two sums to obtain

$$S_2 = \sum_{q=1}^k \sum_{\substack{i_1, i_2, \dots, i_{q+1} \geq 0 \\ \sum_{l=1}^{q+1} i_l \leq k - q}} \prod_{l=1}^{q+1} B_\mu(i_l).$$

Finally, we change $q + 1 \rightarrow q$

$$S_2 = \sum_{q=2}^{k+1} \sum_{\substack{i_1, i_2, \dots, i_q \geq 0 \\ \sum_{l=1}^q i_l \leq k+1-q}} \prod_{l=1}^q B_\mu(i_l),$$

and after summing S_1 and S_2 , (3.14) is obtained for $k + 1$. \square

Now we present an alternative proof without using induction principle.

Alternative proof of Proposition 3.4. If $a = 0$, then by (3.12), the statement is proved. From now on, we assume that $a \neq 0$. By using the formula [19, Problem 7, p. 15]

$$\lim_{n \rightarrow \infty} \frac{\Gamma(n + \mu)}{\Gamma(n + 1)} n^{1-\mu} = 1, \quad (3.16)$$

it is clear that

$$\frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{|B_\mu(n)|}} = 1.$$

So the power series

$$\Delta_\mu(x) = \sum_{n=0}^{\infty} B_\mu(n) x^n$$

has the radius of convergence 1. Furthermore, (3.16) gives that

$$\frac{1}{a} \sum_{n=0}^{\infty} B_\mu(n) = +\infty,$$

and using $\frac{B_\mu(n)}{a} > 0$ we see that

$$\frac{1}{a} \lim_{x \rightarrow 1^-} \Delta_\mu(x) = +\infty. \quad (3.17)$$

Next, we know that the sequence $\{|B_\mu(n)|\}_{n=0}^{\infty}$ is decreasing (see the proof of Lemma 4.6) with $\lim_{n \rightarrow \infty} B_\mu(n) = 0$ (see (3.16)). So the Leibnitz criterion implies the convergence of the series

$$\sum_{n=0}^{\infty} B_\mu(n) (-1)^n,$$

and the Abel theorem [23, p. 9] implies

$$\lim_{x \rightarrow -1^+} \Delta_\mu(x) = \lim_{x \rightarrow 1^-} \sum_{n=0}^{\infty} B_\mu(n) (-1)^n x^n = \sum_{n=0}^{\infty} B_\mu(n) (-1)^n = \Delta_\mu(-1).$$

Next, Lemma 4.6 implies

$$|u(k)| \leq |u_0| (1 + |a|)^k, \quad k \in \mathbb{N}_0.$$

Hence

$$\frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{|u(n)|}} \geq \frac{1}{1 + |a|}, \quad (3.18)$$

thus the power series

$$U(x) = \sum_{n=0}^{\infty} u(n) x^n$$

has the radius of convergence R_U greater than or equal to $\frac{1}{1+|a|}$. Consequently, we start with $0 \neq |x| < \frac{1}{1+|a|}$. Then using (3.13), we derive

$$\begin{aligned}
 U(x)\Delta_\mu(x) &= \left(\sum_{i=0}^{\infty} u(i)x^i \right) \left(\sum_{j=0}^{\infty} B_\mu(j)x^j \right) \\
 &= \sum_{k=1}^{\infty} \sum_{i+j=k-1} u(i)B_\mu(j)x^{k-1} = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} u(j)B_\mu(k-1-j)x^{k-1} \\
 &= \sum_{k=1}^{\infty} (u(k) - u_0)x^{k-1} = \frac{\sum_{k=0}^{\infty} u(k)x^k - u(0)}{x} - \frac{u_0}{1-x} \\
 &= \frac{U(x)}{x} - \frac{u_0}{x(1-x)}.
 \end{aligned} \tag{3.19}$$

Solving (3.19) we obtain

$$U(x) = \frac{u_0}{(1-x)(1-x\Delta_\mu(x))}. \tag{3.20}$$

From (3.20) we obtain

$$\sum_{n=0}^{\infty} u(n)x^n = u_0 \left(\sum_{j=0}^{\infty} x^j \right) \left(\sum_{i=0}^{\infty} x^i \left(\sum_{k=0}^{\infty} B_\mu(k)x^k \right)^i \right). \tag{3.21}$$

By expanding the right-hand side of (3.21) and comparing the powers of x , we immediately get (3.14). \square

If $u_0 = 0$ then $U(x) = 0$. So we suppose that $u_0 \neq 0$. Next, using (3.16) we see that

$$B_\mu(n) \sim \frac{a}{\Gamma(\mu)n^{1-\mu}}$$

as $n \rightarrow \infty$. So by results of [23, pp. 224–225], we have

$$(1-x)\Delta_\mu(x) \sim (1-x) \sum_{n=1}^{\infty} \frac{a}{\Gamma(\mu)n^{1-\mu}} x^n \sim a(1-x)^{1-\mu}$$

as $x \rightarrow 1^-$, so

$$\lim_{x \rightarrow 1^-} (1-x)\Delta_\mu(x) = 0,$$

while we recall (3.17). Then clearly

$$\lim_{x \rightarrow 1^-} |U(x)| = +\infty$$

due to (3.20), thus the radius of convergence R_U of $U(x)$ is less than or equal to 1, and we get

$$\frac{1}{1+|a|} \leq R_U \leq 1. \tag{3.22}$$

On the other hand, we know

$$\begin{aligned}
 |\Delta_\mu(x)| &\leq |a| \left[1 + \frac{1}{\Gamma(\mu)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\mu)}{\Gamma(n+1)} |x|^n \right] \\
 &\leq |a| \left[1 + \frac{1}{\Gamma(\mu)} \sum_{n=1}^{\infty} \frac{|x|^n}{n^{1-\mu}} \right] = |a| \left[1 + \frac{1}{\Gamma(\mu)} \text{Li}_{1-\mu}(|x|) \right]
 \end{aligned}$$

where we applied the estimation of the ratio of gamma functions

$$\frac{\Gamma(n + \mu)}{\Gamma(n + 1)} \leq \frac{1}{n^{1-\mu}}, \quad n \in \mathbb{N}, \mu \in (0, 1)$$

from [17], and used the notation

$$\text{Li}_\nu(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{xt^{\nu-1}}{e^t - x} dt, \quad x, \nu \in (0, 1)$$

for the polylogarithm function [20, Section 7.12]. Consequently,

$$|1 - x\Delta_\mu(x)| \geq 1 - |x|\Delta_\mu(x) \geq 1 - F(|x|) \quad (3.23)$$

for

$$F(x) = x|a| \left[1 + \frac{1}{\Gamma(\mu)} \text{Li}_{1-\mu}(x) \right].$$

We note that F is increasing on $(0, 1)$. The right-hand side of (3.23) is positive if and only if $|x| < F^{-1}(1)$. Note that $F^{-1}(1) > \frac{1}{1+|a|}$. To see this, we estimate

$$\text{Li}_{1-\mu} \left(\frac{1}{1+|a|} \right) < \sum_{n=1}^{\infty} \left(\frac{1}{1+|a|} \right)^n = \frac{1}{|a|} < \frac{\Gamma(\mu)}{|a|},$$

i.e.,

$$\frac{|a|}{1+|a|} \left[1 + \frac{1}{\Gamma(\mu)} \text{Li}_{1-\mu} \left(\frac{1}{1+|a|} \right) \right] = F \left(\frac{1}{1+|a|} \right) < 1.$$

This improves (3.22) to

$$F^{-1}(1) \leq R_U \leq 1. \quad (3.24)$$

Summarizing the above arguments, we obtain the next result.

Proposition 3.5. *The solution $u(n)$ of initial value problem (3.10) with $u_0 \neq 0$ fulfills*

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|u(n)|} = \frac{1}{R_U} \quad (3.25)$$

for R_U satisfying (3.24).

To get a better result, we note if $a > 0$ then $x\Delta_\mu(x)$ is increasing on $[0, 1)$ from 0 to $+\infty$. So for any $a > 0$, there is a unique $1 > r_{a,\mu} > F^{-1}(1)$ solving equation

$$1 = r_{a,\mu}\Delta_\mu(r_{a,\mu}).$$

Then $|1 - x\Delta_\mu(x)| > 0$ for any $|x| < r_{|a|,\mu}$. Hence (3.24) is improved to

$$r_{|a|,\mu} \leq R_U \leq 1. \quad (3.26)$$

Note that estimation (3.24) as well as (3.26) gives better estimate on the asymptotic property (3.25) than (3.18) derived from Lemma 4.6. Moreover, Proposition 3.5 yields the next corollary.

Corollary 3.6. *If the solution $u(n)$ of (3.10) with $u_0 \neq 0$ satisfies $u(n) \rightarrow 0$ as $n \rightarrow \infty$, then the rate of convergence is slower than any exponential one, i.e., there are no constants $c_1 > 0$ and $\omega \in (0, 1)$ so that $|u(n)| \leq c_1\omega^n$ for any $n \in \mathbb{N}_0$.*

Proof. In contrary, if $c_1 > 0$, $\varpi \in (0, 1)$ are such that $|u(n)| \leq c_1 \varpi^n \forall n \in \mathbb{N}_0$, then

$$R_U = \frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{|u(n)|}} \geq \frac{1}{\varpi} > 1$$

what contradicts (3.26). □

On the other hand, if $a > 0$, $u_0 \neq 0$ then

$$\frac{1}{u_0} \lim_{x \rightarrow r_{a,\mu}^-} U(x) = +\infty,$$

so $R_U = r_{a,\mu}$, and thus $\{u(n)\}_{n \in \mathbb{N}_0}$ is unbounded satisfying (3.25). A related result is derived in [7, Theorem 5.1].

Remark 3.7. Taking limit $\mu \rightarrow 1^-$ in (3.10) gives

$$\begin{aligned} \nabla u(k+1) &= au(k), \quad k \in \mathbb{N}_0 \\ u(0) &= u_0 \end{aligned}$$

which has solution $u(k) = u_0(1+a)^k$. On the other hand, from (3.11), we get

$$\begin{aligned} \lim_{\mu \rightarrow 1^-} u(k)c &= u_0 \left[1 + \sum_{j=1}^k a^j \sum_{\substack{i_1, i_2, \dots, i_j \geq 0 \\ \sum_{l=1}^j i_l \leq k-j}} 1 \right] \\ &= u_0 \left[1 + \sum_{j=1}^k a^j \sum_{m=0}^{k-j} \sum_{\substack{i_1, i_2, \dots, i_j \geq 0 \\ \sum_{l=1}^j i_l = m}} 1 \right] = u_0 \left[1 + \sum_{j=1}^k a^j \sum_{m=0}^{k-j} \binom{m+j-1}{j-1} \right] \\ &= u_0 \left[1 + \sum_{j=1}^k \binom{k}{j} a^j \right] = u_0(1+a)^k. \end{aligned}$$

So we call the bracket in (3.11), the generalized binomial. Note that linear Riemann–Liouville fractional difference equations are solved in [7] leading to discrete Mittag-Leffler functions.

Next, we derive a formula for a Green function satisfying (3.4) with constant a .

Proposition 3.8. Let $\mu \in (0, 1)$, $a \in \mathbb{R}$. Then for any $i \in \mathbb{N}_0$, the Green function $\{g_i^c(k)\}_{k \in \mathbb{N}_0}$ satisfying

$$\begin{aligned} \nabla_*^\mu g_i^c(k+1) &= ag_i^c(k) + \delta_i(k), \quad k \in \mathbb{N}_0, \\ g_i^c(0) &= 0 \end{aligned} \tag{3.27}$$

has the form

$$g_i^c(k) = \sum_{j=1}^{k-i} a^{j-1} \sum_{\substack{i_1, i_2, \dots, i_j \geq 0 \\ \sum_{l=1}^j i_l = k-i-j}} \prod_{l=1}^j \frac{\Gamma(i_l + \mu)}{\Gamma(\mu)\Gamma(i_l + 1)}, \quad k \in \mathbb{N}_0. \tag{3.28}$$

Proof. Let $i \in \mathbb{N}_0$ be arbitrary and fixed. As in the proof of Proposition 3.4, applying Lemma 3.1, one can see that

$$g_i^c(k) = \sum_{j=0}^{k-1} (B_\mu(k-1-j)g_i^c(j) + A_\mu(k-1,j)\delta_i(j)), \quad k \in \mathbb{N}_0. \quad (3.29)$$

In particular, $g_i^c(0) = 0$ which agrees with (3.28).

Let (3.28) be valid at $0, 1, \dots, k$. Then by (3.29),

$$g_i^c(k+1) = \underbrace{\sum_{j=0}^k B_\mu(k-j) \sum_{q=1}^{j-i} a^{q-1} \sum_{\substack{i_1, i_2, \dots, i_q \geq 0 \\ \sum_{l=1}^q i_l = j-i-q}} \prod_{l=1}^q C_\mu(i_l)}_{=:S} + \sum_{j=0}^k C_\mu(k-j)\delta_i(j) \quad (3.30)$$

where $C_\mu(k-j) = A_\mu(k, j)$. There are three possible cases. If $i \notin \{0, 1, \dots, k\}$, $g_i^c(k+1) = 0$ due to the empty sum property. The same is true for (3.28).

If $i = k$, then $S = 0$ and (3.30) implies

$$g_i^c(k+1) = \sum_{j=0}^k C_\mu(k-j)\delta_i(j) = C_\mu(0) = 1.$$

The same holds for (3.28).

Finally, if $i \in \{0, 1, \dots, k-1\}$, then

$$\begin{aligned} S &= \sum_{j=i+1}^k \sum_{q=1}^{j-i} a^q C_\mu(k-j) \sum_{\substack{i_1, i_2, \dots, i_q \geq 0 \\ \sum_{l=1}^q i_l = j-i-q}} \prod_{l=1}^q C_\mu(i_l) \\ &= \sum_{q=1}^{k-i} \sum_{j=q+i}^k a^q C_\mu(k-j) \sum_{\substack{i_1, i_2, \dots, i_q \geq 0 \\ \sum_{l=1}^q i_l = j-i-q}} \prod_{l=1}^q C_\mu(i_l). \end{aligned}$$

Note that $k-j$ takes the values $0, 1, \dots, k-q-i$. So we denote $i_{q+1} = k-j$ and write

$$S = \sum_{q=1}^{k-i} a^q \sum_{\substack{i_1, i_2, \dots, i_{q+1} \geq 0 \\ \sum_{l=1}^{q+1} i_l = k-i-q}} \prod_{l=1}^{q+1} C_\mu(i_l).$$

On substitute $q+1 \rightarrow q$, we get

$$S = \sum_{q=2}^{k+1-i} a^{q-1} \sum_{\substack{i_1, i_2, \dots, i_q \geq 0 \\ \sum_{l=1}^q i_l = k+1-i-q}} \prod_{l=1}^q C_\mu(i_l)$$

and (3.30) becomes (3.28). □

4 Fractional difference inequalities

In this section, we explain where the problem lies of the fractional Gronwall inequality established in [11]. Then we propose our alternative fractional difference inequalities of the

Gronwall type that can be used instead of the original one. For a better clarity, we use quotation marks when recalling the false result. At the end of the section, an example of linear fractional difference equation with a given initial condition from [14], is given. On this example, we illustrate different estimates of its solution.

First, we recall a discrete Gronwall lemma (see e.g. [22, Lemma 1.4.2]).

Lemma 4.1. *Let u, p, q be real functions defined on \mathbb{N}_0 , and p be nonnegative. For any $n \in \mathbb{N}_0$, if*

$$u(k) \leq u(0) + \sum_{j=0}^{k-1} (p(j)u(j) + q(j)), \quad \forall k = 0, 1, \dots, n,$$

then

$$u(k) \leq u(0) \prod_{j=0}^{k-1} (1 + p(j)) + \sum_{j=0}^{k-1} q(j) \prod_{l=j+1}^{k-1} (1 + p(l)), \quad \forall k = 0, 1, \dots, n.$$

Now, we recall the questionable “lemma” from [11].

“Lemma” 4.2. *Let $\mu \in (0, 1)$, and u, a, b be real nonnegative functions defined on \mathbb{N}_0 . If*

$$\nabla_*^\mu u(k+1) \leq a(k)u(k) + b(k), \quad k \in \mathbb{N}_0, \quad (4.1)$$

then

$$\begin{aligned} u(k) &\leq u(0) \prod_{j=0}^{k-1} (1 + A_\mu(k-1, j)a(j)) \\ &\quad + \sum_{j=0}^{k-1} A_\mu(k-1, j)b(j) \prod_{l=j+1}^{k-1} (1 + A_\mu(k-1, l)a(l)), \quad k \in \mathbb{N}_0. \end{aligned}$$

Originally, inequality (4.1) is transformed to a corresponding sum equation using the properties of operator ∇_*^μ . However, from Section 2 we know that this does not have to be valid. Next, $A_\mu(k, j)$ is considered only as a function of j and Lemma 4.1 is applied with $p(j) = A_\mu(k-1, j)a(j)$ and $q(j) = A_\mu(k-1, j)b(j)$. This can cause another problem, since in general, the discrete Gronwall inequality does not hold with p, q depending on k . To prove that, it really makes a problem and “Lemma” 4.2 is not correct, we provide the following counterexample.

Example 4.3. Let $u(0) = u_0 > 0$, $u(1) = 2u_0$ and $u(2) = (3 + \mu)u_0$ for $\mu \in (0, 1)$. Consider $\{u(k)\}_{k \in \mathbb{N}_3}$ such that (4.1) holds for $k \in \mathbb{N}_2$. For $k = 0$, we have

$$\begin{aligned} [\nabla_*^\mu u(k)]_{k=1} &= \left[\sum_{j=0}^{k-1} \binom{j-\mu}{j} \nabla_k u(k-j) \right]_{k=1} \\ &= \binom{-\mu}{0} ((u(1) - u(0)) = u_0 \leq u(0), \end{aligned}$$

and for $k = 1$,

$$\begin{aligned} [\nabla_*^\mu u(k)]_{k=2} &= \left[\sum_{j=0}^{k-1} \binom{j-\mu}{j} \nabla_k u(k-j) \right]_{k=2} \\ &= \binom{-\mu}{0} ((u(2) - u(1)) + \binom{1-\mu}{1} (u(1) - u(0))) \\ &= (1 + \mu)u_0 + \frac{\Gamma(2-\mu)}{\Gamma(2)\Gamma(1-\mu)} u_0 = 2u_0 \leq u(1). \end{aligned}$$

Thus (4.1) is satisfied for each $k \in \mathbb{N}_0$ with $a(0) = a(1) = 1$, $b(0) = b(1) = 0$, and all the assumptions of “Lemma” 4.2 are satisfied. But the statement does not hold for $k = 2$, since

$$\begin{aligned} u_0 \prod_{j=0}^1 (1 + A_\mu(1, j)) &= u_0 \left(1 + \binom{\mu}{1}\right) \left(1 + \binom{\mu-1}{0}\right) \\ &= (2 + 2\mu)u_0 < (3 + \mu)u_0 = u(2). \end{aligned}$$

Instead of “Lemma” 4.2, we provide the following fractional difference inequalities of the Gronwall type.

Lemma 4.4. *Let $\mu \in (0, 1)$, u, a, b be real functions defined on \mathbb{N}_0 , and a be nonnegative. For any $n \in \mathbb{N}$, if*

$$\nabla_*^\mu u(k+1) \leq a(k)u(k) + b(k), \quad \forall k = 0, 1, \dots, n-1,$$

then

$$u(k) \leq u(0)h(k) + \sum_{j=0}^{k-1} g_j(k)b(j), \quad k = 0, 1, \dots, n$$

where h solves (3.3) and g_j is a solution of (3.4).

Proof. Let $v(k)$ denote a solution of the initial value problem

$$\begin{aligned} \nabla_*^\mu v(k+1) &= a(k)v(k) + b(k), \quad k \in \mathbb{N}_0 \\ v(0) &= u(0). \end{aligned}$$

Then by Lemma 3.3,

$$v(k) = u(0)h(k) + \sum_{j=0}^{k-1} g_j(k)b(j)$$

for $k = 0, 1, \dots, n$. So it is sufficient to prove $u(k) \leq v(k)$ for each $k = 0, 1, \dots, n$. Clearly, it is true for $k = 0$. Let it be valid for $1, 2, \dots, k$. Then by Corollary 3.2 and Lemma 3.1,

$$\begin{aligned} u(k+1) &\leq u(0) + \sum_{j=0}^k A_\mu(k, j)(a(j)u(j) + b(j)) \\ &\leq u(0) + \sum_{j=0}^k A_\mu(k, j)(a(j)v(j) + b(j)) = v(k+1). \end{aligned}$$

Hence the lemma is proved. □

Lemma 4.5. *Let $\mu \in (0, 1)$, $a > 0$, u, b be real functions defined on \mathbb{N}_0 . For any $n \in \mathbb{N}$, if*

$$\nabla_*^\mu u(k+1) \leq au(k) + b(k), \quad \forall k = 0, 1, \dots, n-1,$$

then

$$\begin{aligned} u(k) &\leq u(0) \left[1 + \sum_{j=1}^k \sum_{\substack{i_1, i_2, \dots, i_j \geq 0 \\ \sum_{l=1}^j i_l \leq k-j}} \prod_{l=1}^j \frac{a\Gamma(i_l + \mu)}{\Gamma(\mu)\Gamma(i_l + 1)} \right] \\ &\quad + \sum_{i=0}^{k-1} b(i) \sum_{j=1}^{k-i} a^{j-1} \sum_{\substack{i_1, i_2, \dots, i_j \geq 0 \\ \sum_{l=1}^j i_l = k-i-j}} \prod_{l=1}^j \frac{\Gamma(i_l + \mu)}{\Gamma(\mu)\Gamma(i_l + 1)}, \quad k = 0, 1, \dots, n \end{aligned}$$

Proof. The lemma can be proved as Lemma 4.4 using Propositions 3.4 and 3.8, and Lemma 3.3. \square

Next, we state an estimation independent of μ .

Lemma 4.6. *If the assumptions of Lemma 4.4 are satisfied, then*

$$u(k) \leq u(0) \prod_{j=0}^{k-1} (1 + a(j)) + \sum_{j=0}^{k-1} b(j) \prod_{l=j+1}^{k-1} (1 + a(l)), \quad \forall k = 0, 1, \dots, n.$$

Proof. Corollary 3.2 is applied to obtain the sum inequality

$$u(k) \leq u(0) + \sum_{j=0}^{k-1} A_{\mu}(k-1, j)(a(j)u(j) + b(j)), \quad \forall k = 0, 1, \dots, n. \quad (4.2)$$

Let us define a function g as

$$g(x) = \frac{\Gamma(x + \mu)}{\Gamma(x + 1)}$$

for $x \geq 0$. We claim that g is decreasing on $[0, \infty)$. Clearly, g is positive on the whole domain. Next, $g'(x) = (\Psi(x + \mu) - \Psi(x + 1))g(x)$ where Ψ is a logarithmic derivative of gamma function (psi function or digamma function, see e.g. [19]) which satisfies [2, 6.4.10]

$$\Psi'(x) = \sum_{j=0}^{\infty} \frac{1}{(x+j)^2} > 0, \quad x > 0.$$

Thus indeed, $g'(x) < 0$ for $x \geq 0$. Consequently, since $A_{\mu}(k-1, j) = \frac{g^{(k-1-j)}}{\Gamma(\mu)}$, then $A_{\mu}(\cdot, j)$ is decreasing on $[j, \infty)$, and $A_{\mu}(k-1, \cdot)$ is increasing on $[1, k-1]$. So we get two estimates

$$A_{\mu}(k-1, j) \leq A_{\mu}(j, j), \quad A_{\mu}(k-1, j) \leq A_{\mu}(k-1, k-1)$$

for $1 \leq j \leq k-1$. It does not matter which we use, as both give the same result $A_{\mu}(j, j) = A_{\mu}(k-1, k-1) = 1$. Hence by (4.2),

$$u(k) \leq u(0) + \sum_{j=0}^{k-1} (a(j)u(j) + b(j)), \quad \forall k = 0, 1, \dots, n.$$

Finally, the statement follows by Lemma 4.1. \square

Remark 4.7. Lemmas 4.4, 4.5 and 4.6 remain valid when \leq is replaced with \geq .

Example 4.8. Consider the following initial value problem

$$\begin{aligned} \nabla_{*}^{\frac{1}{2}} u(k+1) &= u(k), \quad k \in \mathbb{N}_0 \\ u(0) &= 1. \end{aligned} \quad (4.3)$$

By Lemma 3.1,

$$u(k) = u(0) + \sum_{j=0}^{k-1} A_{\frac{1}{2}}(k-1, j)u(j), \quad k \in \mathbb{N}_0. \quad (4.4)$$

In Figure 4.1, one can compare values of the solution of (4.3), its estimate

$$u(k) \leq 2^k, \quad k \in \mathbb{N}_0 \quad (4.5)$$

obtained from Lemma 4.6, the false estimate

$$u(k) \leq \prod_{j=0}^{k-1} (1 + A_{\frac{1}{2}}(k-1, j)), \quad k \in \mathbb{N}_0 \quad (4.6)$$

given by “Lemma” 4.2, and the following one presented in [14]

$$u(k) \leq \exp \left(\sum_{j=0}^{k-1} A_{\frac{1}{2}}(k-1, j) \right), \quad k \in \mathbb{N}_0. \quad (4.7)$$

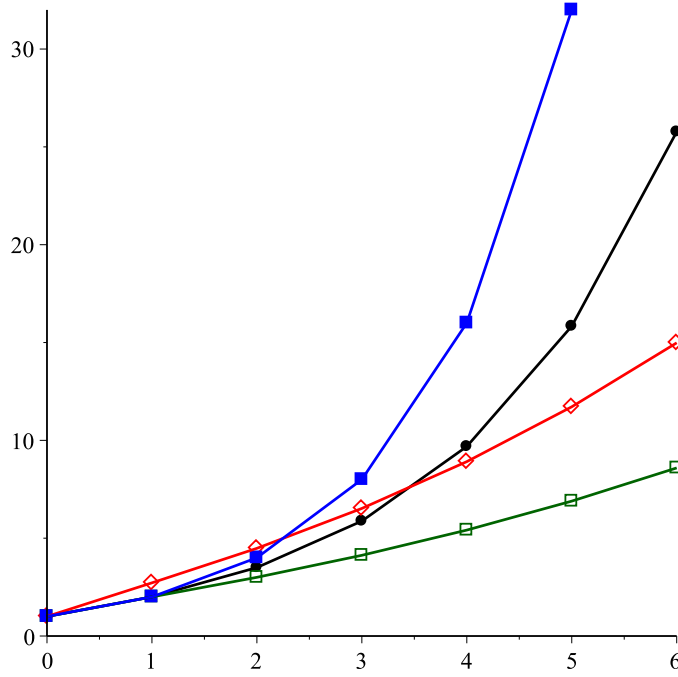


Figure 4.1: Solution of (4.3) – black with points, estimate (4.5) – blue with solid boxes, estimate (4.6) – green with boxes, estimate (4.7) – red with diamonds.

By the way, the solution of (4.3) is in [14] stated as

$$u(k) = \prod_{j=0}^{k-1} (1 + A_{\frac{1}{2}}(k-1, j)),$$

which obviously does not satisfy (4.4) (one can set $k = 2$ and see the difference). Nevertheless, because of the simple form of (4.3), its solution can be explicitly calculated using Proposition 3.4 to get

$$u(k) = 1 + \sum_{j=1}^k \sum_{\substack{i_1, i_2, \dots, i_j \geq 0 \\ \sum_{l=1}^j i_l \leq k-j}} \prod_{l=1}^j \frac{\Gamma(i_l + \frac{1}{2})}{\sqrt{\pi} \Gamma(i_l + 1)}, \quad k \in \mathbb{N}_0.$$

Remark 4.9. We recall that Gronwall type inequalities are derived in [1, 8, 16] for different fractional difference inequalities from our paper. More general inequalities are presented in [4].

5 Conclusion

In this note, we showed by Example 4.3 that the Caputo like ∇ -based fractional Gronwall lemma (“Lemma” 4.2) is false. By this, many proofs of the recent results of Deekshitulu and Mohan are discarded, as they are based on the “lemma”. This should motivate authors to carefully and critically approach any results published in more or less esteemed journals.

One can try to find simple counterexamples to show that the inequalities from [12, 13, 15] are false as well, or use the proposed Lemma 4.6 to obtain valid results.

We note that in [10], the authors made a similar fault as in [11] – they neglected the dependence of a summed function on independent variable in discrete Langenhop inequality without any explanation. So one can doubt the correctness of this step. So we hope that our approach and results could help these authors improve their results.

On the other hand, their results were stimulations for us to propose new ∇ -based fractional difference inequalities of the Gronwall type for constant or variable coefficients, which have been not yet studied. We also dealt with related linear fractional difference equations.

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